EE5138R: Simplified Proof of Slater's Theorem for Strong Duality

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March 6, 2015

In this document, we provided a simplified proof of Slater's theorem that ensures strong duality holds for convex problems.

Consider the convex primal problem

$$\min_{x\in\mathbb{R}^n} \; f_0(x) \quad \text{s.t.} \quad f_i(x) \le 0, i \in [m]. \tag{1}$$

We consider no equality constraints because if there were, each equality constraint can be cast as two separate inequality constraints. We assume that

1. The objective f_0 and all the constraint functions $f_i, i \in [m]$ are convex;
2. The optimal value $p^* = \inf_x \{f_0(x) : f_i(x) \le 0, i \in [m]\}$ is finite.

Slater's condition states that there exists a vector $\bar{x} \in \mathbf{dom}\, f_0$ (called a *Slater vector*) such that

$$f_i(\bar{x}) < 0, \quad \forall\, i \in [m] \tag{2}$$

Theorem 1. *Let Assumptions 1 and 2 as well as Slater's condition's hold. Then:*

1. *There is no duality gap, i.e.,* $d^* = p^*$*;*
2. *The set of dual optimal solutions is nonempty and bounded.*

Recall that the Lagrangian $L : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}$ is

$$L(x, \lambda) := f_0(x) + \sum_{i=1}^m \lambda_i f_i(x) \tag{3}$$

and the Lagrange dual function is

$$g(\lambda) := \inf_x L(x, \lambda). \tag{4}$$

The Lagrange dual problem is

$$\max_{\lambda} \; g(\lambda) \quad \text{s.t.} \quad \lambda \succeq 0. \tag{5}$$

Slater's theorem says that under Assumptions 1 and 2 and the existence of a Slater vector that the optimal values of the primal in (1) and the dual in (5) are equal. By weak duality, we always have that $d^* \le p^*$.

Proof. Consider the set $\mathcal{V} \subset \mathbb{R}^m \times \mathbb{R}$ given by

$$\mathcal{V} := \{(u, w) \in \mathbb{R}^m \times \mathbb{R} : f_0(x) \le w, f_i(x) \le u_i, \forall\, i \in [m], \forall\, x\}. \tag{6}$$

This set has several properties including: (i) it is convex; (ii) if $(u, w) \in \mathcal{V}$, then $(u', w') \in \mathcal{V}$ for any $(u', w') \succeq (u, w)$. Convexity of $\mathcal{V}$ follows from the convexity of the $f_i, i \in \{0\} \cup [m]$. The second property follows directly from the definition of $\mathcal{V}$.

We first claim that the vector $(0, p^*)$ is not in the interior of the set $\mathcal{V}$. Suppose it is, i.e., $(0, p^*) \in \text{int}(\mathcal{V})$. Then, there exists an $\varepsilon > 0$ such that $(0, p^* - \varepsilon) \in \text{int}(\mathcal{V})$, thus clearly contradicting the optimality of p^*.

Thus, either $(0, p^*) \in \text{bd}(\mathcal{V})$ or $(0, p^*) \notin \mathcal{V}$. By the Supporting Hyperplane Theorem, there exists a hyperplane passing through $(0, p^*)$ and supporting the set $\mathcal{V}$. In other words, there exists $(\lambda, \lambda_0) \in \mathbb{R}^m \times \mathbb{R}$ with $(\lambda, \lambda_0) \neq 0$ such that

$$(\lambda, \lambda_0)^T (u, w) = \lambda^T u + \lambda_0 w \geq \lambda_0 p^*, \qquad \forall\, (u, w) \in \mathcal{V}. \tag{7}$$

This relation means that $\lambda \succeq 0$ and $\lambda_0 \geq 0$ because if there *were* one negative component in (λ, λ_0), we could make the corresponding component of (u, w) arbitrarily large and still in $\mathcal{V}$ (property (ii) of $\mathcal{V}$) and hence contradicting (7). Now we consider two different cases: (i) $\lambda_0 = 0$; and (ii) $\lambda_0 > 0$.

- Case (i): The relation in (7) and $\lambda \neq 0$ implies that

$$\inf_{(u,w)\in\mathcal{V}} \lambda^T u = 0. \tag{8}$$

  On the other hand, by the definition of the set $\mathcal{V}$, since $\lambda \succeq 0$ and $\lambda \neq 0$, we have

$$\inf_{(u,w)\in\mathcal{V}} \lambda^T u = \inf_x \sum_{i=1}^m \lambda_i f_i(x) \leq \sum_{i=1}^m \lambda_i f_i(\bar{x}) < 0 \tag{9}$$

  where $\bar{x}$ is the Slater vector and the last inequality is due to Slater's condition. This contradicts (8) so $\lambda_0 = 0$ is not possible.

- Case (ii): Hence, the only possibility is $\lambda_0 > 0$. Now we may divide (7) by λ_0 yielding

$$\inf_{(u,w)\in\mathcal{V}} \left\{ \tilde{\lambda}^T u + w \right\} \geq p^* \tag{10}$$

  with $\tilde{\lambda} := \lambda/\lambda_0 \succeq 0$. Therefore,

$$g(\tilde{\lambda}) = \inf_x \left\{ f_0(x) + \sum_{i=1}^n \tilde{\lambda}_i f_i(x) \right\} \geq p^*. \tag{11}$$

  Now if we maximize the LHS over all $\tilde{\lambda} \succeq 0$, we obtain

$$d^* \geq p^* \tag{12}$$

  which completes the proof of strong duality by weak duality $d^* \leq p^*$.

Now we show that the set of dual optimal solutions is bounded. For any dual optimal $\tilde{\lambda} \succeq 0$, we have

$$d^* = g(\tilde{\lambda}) = \inf_x \left\{ f_0(x) + \sum_{i=1}^m \tilde{\lambda}_i f_i(x) \right\} \tag{13}$$

$$\leq f_0(\bar{x}) + \sum_{i=1}^m \tilde{\lambda}_i f_i(\bar{x}) \tag{14}$$

$$\leq f_0(\bar{x}) + \max_{1\leq i\leq m} \{f_i(\bar{x})\} \left[\sum_{i=1}^m \tilde{\lambda}_i \right] \tag{15}$$

Therefore

$$\min_{1\leq i\leq m} \{-f_i(\bar{x})\} \left[\sum_{i=1}^m \tilde{\lambda}_i \right] \leq f_0(\bar{x}) - d^* \tag{16}$$

implying that

$$\|\tilde{\lambda}\| \leq \sum_{i=1}^m \tilde{\lambda}_i \leq \frac{f_0(\bar{x}) - d^*}{\min_{1\leq i\leq m}\{-f_i(\bar{x})\}} < \infty \tag{17}$$

where the final equality is again due Slater's condition. □