The $\varepsilon$-Capacity Region of AWGN Multiple Access Channels with Feedback

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(Joint work with Lan V. Truong and Silas L. Fong)

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Shannon Centenary:

Shannon abstracted away information meaning, "semantics" • treat all data equally — bits as a "universal currency" • crucial abstraction for modern communication and computing systems

Also relaxed computation and delay constraints to discover a fundamental limit: capacity, providing a goal-post to work toward

For a channel \( \{ p(y|x) : x \in X, y \in Y \} \), we can transmit information with rates up to the capacity [Shannon (1948)]

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Vincent Tan (NUS)
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“Feedback doesn’t increase capacity” [Shannon (1956)]
At time $i = 1, 2, ..., n$, the channel input and output are related by $Y_i = gX_i + Z_i$, where $Z_i \sim \mathcal{N}(0, 1)$.

Send $M$ messages encoded as codewords $\{X_n^m\} : m = 1, ..., M$.

Peak power constraint:
$$\sum_{i=1}^{n} X_i^2(m) \leq P, \quad \forall m \in \{1, ..., M\}$$

Expected or Long-Term power constraint:
$$\sum_{m=1}^{M} \left( \sum_{i=1}^{n} X_i^2(m) \right) \leq P.$$
At time $i = 1, 2, ..., n$, the channel input and output are related by:

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Define

$$M^*_{PP}(n, P, \varepsilon) := \max \left\{ M \in \mathbb{N} : \exists \text{ length-}n \text{ code with} \right.$$

$$M \text{ codewords and } P_e^{(n)} \leq \varepsilon \text{ under the PP constraint} \left. \right\}$$
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Define

$$M^*_\text{LT}(n, P, \varepsilon) := \max \left\{ M \in \mathbb{N} : \exists \text{ length-}n \text{ code with} \right. \left. M \text{ codewords and } P_e^{(n)} \leq \varepsilon \text{ under the LT constraint} \right\}$$
Let

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First-Order Results

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\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log M^*_\text{LT}(n, P, \varepsilon) = C(P).
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  \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log \mathcal{M}_{PP}^*(n, P, \varepsilon) = C(P), \\
  \lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \frac{1}{n} \log \mathcal{M}_{LT}^*(n, P, \varepsilon) = C(P).
  \]

- In \( n \) channel uses, can send up to \( nC(P) \) nats over \( p(y|x) \) reliably.
If we do not demand that the avg error prob. vanishes 
[Yoshihara (1964), Polyanskiy-Poor-Verdú (2010)],

\[
\lim_{n \to \infty} \frac{1}{n} \log M^*_{PP}(n, P, \varepsilon) = C(P)
\]

\[
\lim_{n \to \infty} \frac{1}{n} \log M^*_{LT}(n, P, \varepsilon) = C\left(\frac{P}{1 - \varepsilon}\right), \quad \forall \varepsilon \in (0, 1).
\]
If we do not demand that the avg error prob. vanishes \newline \cite{Yoshihara1964, Polyanskiy2010},

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First-Order Results

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- The above limits are known as the **\(\varepsilon\)-capacities**

- Since for **peak-power**, the **\(\varepsilon\)**-capacity does not depend on \(\varepsilon\), the strong converse holds
First-Order Results

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- Since for long-term, the $\varepsilon$-capacity depends on $\varepsilon$, the strong converse does not hold
Strong Converse?

\[ \varepsilon = \lim_{n \to \infty} P_e^{(n)}, \quad R = \lim_{n \to \infty} \frac{1}{n} \log M \]
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Higher-Order Results

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- Third-order [Polyanskiy-Poor-Verdú (2010), T.-Tomamichel (2015)],

\[
\log M^*_{PP}(n, P, \varepsilon) = nC(P) + \sqrt{nV(P)}\Phi^{-1}(\varepsilon) + \frac{1}{2} \log n + O(1)
\]

where the channel dispersion is

\[
V(x) := \frac{x(x + 2)}{2(x + 1)^2}
\]

squared nats per ch. use

and

\[
\Phi(a) := \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \, dt.
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Higher-Order Results

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  \]
- Second-order [Yang-Caire-Durisi-Polyanskiy (2015)]
  \[
  \log M^*_\text{LT}(n, P, \varepsilon) = nC\left(\frac{P}{1 - \varepsilon}\right) - \sqrt{V\left(\frac{P}{1 - \varepsilon}\right)}\sqrt{n \log n} + o(\sqrt{n}).
  \]
Feedback helps to simplify coding schemes
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Long-term power constraint under feedback

\[
\frac{1}{M} \sum_{m=1}^{M} \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} [X_i^2(m, Y^{i-1})] \right) \leq P.
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Non-asymptotic fundamental limit

\[
M_{FB}^*(n, P, \varepsilon) := \max \left\{ M \in \mathbb{N} : \exists \text{ length-}n \text{ code with } M \text{ codewords and } P_e^{(n)} \leq \varepsilon \text{ under the LT-FB constraint} \right\}
\]
Feedback: Existing Results

- First-order [Shannon (1956)]

\[
\lim_{{\varepsilon \downarrow 0}} \lim_{{n \to \infty}} \frac{1}{n} \log M^*_{{FB}}(n, P, \varepsilon) = C(P).
\]

Schalkwijk and Kailath (1966) demonstrated a simple coding scheme based on estimation-theoretic ideas to show that

\[
P(n) \leq 2 \exp \left( -\frac{2}{n} (C(P) - R) \right),
\]

for

\[
R = \frac{1}{n} \log M^*_{{FB}}(n, P, \varepsilon).
\]

Error exponent is infinity.

Suggests that the fixed-error results can also be drastically improved.
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- Schalkwijk and Kailath (1966) demonstrated a simple coding scheme based on estimation-theoretic ideas to show that

\[
P_e^{(n)}(R) \leq 2 \exp \left( - \frac{2^{2n(C(P) - R)}}{2n} \right), \quad \text{for} \quad R = \frac{1}{n} \log M < C(P).
\]
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- Suggests that the fixed-error results can also be **drastically improved**
Theorem (Truong-Fong-T. (ISIT 2016))

For the direct part,

\[
\log M_{FB}^*(n, P, \varepsilon) \geq nC \left( \frac{P}{1 - \varepsilon} \right) - \log \log n + O(1).
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For the converse part

\[
\log M_{FB}^*(n, P, \varepsilon) \leq nC\left(\frac{P}{1 - \varepsilon}\right) + \sqrt{V\left(\frac{P}{1 - \varepsilon}\right)} \sqrt{n \log n} + O(\sqrt{n}).
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$$\log M^*_{FB}(n, P, \varepsilon) \leq nC\left(\frac{P}{1 - \varepsilon}\right) + \sqrt{V\left(\frac{P}{1 - \varepsilon}\right)}\sqrt{n\log n} + O(\sqrt{n}).$$

From these results, the $\varepsilon$-capacity is

$$\lim_{n \to \infty} \frac{1}{n} \log M^*_{FB}(n, P, \varepsilon) = C\left(\frac{P}{1 - \varepsilon}\right).$$
\[ \lim_{n \to \infty} \frac{1}{n} \log M^*_\text{FB}(n, P, \varepsilon) = C\left(\frac{P}{1 - \varepsilon}\right). \]

Feedback doesn’t improve the first-order term since

\[ \lim_{n \to \infty} \frac{1}{n} \log M^*_\text{LT}(n, P, \varepsilon) = C\left(\frac{P}{1 - \varepsilon}\right) \]
AWGN Channels with Feedback: Remarks

\[
\lim_{n \to \infty} \frac{1}{n} \log M_{FB}^*(n, P, \varepsilon) = C \left( \frac{P}{1 - \varepsilon} \right).
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\lim_{n \to \infty} \frac{1}{n} \log M_{LT}^*(n, P, \varepsilon) = C \left( \frac{P}{1 - \varepsilon} \right)
\]

With feedback, second-order term is at least

\[- \log \log n + O(1).\]

This is a great improvement over without feedback where the second-order term is [Yang-Caire-Durisi-Polyanskiy (2015)]

\[- \sqrt{\mathcal{V} \left( \frac{P}{1 - \varepsilon} \right)} \sqrt{n \log n} + o(\sqrt{n}).\]
Proof Idea for the Direct Part

- Partition msg set \( \{1, \ldots, M\} \) into \( A_1 \uplus A_2 \).
Proof Idea for the Direct Part

- Partition msg set \( \{1, \ldots, M\} \) into \( A_1 \cup A_2 \).
- \( A_1 \): Send \((0, 0, \ldots, 0) \in \mathbb{R}^n\)
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- \( A_2 \): Schalkwijk-Kailath (1966) scheme \( M' = |A_2| \approx (1 - \varepsilon)M \) msg

\[
P_{e}^{(n)}(R'_{n} | A_2) \leq \frac{1}{n}, \quad \text{where} \quad R'_n := \frac{1}{n} \log M'.
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P_e^{(n)}(R_n' \mid A_2) \leq \frac{1}{n}, \text{ where } R_n' := \frac{1}{n} \log M'.
\]

Choose \( \log M' = nC \left( \frac{P}{1 - \varepsilon} \right) - \log \log n + O_\varepsilon(1) \)

where \( -\log \log n \) because of double exponential decay of \( P_e^{(n)}(R) \)
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- Hence,

\[
\Pr_e^n = \Pr(A_1)\Pr_e^n(A_1) + \Pr(A_2)\Pr_e^n(A_2) \leq \varepsilon \cdot 1 + (1 - \varepsilon)\frac{1}{n} \approx \varepsilon.
\]
Proof Idea for the Converse Part

- Convert **expected long-term power** to a **peak-power** code.
Proof Idea for the Converse Part

- Convert **expected long-term power** to a **peak-power code**.

- **Key observation**

  \[ \exists \text{ LT-FB code } \{X_i(\cdot , \cdot)\}_{i=1}^n \text{ with } M \text{ msgs and } P_e^{(n)} \leq \varepsilon \]

  \[ \implies \exists \text{ PP-FB code } \{X_i'(\cdot , \cdot)\}_{i=1}^n \text{ with } M \text{ msgs and } P_e^{(n)} \leq 1 - \frac{1}{\sqrt{n}} \]
Proof Idea for the Converse Part

■ Convert expected long-term power to a peak-power code.

■ Key observation

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with

\[ \frac{1}{n} \sum_{i=1}^{n} (X'_i(M, Y^{i-1}))^2 \leq \frac{P}{1 - \varepsilon - \frac{1}{\sqrt{n}}} \quad \text{a.s.} \]
Proof Idea for the Converse Part

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\[ \exists \text{ LT-FB code } \{X_i(\cdot, \cdot)\}_{i=1}^n \text{ with } M \text{ msges and } P_e^{(n)} \leq \varepsilon \]

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\[ \frac{1}{n} \sum_{i=1}^{n} \left( X'_i(M, Y^{i-1}) \right)^2 \leq \frac{P}{1 - \varepsilon - \frac{1}{\sqrt{n}}} \quad \text{a.s.} \]

- Exploit connection between binary hypothesis testing and channel coding with feedback under peak-power constraint

[Polyanskiy-Poor-Verdú (2011)] [Fong-T. (2015)]
MACs and Gaussian MACs

- The multiple access channel (MAC)

![Diagram of multiple access channel](image-url)
The multiple access channel (MAC)

\[ M_1 \rightarrow \text{Encoder 1} \rightarrow X_1^n \rightarrow p(y|x_1, x_2) \rightarrow Y^n \rightarrow \text{Decoder} \rightarrow (\hat{M}_1, \hat{M}_2) \]

The Gaussian multiple access channel

Again assume \( g_1 = g_2 = 1 \).
Capacity Region for the Gaussian MAC

\[ R_1 \leq C(P_1) \]

\[ R_2 \leq C(P_2) \]

\[ R_1 + R_2 \leq C(P_1 + P_2) \]
Gaussian MAC with Feedback

Consider Gaussian version with expected long-term power constraints

\[ E[X_1^2(M_1, Y_{i-1})] \leq P_1 \]

\[ E[X_2^2(M_2, Y_{i-1})] \leq P_2 \]

Vincent Tan (NUS)

AWGN MACs with Feedback

SPCOM 2016
Consider Gaussian version with expected long-term power constraints

$$\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ X_{1i}^2(M_1, Y_{i-1}) \right] \leq P_1, \quad \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ X_{2i}^2(M_2, Y_{i-1}) \right] \leq P_2.$$
Ozarow (1984) showed that the capacity region is

\[ \mathcal{R}_{\text{Ozarow}}(P_1, P_2) := \bigcup_{0 \leq \rho \leq 1} \left\{ (R_1, R_2) \mid \begin{array}{l}
R_1 \leq C((1 - \rho^2)P_1), \\
R_2 \leq C((1 - \rho^2)P_2), \\
R_1 + R_2 \leq C\left(P_1 + P_2 + 2\rho\sqrt{P_1P_2}\right) \end{array} \right\}. \]
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With feedback, capacity region is enlarged!
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\]

- With feedback, capacity region is **enlarged**!
- It appears that transmitters can **cooperate**!
Ozarow (1984) showed that the capacity region is

$$\mathcal{R}_{\text{Ozarow}}(P_1, P_2) := \bigcup_{0 \leq \rho \leq 1} \left\{ (R_1, R_2) \mid \begin{align*}
R_1 &\leq C((1 - \rho^2)P_1), \\
R_2 &\leq C((1 - \rho^2)P_2), \\
R_1 + R_2 &\leq C(P_1 + P_2 + 2\rho\sqrt{P_1P_2})
\end{align*} \right\}. $$

With feedback, capacity region is enlarged!

It appears that transmitters can cooperate!

Direct part is an extension of the Schalkwijk and Kailath coding scheme.
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

No feedback
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

$\rho = 0$

$R_{CW}$

$R_1$

$R_2$
CR of the G-MAC with Feedback \( P_1 = P_2 = 1 \)

\[ \rho = 0.1 \]
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

$\rho = 0.2$

$R_{CW}$
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

\[ \rho = 0.3 \]
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

\[ \rho = 0.4 \]
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

\[
\rho = 0.5
\]
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

$\rho = 0.6$
CR of the G-MAC with Feedback $P_1 = P_2 = 1$

\[ \rho = 1.0 \]
The Ozarow region
Similarly to the single-user case, extend to non-vanishing errors.
- Capacity Region of the G-MAC with Feedback

- Similarly to the single-user case, extend to non-vanishing errors

- \((R_1, R_2)\) is \(\varepsilon\)-achievable

\[
\lim_{n \to \infty} \frac{1}{n} \log M_1 \geq R_1 \quad \lim_{n \to \infty} \frac{1}{n} \log M_2 \geq R_2,
\]

and the average probability of error

\[
\lim_{n \to \infty} P_e^{(n)} \leq \varepsilon.
\]
Similarly to the single-user case, extend to non-vanishing errors.

\((R_1, R_2)\) is \(\varepsilon\)-achievable if there exists a sequence of codes with \((M_1, M_2)\) messages such that

\[
\lim_{n \to \infty} \frac{1}{n} \log M_1 \geq R_1 \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log M_2 \geq R_2,
\]

and the average probability of error

\[
\lim_{n \to \infty} P_e^{(n)} \leq \varepsilon.
\]

\(C_\varepsilon(P_1, P_2)\) is the set of all \(\varepsilon\)-achievable \((R_1, R_2)\).
The $\varepsilon$-capacity region is

$$C_\varepsilon(P_1, P_2) = R_{\text{Ozarow}} \left( \frac{P_1}{1 - \varepsilon}, \frac{P_2}{1 - \varepsilon} \right), \text{ for all } \varepsilon \in [0, 1).$$
**Theorem (Truong-Fong-T. (arXiv 2015))**

The $\varepsilon$-capacity region is

$$C_\varepsilon(P_1, P_2) = R_{Ozarow}\left(\frac{P_1}{1 - \varepsilon}, \frac{P_2}{1 - \varepsilon}\right), \quad \text{for all } \varepsilon \in [0, 1).$$

If we can tolerate an error of $\leq \varepsilon$, we can operate at $(R_1, R_2)$ satisfying

$$R_1 \leq C\left(\frac{(1 - \rho^2)P_1}{1 - \varepsilon}\right)$$

$$R_2 \leq C\left(\frac{(1 - \rho^2)P_2}{1 - \varepsilon}\right), \quad \text{for any } 0 \leq \rho \leq 1.$$

$$R_1 + R_2 \leq C\left(\frac{P_1 + P_2 + 2\rho\sqrt{P_1P_2}}{1 - \varepsilon}\right)$$

This is optimal.
\( \varepsilon = 0 \) recovers Ozarow's result

\[
C(P_1, P_2) = C_0(P_1, P_2) = R_{Ozarow}(P_1, P_2).
\]
\( \varepsilon = 0 \) recovers Ozarow’s result

\[
\mathcal{C}(P_1, P_2) = \mathcal{C}_0(P_1, P_2) = \mathcal{R}_{\text{Ozarow}}(P_1, P_2).
\]

Again \( \mathcal{C}_\varepsilon \) depends on \( \varepsilon \)

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\mathcal{C}_\varepsilon(P_1, P_2) = \mathcal{R}_{\text{Ozarow}}\left(\frac{P_1}{1 - \varepsilon}, \frac{P_2}{1 - \varepsilon}\right), \quad \text{for all} \quad \varepsilon \in [0, 1).
\]
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\[ C(P_1, P_2) = C_0(P_1, P_2) = R_{Ozarow}(P_1, P_2). \]

Again \( C_\varepsilon \) depends on \( \varepsilon \)

\[ C_\varepsilon(P_1, P_2) = R_{Ozarow}\left(\frac{P_1}{1 - \varepsilon}, \frac{P_2}{1 - \varepsilon}\right), \quad \text{for all} \quad \varepsilon \in [0, 1). \]

Strong converse doesn't hold
\( \varepsilon \)-Capacity of the G-MAC with Feedback : Remarks

- \( \varepsilon = 0 \) recovers Ozarow’s result

\[ C(P_1, P_2) = C_0(P_1, P_2) = \mathcal{R}_{\text{Ozarow}}(P_1, P_2). \]

- Again \( C_\varepsilon \) depends on \( \varepsilon \)

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- **Strong converse doesn’t hold**

- We have bounds on the “second-order” terms but they are quite loose
$\epsilon$-Capacity of the G-MAC with Feedback: Remarks

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- Again $C_\epsilon$ depends on $\epsilon$

$$C_\epsilon(P_1, P_2) = R_{Ozarow}\left(\frac{P_1}{1-\epsilon}, \frac{P_2}{1-\epsilon}\right), \quad \text{for all } \epsilon \in [0, 1).$$

- Strong converse doesn’t hold

- We have bounds on the “second-order” terms but they are quite loose

- Direct part follows similarly to the single-user case
Proof Idea for the Converse: Step 1

Start with an information-spectrum bound somewhat similar to Chen-Alajaji (1995) and Han (1998)
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Start with an information-spectrum bound somewhat similar to Chen-Alajaji (1995) and Han (1998)

Lemma (Information-Spectrum Bounds)

Fix a MAC $W^n(y^n|x_1^n, x_2^n)$ with feedback and error prob. $\leq \varepsilon$. 
Proof Idea for the Converse : Step 1

Start with an information-spectrum bound somewhat similar to Chen-Alajaji (1995) and Han (1998)

**Lemma (Information-Spectrum Bounds)**

*Fix a MAC $W^n(y^n|x^n_1, x^n_2)$ with feedback and error prob. $\leq \varepsilon$. For any $\gamma_1, \gamma_2, \gamma_3 > 0$ and any $\{(Q_{Y_i}|X_{1i}, Q_{Y_i}|X_{2i}, Q_{Y_i})\}_{i=1}^n$*
Proof Idea for the Converse: Step 1

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Lemma (Information-Spectrum Bounds)

Fix a MAC $W^n(y^n|x^n_1, x^n_2)$ with feedback and error prob. $\leq \varepsilon$.

For any $\gamma_1, \gamma_2, \gamma_3 > 0$ and any $\{(Q_{Y_i|X_1i}, Q_{Y_i|X_2i}, Q_{Y_i})\}_{i=1}^n$,

$$\log M_1 \leq \gamma_1 - \log^+ \left[ 1 - \varepsilon - \Pr \left( \sum_{i=1}^n \log \frac{W(Y_i|X_1i, X_2i)}{Q_{Y_i|X_2i}(Y_i|X_2i)} \geq \gamma_1 \right) \right]$$

$$\log M_2 \leq \gamma_2 - \log^+ \left[ 1 - \varepsilon - \Pr \left( \sum_{i=1}^n \log \frac{W(Y_i|X_1i, X_2i)}{Q_{Y_i|X_1i}(Y_i|X_1i)} \geq \gamma_2 \right) \right]$$

$$\log(M_1M_2) \leq \gamma_3 - \log^+ \left[ 1 - \varepsilon - \Pr \left( \sum_{i=1}^n \log \frac{W(Y_i|X_1i, X_2i)}{Q_{Y_i}(Y_i)} \geq \gamma_3 \right) \right]$$
Given a code generating symbols \(\{(X_{1i}(M_1, Y^{i-1}), X_{2i}(M_2, Y^{i-1}))\}_{i=1}^n\), let
Proof Idea for the Converse Part : Step 2

Given a code generating symbols \( \{(X_{1i}(M_1, Y_i^{-1}), X_{2i}(M_2, Y_i^{-1}))\}_{i=1}^n \), let

\[
P_{1i} := \mathbb{E}[X_{1i}^2], \quad P_{2i} := \mathbb{E}[X_{2i}^2], \quad \rho_i := \frac{\mathbb{E}[X_{1i}X_{2i}]}{\sqrt{P_{1i}P_{2i}}},
\]

Define

\[
\rho := \frac{\sum_{i=1}^n \rho_i \sqrt{P_{1i}P_{2i}}}{n \sqrt{P_1P_2}}
\]
Proof Idea for the Converse Part : Step 2

Given a code generating symbols \( \{ (X_1(1, Y^{i-1}), X_2(M_2, Y^{i-1})) \}_{i=1}^n \), let

\[
P_{1i} := \mathbb{E}[X_1^2], \quad P_{2i} := \mathbb{E}[X_2^2], \quad \rho_i := \frac{\mathbb{E}[X_1X_2]}{\sqrt{P_{1i}P_{2i}}}.
\]

Define

\[
\rho := \frac{\sum_{i=1}^n \rho_i \sqrt{P_{1i}P_{2i}}}{n \sqrt{P_1P_2}}
\]

Lemma ("Single-Letterization")

\[
|\rho| \leq 1,
\]

\[
\sum_{i=1}^n \left( P_{1i}(1 - \rho_i^2) \right) \leq nP_1(1 - \rho^2), \quad \text{and}
\]

\[
\sum_{i=1}^n \left( P_{1i} + P_{2i} + 2\rho_i \sqrt{P_{1i}P_{2i}} \right) \leq n \left( P_1 + P_2 + 2\rho \sqrt{P_1P_2} \right).
\]
Proof Idea for the Converse Part : Step 3

Finally, we need to bound the probabilities. We do so using Chebyshev.
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Lemma

For any $T > 1$, choose

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\gamma_1 := nC(P_1(1 - \rho^2)T) + n^{2/3}
\]

\[
\gamma_3 := nC((P_1 + P_2 + 2\rho\sqrt{P_1P_2})T) + n^{2/3}.
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Finally, we need to bound the probabilities. We do so using Chebyshev.

**Lemma**

For any $T > 1$, choose

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\]

Then, with a good choice of $Q$'s

\[
\Pr\left(\sum_{i=1}^{n} \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \geq \gamma_1\right) \leq \frac{1}{T} + O(n^{-1/3})
\]
\[
\Pr\left(\sum_{i=1}^{n} \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i}(Y_i)} \geq \gamma_3\right) \leq \frac{1}{T} + O(n^{-1/3}).
\]
Proof Idea for the Converse Part : Completion

Recall that

$$\log M_1 \leq \gamma_1 - \log^+ \left[1 - \varepsilon - \Pr \left( \sum_{i=1}^{n} \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \geq \gamma_1 \right) \right]$$

Conclusion:

$$\log M_1 \leq n \mathbb{C} \left( P_1 \left( 1 - \rho_2 \right) 1 - \varepsilon \right) + O \left( \frac{n^2}{3} \right).$$

By product: Second-order term is upper bounded by $$O \left( \frac{n^2}{3} \right).$$
Recall that

\[
\log M_1 \leq \gamma_1 - \log^+ \left[ 1 - \varepsilon - \Pr \left( \sum_{i=1}^{n} \log \frac{W(Y_i|X_{1i}, X_{2i})}{Q_{Y_i|X_{2i}}(Y_i|X_{2i})} \geq \gamma_1 \right) \right]
\]

Probability term satisfies

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\]
Proof Idea for the Converse Part : Completion

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- Choose
  \[ \frac{1}{T} = 1 - \varepsilon - O(n^{-1/3}) \quad \text{so} \quad \gamma_1 = nC \left( \frac{P_1(1 - \rho^2)}{1 - \varepsilon} \right) + O(n^{2/3}). \]
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- Conclusion:

  \[ \log M_1 \leq nC \left( \frac{P_1(1-\rho^2)}{1-\varepsilon} \right) + O(n^{2/3}). \]
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Generalized a result by Ozarow (1984) to non-vanishing $\varepsilon \in [0, 1)$
Wrap Up

- Generalized a result by Ozarow (1984) to non-vanishing \( \varepsilon \in [0, 1) \)
- Established \( \varepsilon \)-capacity region for AWGN-MAC with feedback

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C_\varepsilon(P_1, P_2) = R_{\text{Ozarow}}\left(\frac{P_1}{1 - \varepsilon}, \frac{P_2}{1 - \varepsilon}\right).
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Generalized a result by Ozarow (1984) to non-vanishing $\varepsilon \in [0, 1)$

Established $\varepsilon$-capacity region for AWGN-MAC with feedback

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First step to obtaining higher-order terms in asymptotic expansion
Wrap Up

- Generalized a result by Ozarow (1984) to non-vanishing $\varepsilon \in [0, 1)$
- Established $\varepsilon$-capacity region for AWGN-MAC with feedback
  \[ C_\varepsilon(P_1, P_2) = R_{Ozarow}\left(\frac{P_1}{1 - \varepsilon}, \frac{P_2}{1 - \varepsilon}\right). \]
- First step to obtaining higher-order terms in asymptotic expansion
- Current second-order bounds are loose

http://arxiv.org/abs/1512.05088
Generalized a result by Ozarow (1984) to non-vanishing $\varepsilon \in [0, 1)$

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Lan V. Truong  Silas L. Fong