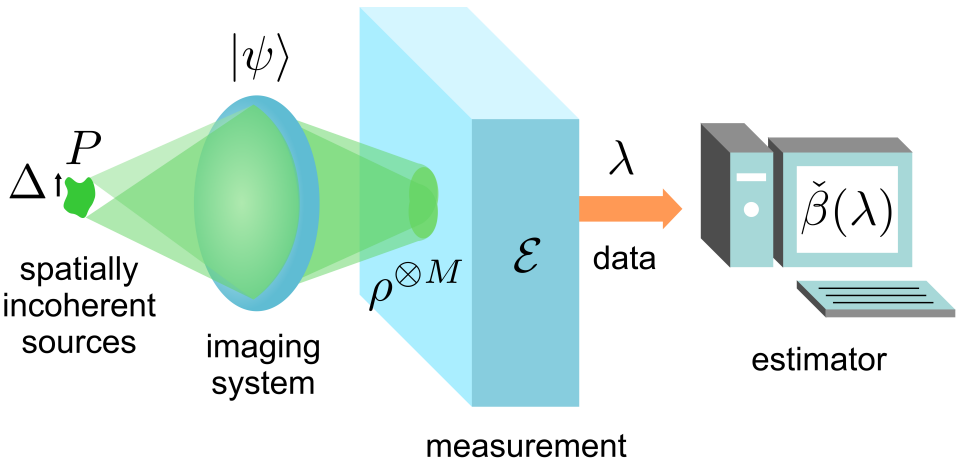
Resolving Starlight: A Quantum Perspective

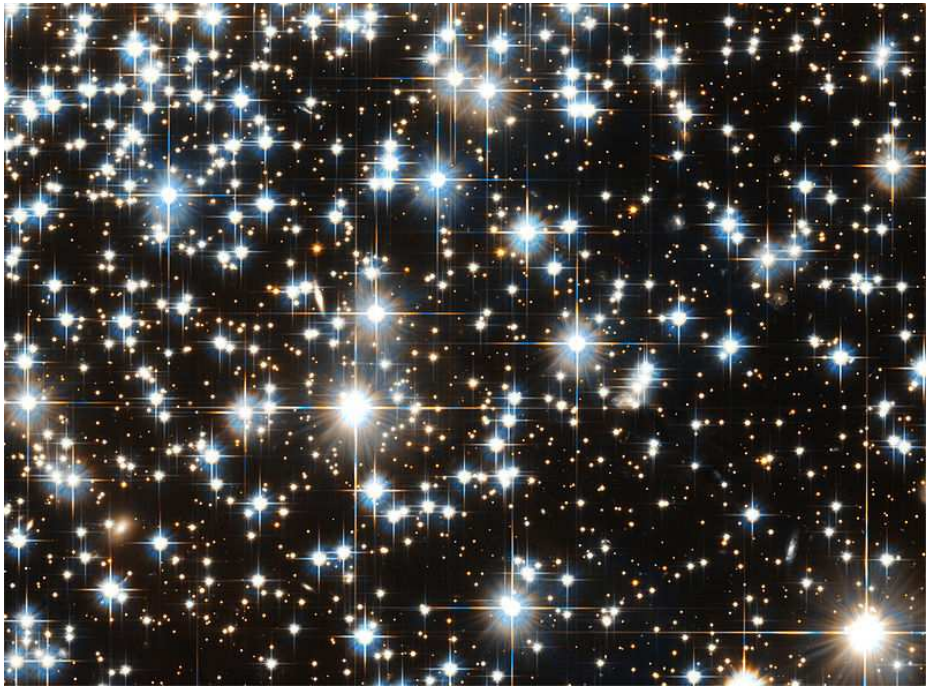
Mankei Tsang

National University of Singapore

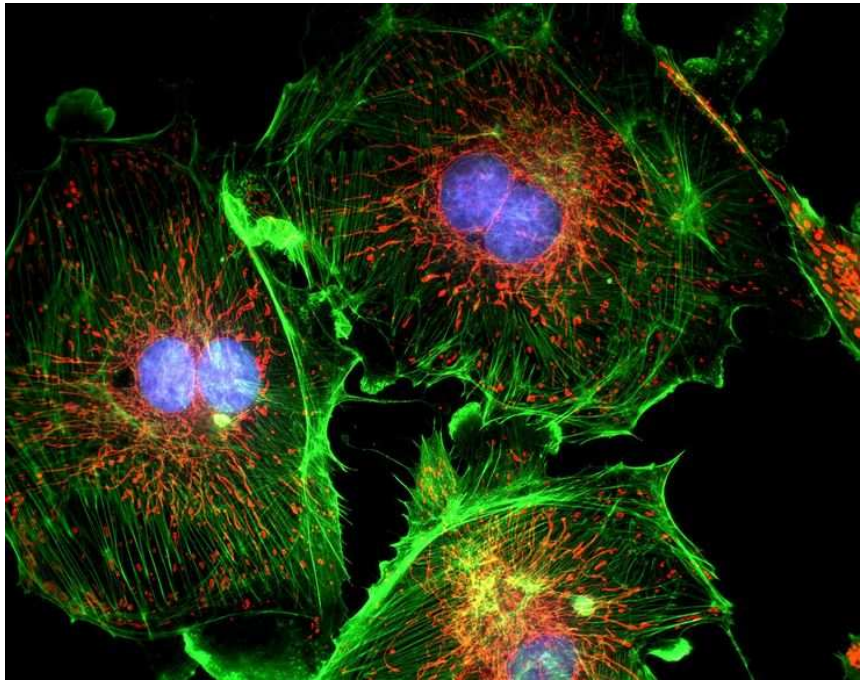
ALP Seminar, Oxford, 2 Dec 2022



Fundamental Resolution of Incoherent Imaging



Observational Astronomy



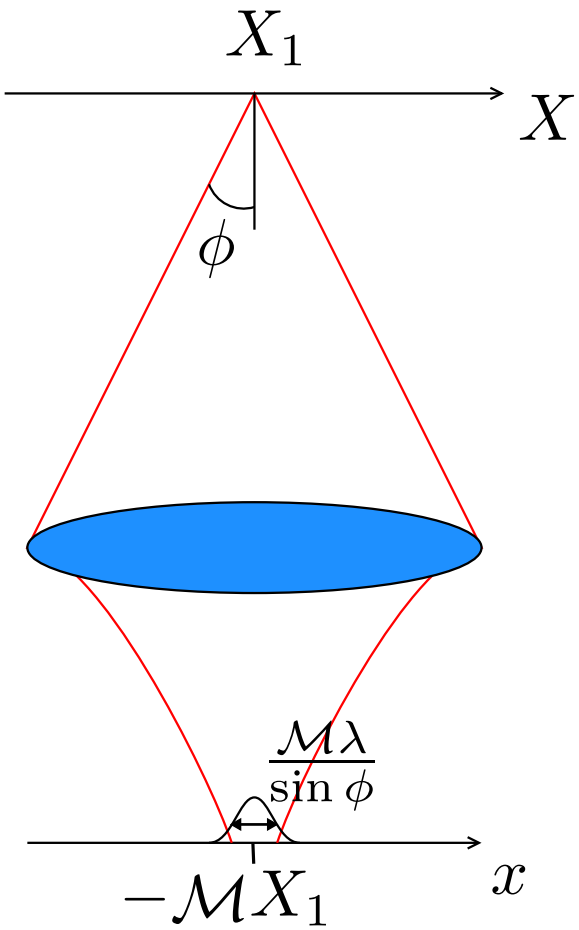
Fluorescence Microscopy

(images from the internet)

- Quantum information theory
- Quantum-inspired measurements

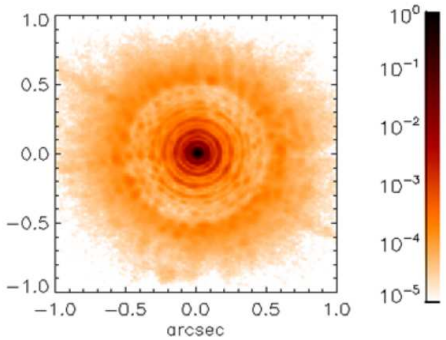
Part I: Two Point Sources

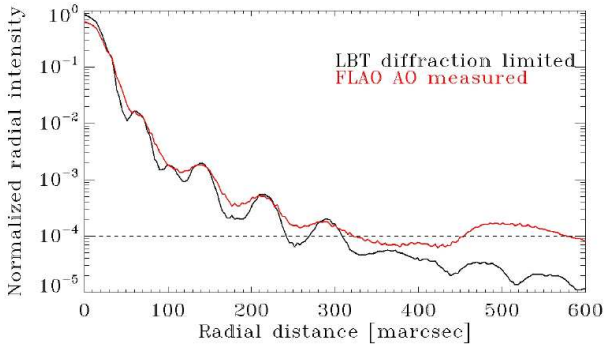
Wave Nature of Light: Diffraction Limit



Diffraction-limited (λ/NA or λ/D)

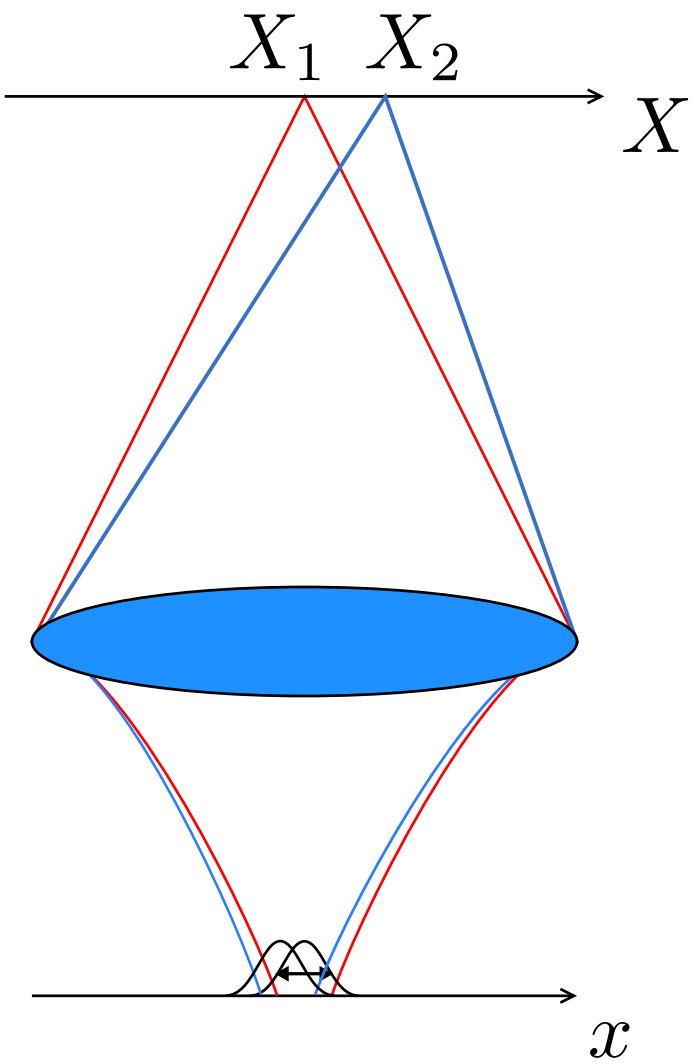
1. Fluorescence microscopy
2. Space telescopes (Webb, \$10 billion)
3. Ground-based telescopes (corrected by **adaptive optics**):
   (a) Large Binocular Telescope (LBT) (Strehl ratio $> 80\%$, \$120 million)
   (b) Giant Magellan Telescope (GMT)
   (c) Thirty Meter Telescope (TMT)
   (d) European Extremely Large Telescope (E-ELT) (>\$1 billion each)



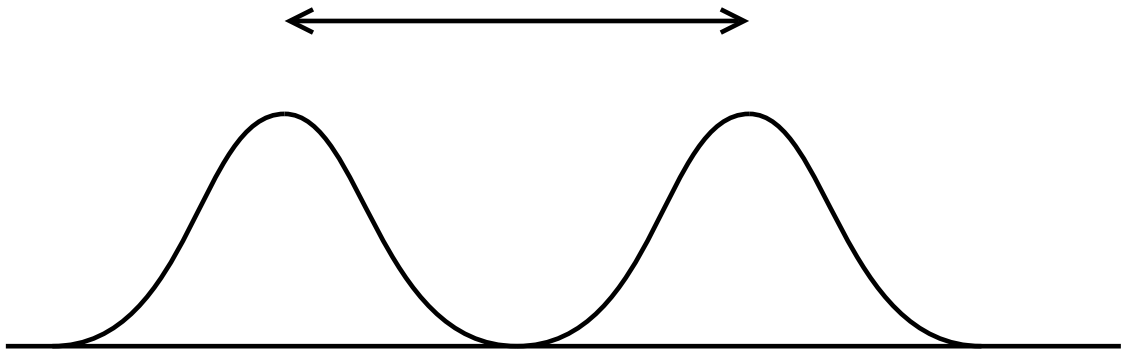


Esposito *et al.*, SPIE **8149**, 814902 (2011).

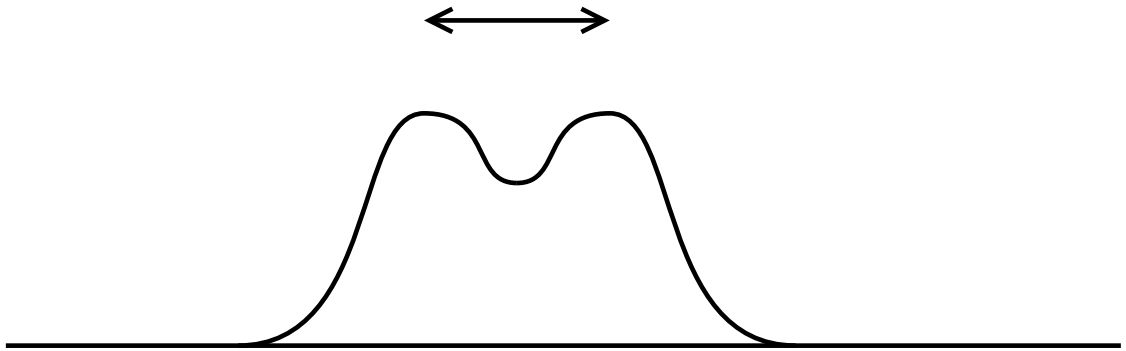
Rayleigh (1879)



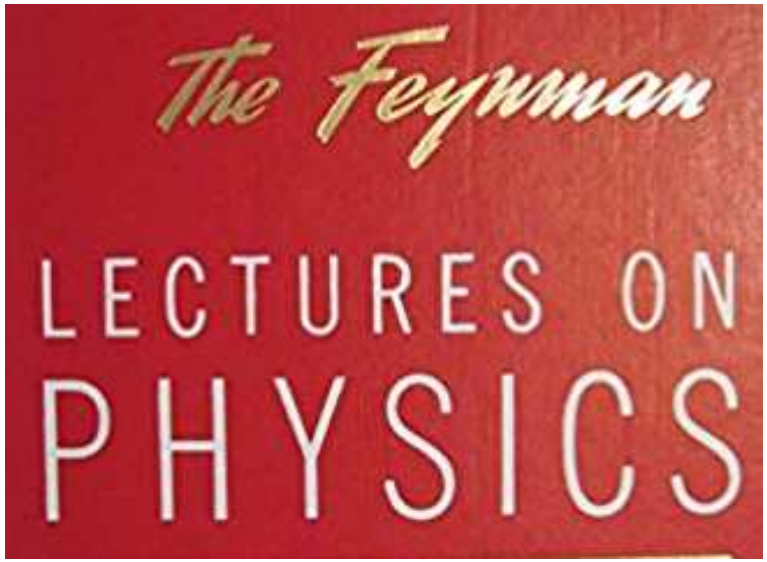
- Resolved:



- **Not resolved**:



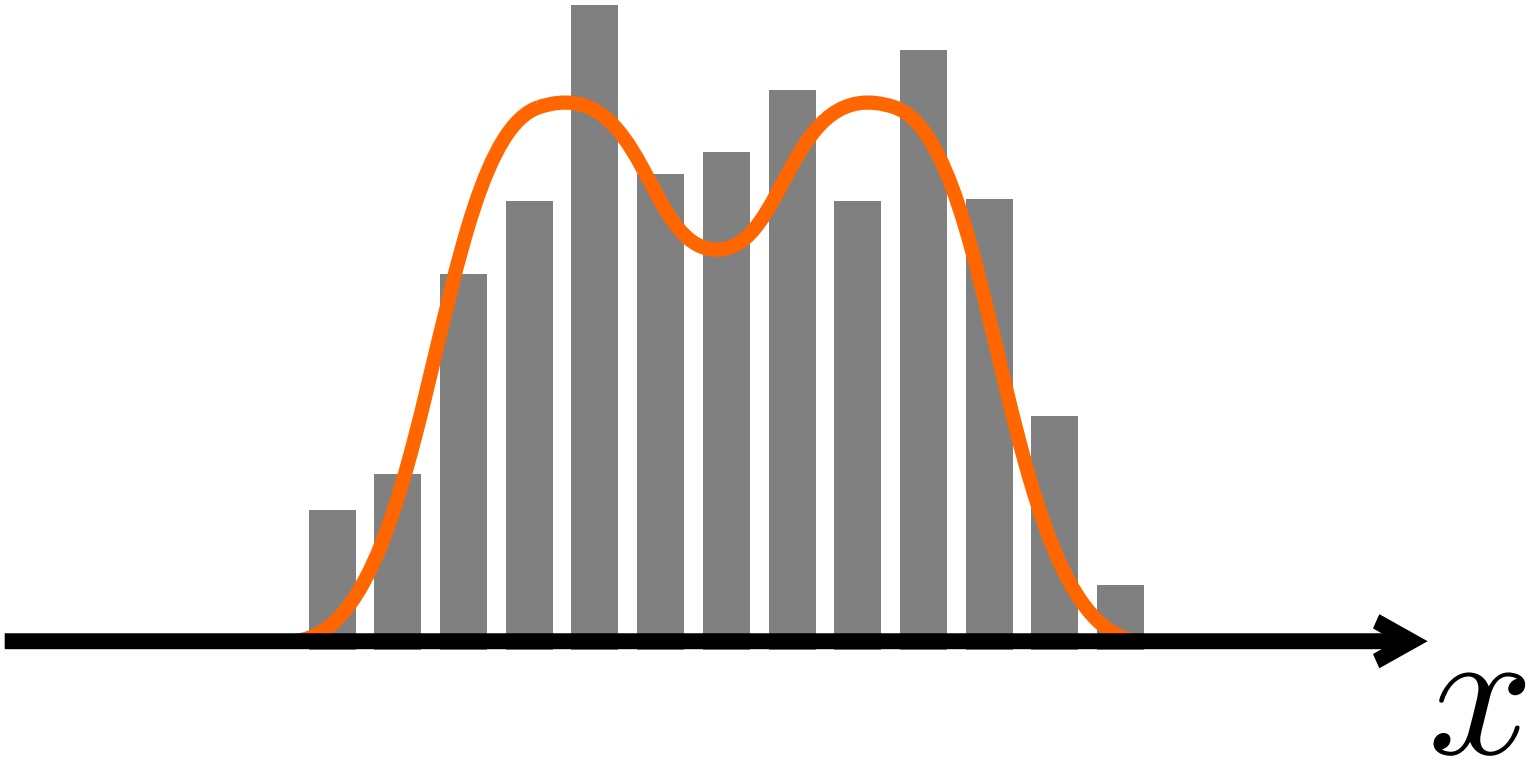
Rough Idea



"Rayleigh's criterion is a **rough idea**..." a better resolution can be achieved "if sufficiently careful **measurements of the exact intensity distribution** over the diffracted image spot can be made."

Chapter 30. *Diffraction*, Feynman Lectures on Physics, Vol. I

Photon Shot Noise



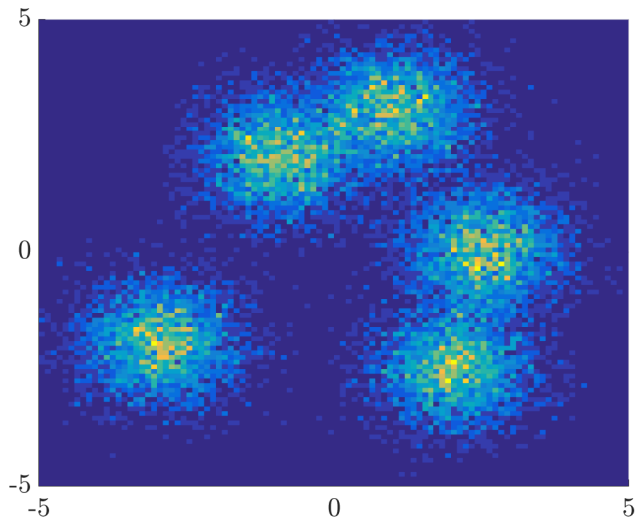
- **Poisson** at optical frequencies
- Bunching (thermal)/anti-bunching (fluor.) negligible in practice
- Random arrival of "photons"

Photon Shot Noise

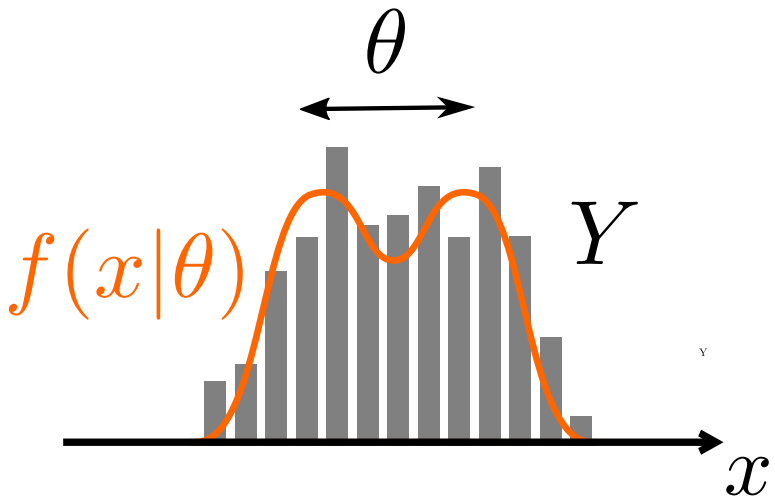
- Thermal sources (stars, etc.)
  - **Poisson**, bunching negligible at optical
  - Goodman, *Statistical Optics*; Zmuidzinas, JOSA A **20**, 218 (2003)

> considerations. If the count degeneracy parameter is much less than 1, it is highly probable that there will be either zero or one counts in each separate coherence interval of the incident classical wave. In such a case the classical intensity fluctuations have a negligible "bunching" effect on the photo-events, for (with high probability) the light is simply too weak to generate multiple events in a single coherence cell. If negligible bunching of the events takes place, the count statistics will be indistinguishable from those produced by stabilized single-mode laser radiation, for which no bunching occurs.

- Fluorophores (GFP, dye molecules, quantum dots, etc.)
  - **Poisson**, negligible anti-bunching
  - Pawley *ed.*, *Handbook of Biological Confocal Microscopy*; Ram, Ober, Ward, PNAS (2006)



Parameter Estimation

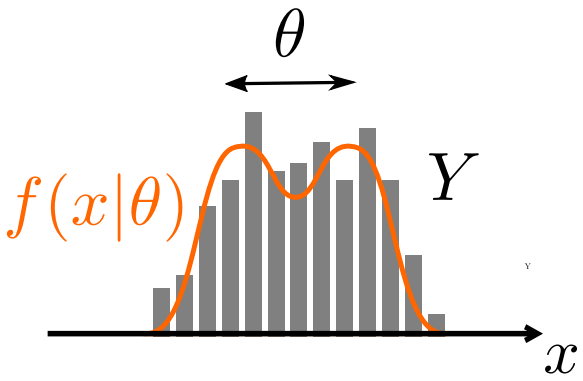


- **Estimator** $\check{\theta}(Y)$: guess θ from noisy data Y
- **Mean-square error**: $\text{MSE} = \mathbb{E}\left[\check{\theta}(Y) - \theta\right]^2$.
- Cramér-Rao bound (unbiased estimators):

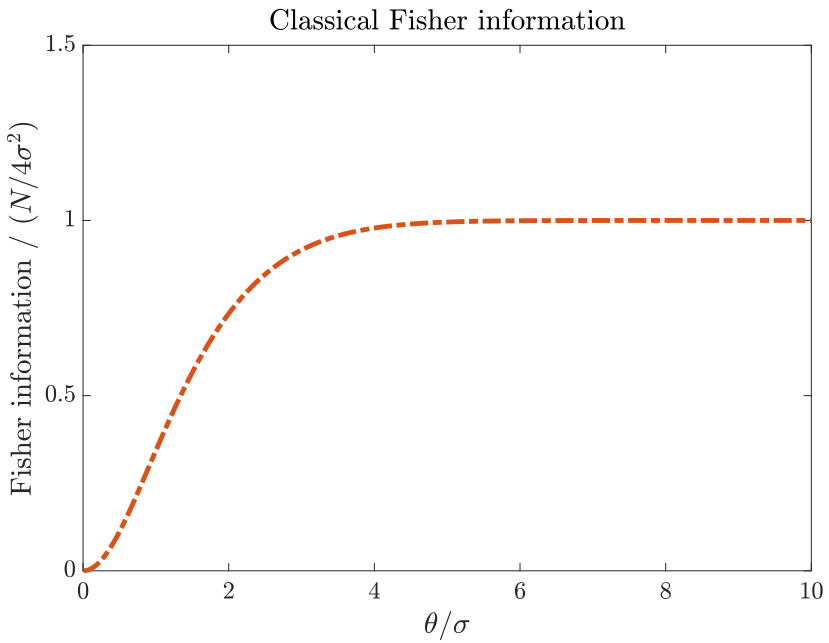
$$\text{MSE}(\theta) \geq J(\theta)^{-1}, \qquad J(\theta) = N \int_{-\infty}^{\infty} dx f(x|\theta) \left[\frac{\partial}{\partial\theta} \ln f(x|\theta)\right]^2.$$

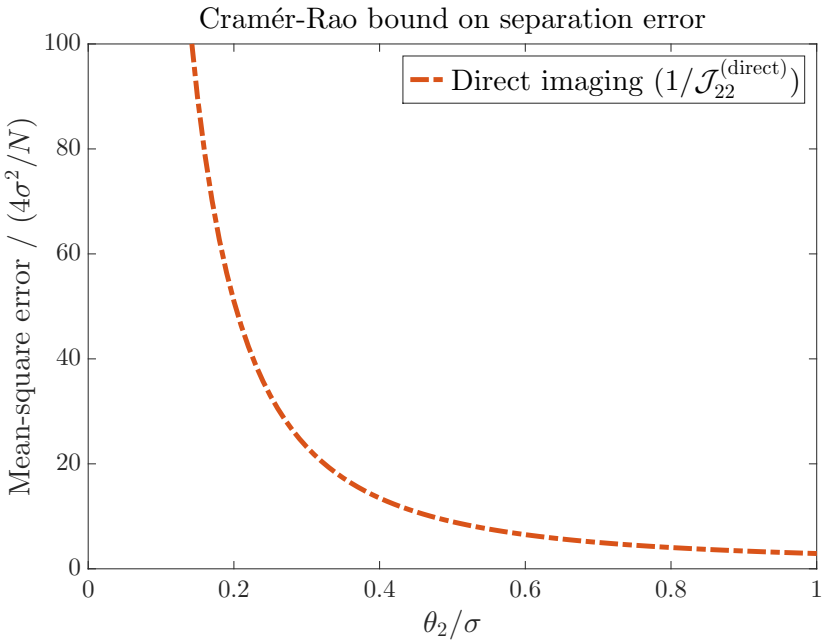
- J: Fisher information
- MSE $\to J(\theta)^{-1}$ via maximum-likelihood.

Estimating Two-Point Separation



- Conventional **direct imaging** (photon counting on image plane, Poisson noise):



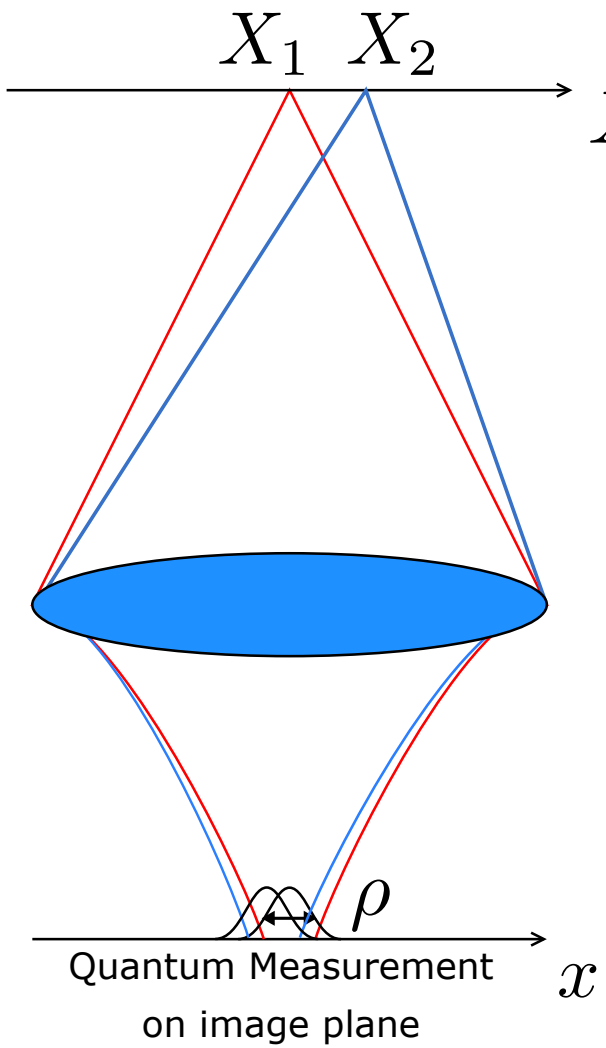


$J(0) = 0,\ J(\infty) = \frac{N}{4\sigma^2},\ \sigma = \frac{\lambda}{\mathsf{NA}}$

$\frac{1}{J(0)} = \infty,\ \frac{1}{J(\infty)} = \frac{4\sigma^2}{N}$

- **"Rayleigh's curse"**
- See, e.g., Ram, Ward, Ober, PNAS **103**, 4457 (2006).

Quantum Limit to Passive Imaging



- Quantum: Direct imaging is just one of the **infinitely many possible measurements.**
- Born's rule: $P(y|\theta) = \operatorname{tr} E(y)\rho(\theta)$
- Helstrom (1967), Nagaoka (1989), etc.: For any POVM,

$$\text{MSE} \geq J^{-1} \boxed{\geq K^{-1}},$$

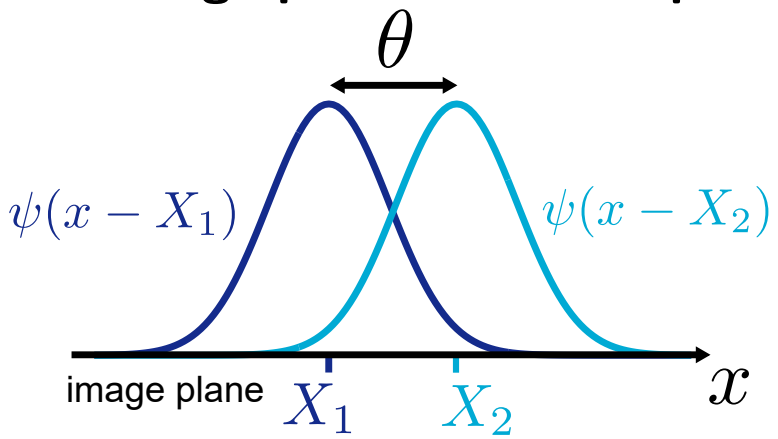
$$K(\rho^{\otimes M}) = M \operatorname{tr} S^2 \rho,$$

$$\frac{\partial \rho}{\partial \theta} = \frac{1}{2}\left(S\rho + \rho S\right).$$

- $K(\rho)$ is the quantum Fisher information, **the ultimate amount of information in the photons.**

Quantum Optics for Incoherent Imaging

- Thermal optical source: **average photon number per mode** $\epsilon \ll 1$



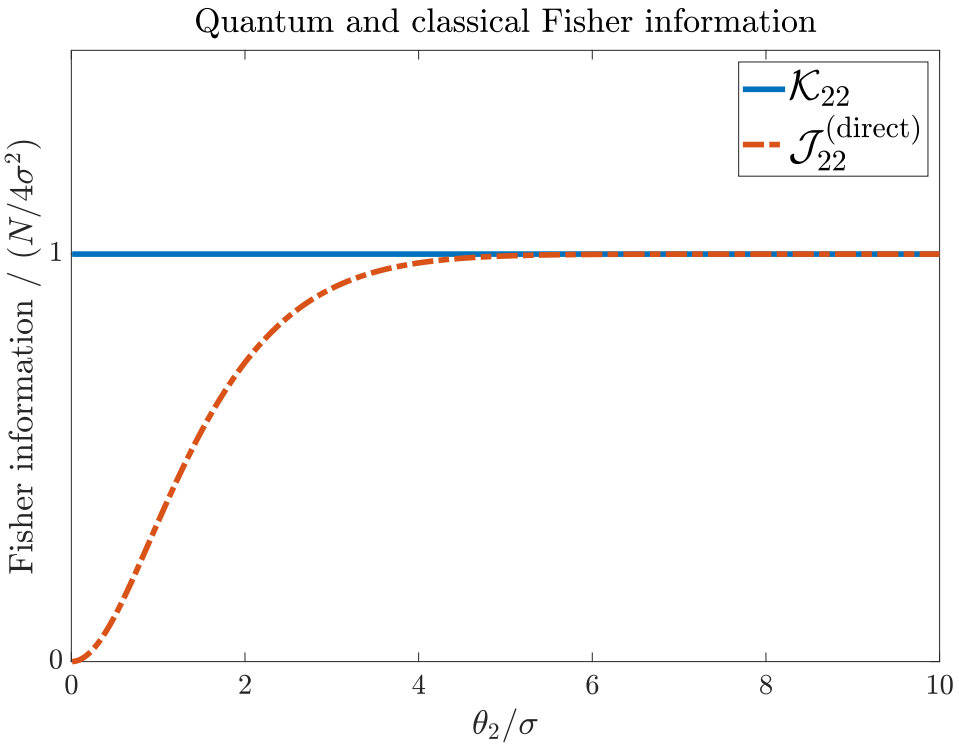
- Quantum state in M spectral modes $= \rho^{\otimes M}$, $N = M\epsilon$

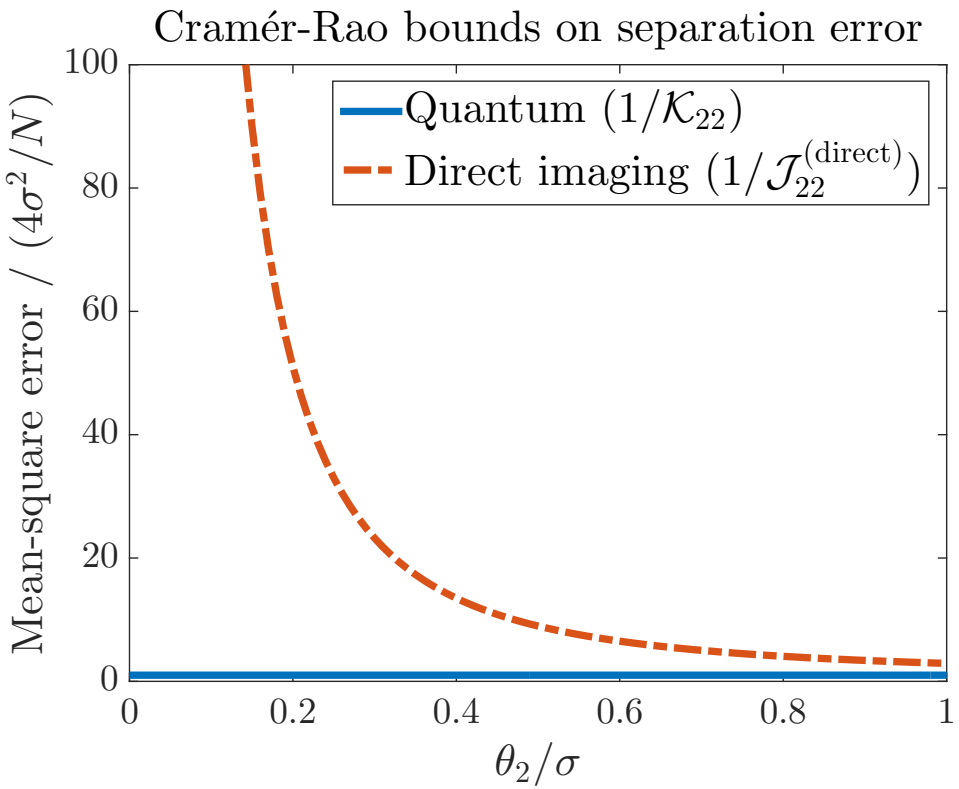
$$\rho = (1-\epsilon)\left|\text{vac}\right\rangle\left\langle\text{vac}\right| + \frac{\epsilon}{2}\left(\left|\psi_1\right\rangle\left\langle\psi_1\right| + \left|\psi_2\right\rangle\left\langle\psi_2\right|\right) + O(\epsilon^2),$$

$$\left|\psi_s\right\rangle \equiv \int_{-\infty}^{\infty} dx \psi(x - X_s)\left|x\right\rangle.$$

- derived from Glauber representation.
- Consistent with **Poisson** counting statistics.
- see, e.g., Tsang, PRL **107**, 270402 (2011); Tsang, Nair, Lu, PRX (2016); Quantum (2021).

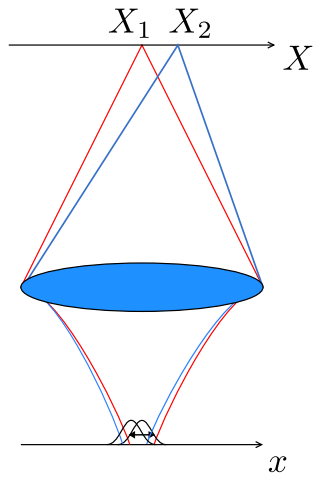
Plenty of Room at the Bottom

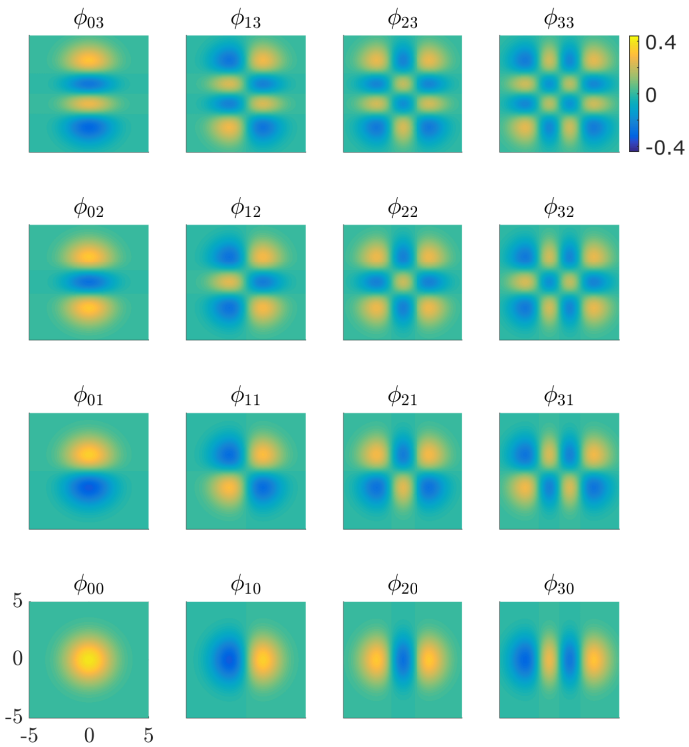




- Tsang, Nair, Lu, PRX **6**, 031033 (2016).

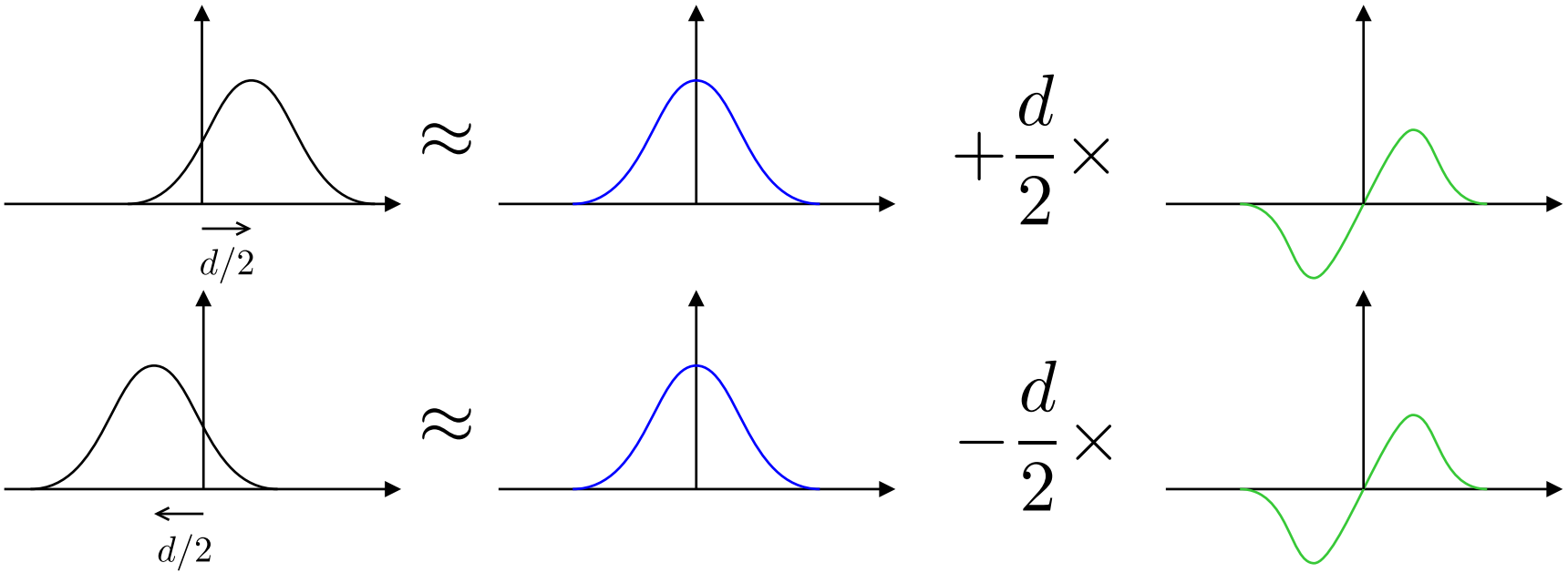
Optimal Measurement





- ☐ Sort the photons in **Hermite-Gaussian (TEM)** modes first, then do photon counting
- ☐ **Spatial-mode demultiplexing (SPADE)**
  - Tsang, Nair, Lu, PRX (2016); Rehacek *et al.*, OL **42**, 231 (2017).
- ☐ Many implementations (in optical comm., photonic circuits, interferometers, spatial light modulators, etc.)
- ☐ **Classical sources**
- ☐ **Linear optics/photon counting**
- ☐ **Killer applications** (astronomy, fluorescence microscopy, etc.)

Semiclassical Explanation

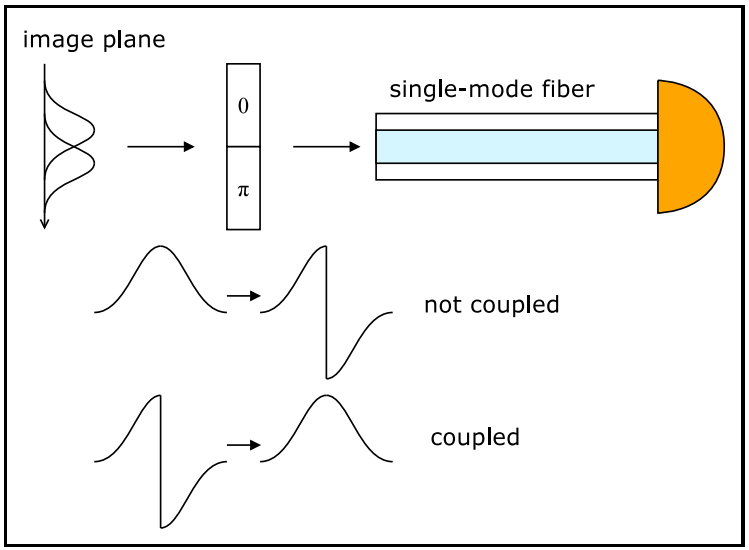


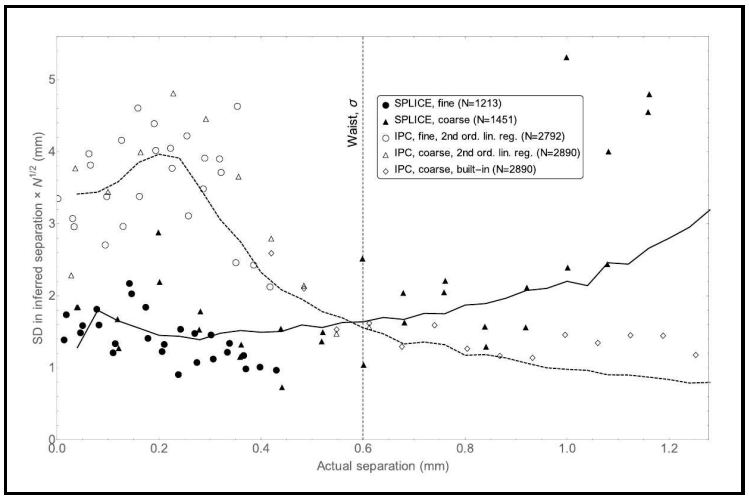
- **Incoherent sources**: energy in **1st-order mode** is

$$\propto \left(\frac{d}{2}\right)^2 + \left(-\frac{d}{2}\right)^2 = \frac{d^2}{2}.$$

- **0th-order mode** is just **background noise**; filtering it improves SNR.
- Incoherence (sources) + Coherence (diffraction) + Signal-dependent noise (Poisson)

Experimental Demonstrations





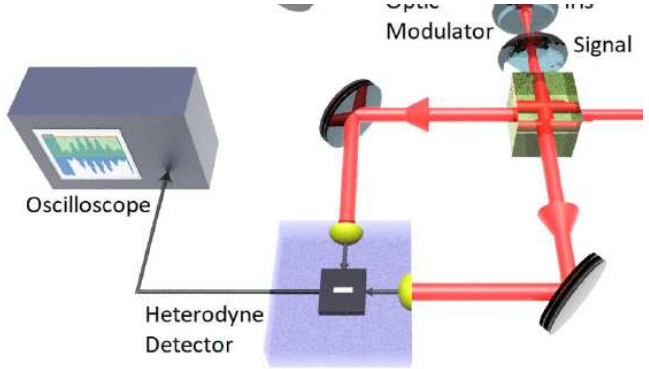
Tham, Ferretti, Steinberg (Toronto), PRL **118**, 070801 (2017).

- ☐ MSE $\sim 5 \times$ QCRB
- ☐ $\sim \frac{1}{9} \times$ MSE of direct imaging (biased estimators)

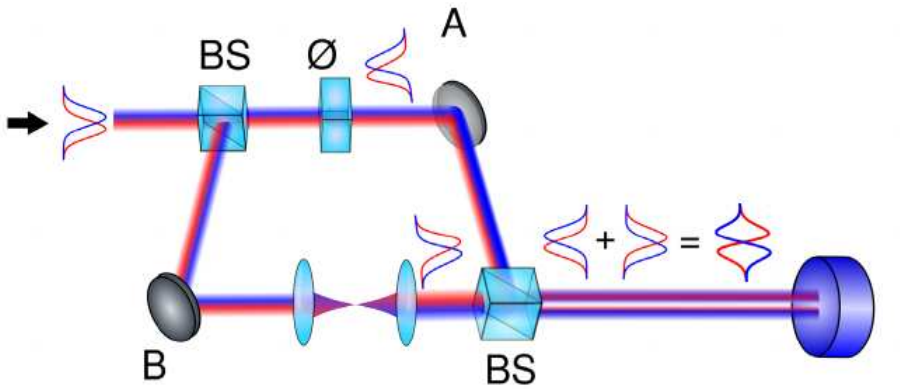
1. Tang, Durak, Ling (CQT Singapore), OE **24**, 22004 (2016).
2. Yang, Taschilina, Moiseev, Simon, Lvovsky (Calgary), Optica **3**, 1148 (2016).
3. Paúr, Stoklasa, Hradil, Sánchez-Soto, Rehacek (Palacky/Madrid/Max Planck), Optica **3**, 1144 (2016).
4. Parniak *et al.* (Warsaw), PRL **121**, 250503 (2018).
5. Donohue *et al.* (Paderborn), PRL **121**, 090501 (2018).
6. Paur *et al.* (Palacky/Madrid/Max Planck/ESA), Optica **5**, 1177 (2018).
7. J. Hassett *et al.* (Rochester), FiO/LS, JW4A.124 (2018).
8. Zhou *et al.* (Rochester), Optica **6**, 534 (2019).
9. Paur *et al.*, OL **44**, 3114 (2019).
10. Wadood *et al.*, (Rochester) Fio/LS, FM3C.7 (2019).
11. Rehacek *et al.*, PRL **123**, 193601 (2019).
12. Salit *et al.* (Honeywell), AO **59**, 5319 (2020).
13. Zhang *et al.* (Stevens), OL **45**, 4968 (2020).
14. Boucher *et al.* (Kastler Brossel), Optica **7**, 1621 (2020).
15. Ansari *et al.* (Paderborn), PRXQ **2**, 010301 (2020).
16. Brecht *et al.* (Paderborn), OSA Quantum 2.0, QW6A.17 (2020).
17. Wadood *et al.*, Optics Express **29**, 22034 (2021).
18. Mouradian *et al.* (Berkeley), PRA **103**, 032419 (2021).
19. De *et al.* (Paderborn), PRR **3**, 033082 (2021).
20. Santra *et al.* (Iowa State), JPCB **125**, 3092 (2021).
21. Pushkina *et al.* (Oxford), PRL **127**, 253602 (2021).
22. Mazelanik *et al.* (Warsaw), arXiv:2106.04450 (2021).
23. Aarav and Fleischer (Princeton), OSA COSI, CF4B.3 (2021).
24. Zanforlin *et al.* (Heriot-Watt/Bari/Sheffield/Macquarie), arXiv:2202.09406 (2022).
25. Santamaria *et al.* (Agenzia Spaziale Italiana/Bari), arXiv:2206.05246 (2022).
26. Qi *et al.* (NUS), CEO 2022, paper JTu3A.22.
27. Greenwood *et al.* (Bristol), QST (2022).

Other Implementations

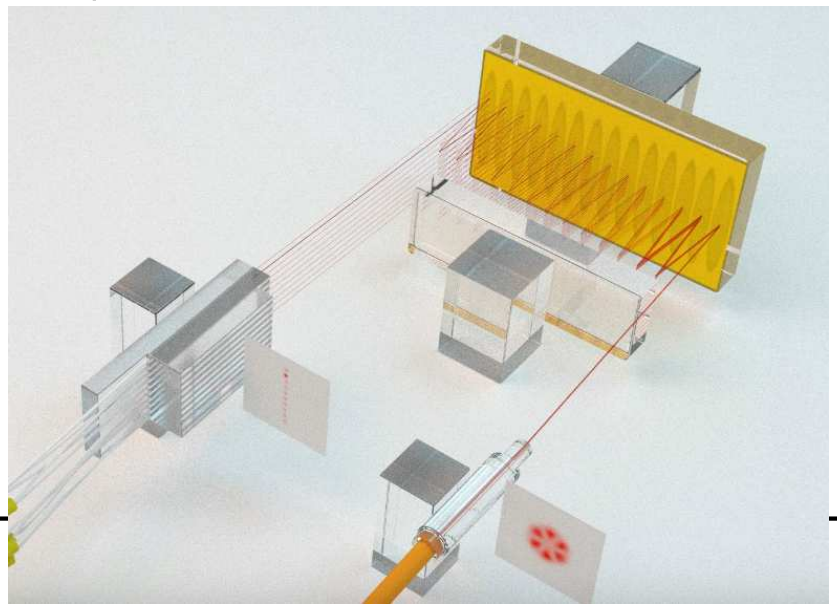
Yang *et al.* (Calgary), Optica (2016);
Pushkina *et al.* (Oxford), PRL (2021)



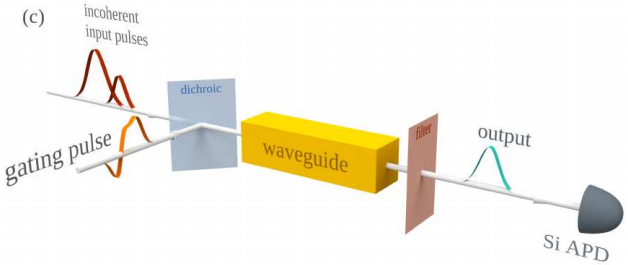
Tang *et al.* (CQT), OE (2016), etc.



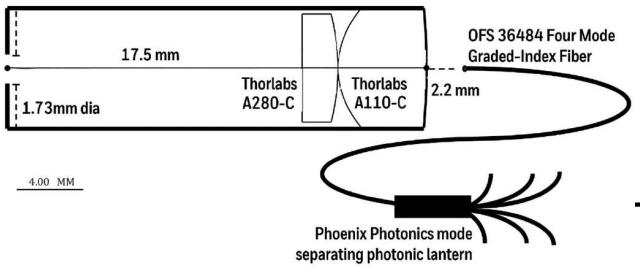
Boucher *et al.* (Cailabs), Optica (2020) https://www.youtube.com/watch?v=fKmXakWRUjU



Ansari *et al.* (Paderborn), PRXQ (2020)

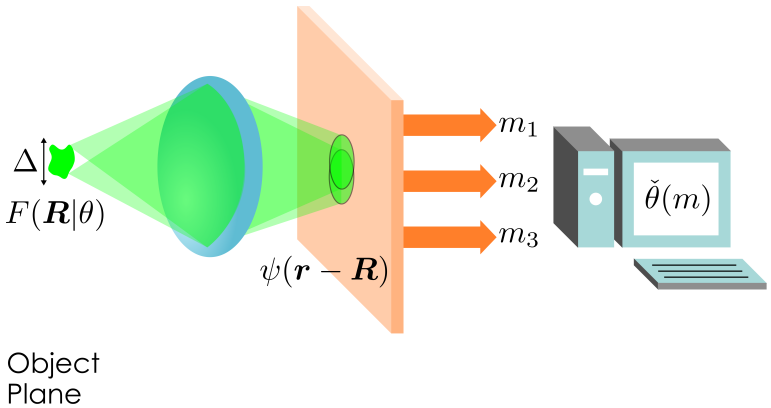


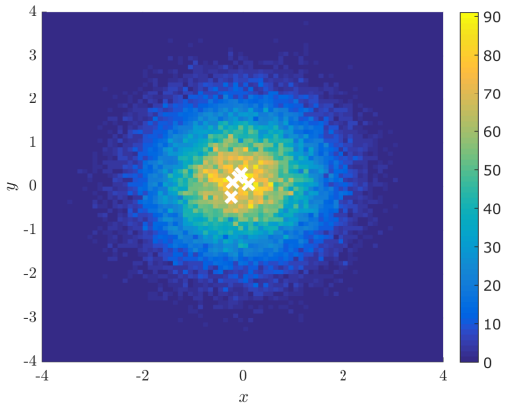
Salit *et al.* (Honeywell), AO (2020).



Part II: Extended Sources

Arbitrary Source Distribution





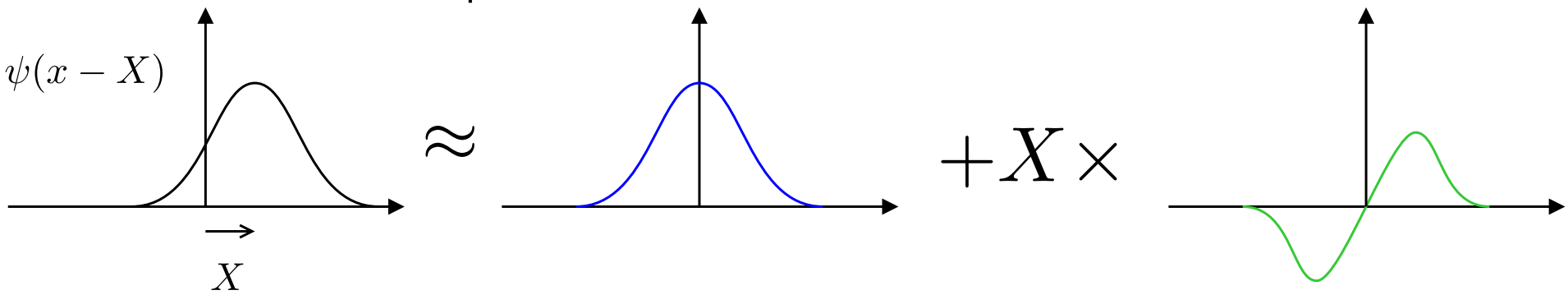
- Source density $= F(X)$. One-photon density operator:

$$\tau_1 = \int e^{-i\hat{k}X} \left|\psi\right\rangle \left\langle\psi\right| e^{i\hat{k}X} F(X) dX.$$

- Multi-parameter/multiple sources: Fiderer *et al.*, PRXQ (2021); Bisketzi *et al.*, NJP (2019); Bonsma-Fisher *et al.*, NJP (2019); Bao *et al.*, OL (2021); Liang *et al.*, PRA (2021); Bearne *et al.*, OE (2021); Pushkina *et al.*, PRL (2021); Matlin and Zipp *et al.*, SR (2022); Lee *et al.*, arXiv (2022); Krovi, arXiv (2022), etc.
- Arbitrary F:
  - **UNKNOWN/INFINITE number of incoherent point sources**
  - **Infinite-dimensional parameter space! (semiparametric)**

SPADE for Even Moment Estimation

- Wavefunction from each point source:



- For one point source,

$$\text{Energy in first-order mode} \propto X^2.$$

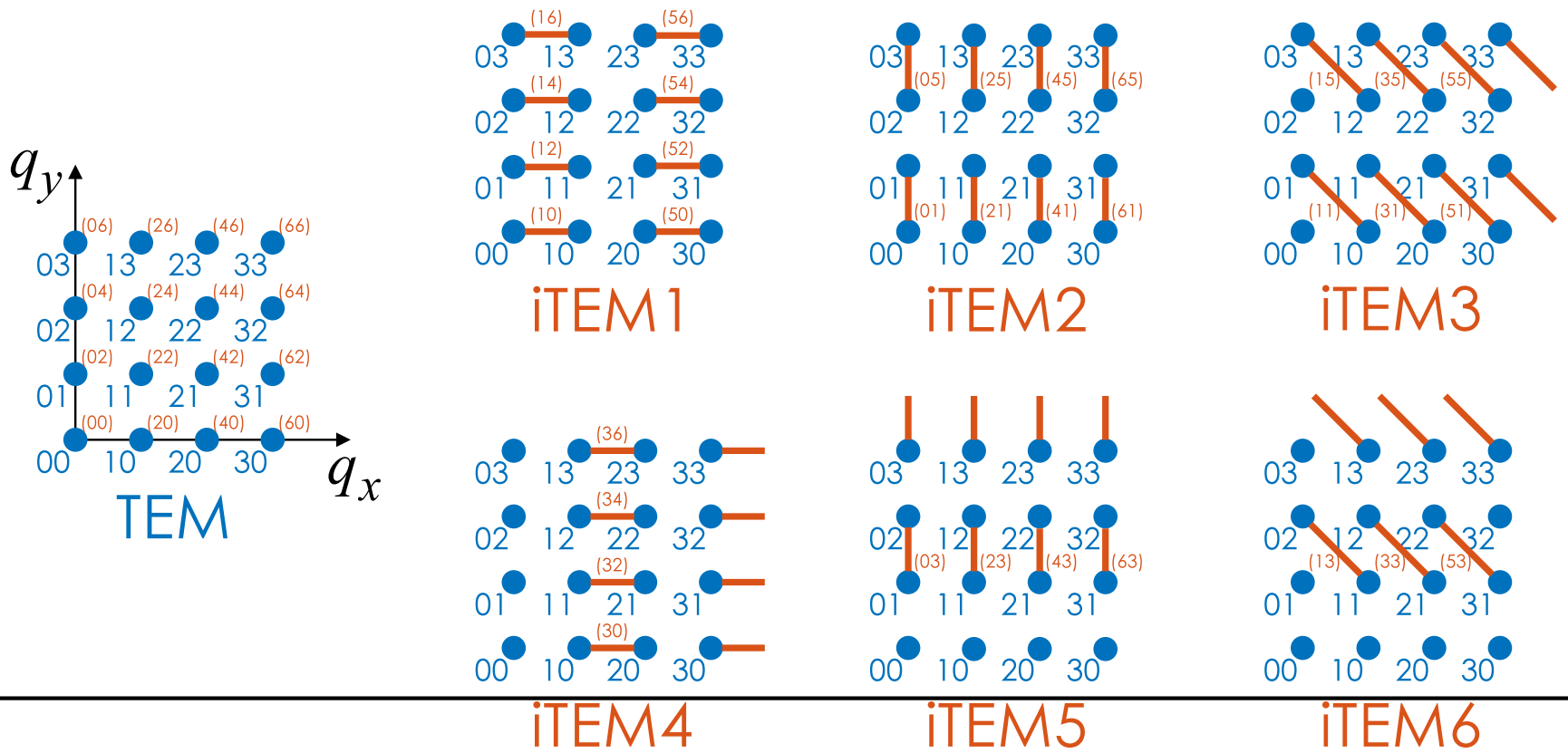
- A distribution of **incoherent** sources:

$$\text{Total energy in first-order mode} \propto \int X^2 F(X) dX.$$

- 2nd mode gives 4th moment, 3rd mode gives 6th moment, etc.

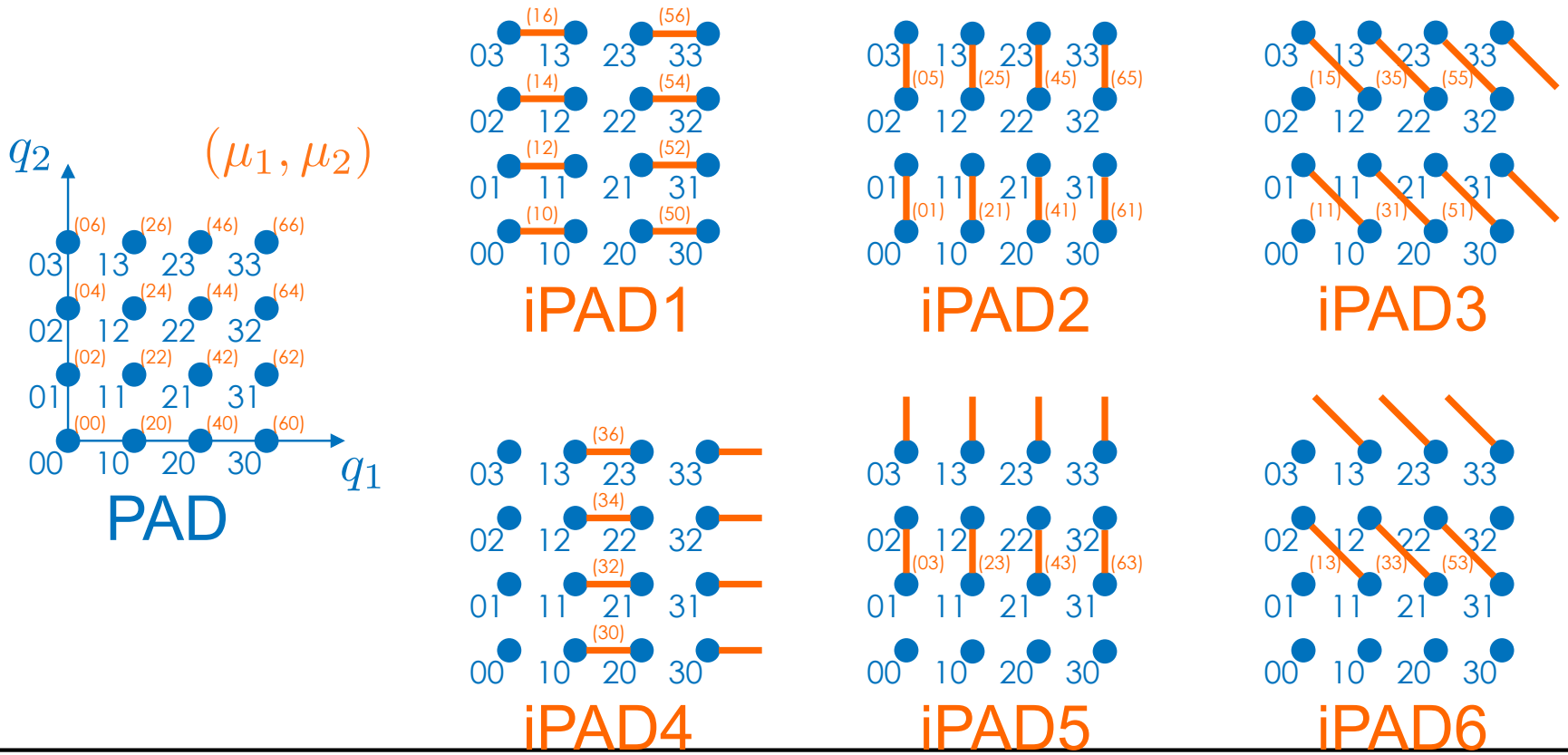
Generalized SPADE for Moment Estimation in 2D Imaging

- Gaussian PSF [Tsang, NJP (2017)]:
  - For moments with even μ_1 & even μ_2: TEM basis
    - See also Yang *et al.*, Optica (2016)
  - For other moments: interference of pairs of TEM modes



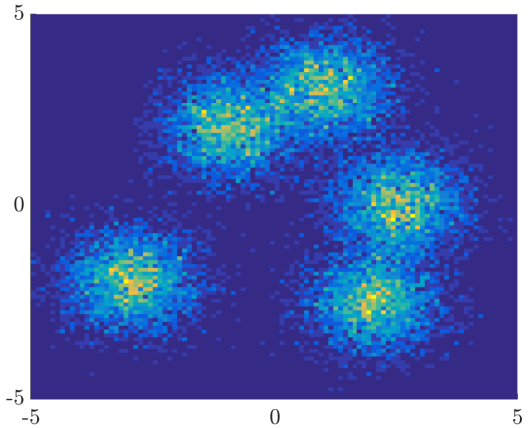
More General PSFs

- Any centrosymmetric and separable PSF [Tsang, PRA (2018)]:
  - For moments with even μ_1, even μ_2: **"PSF-adapted" (PAD) basis** (Rehacek *et al.* OL (2017), generalizes TEM)
  - For other moments: interference of pairs of PAD modes

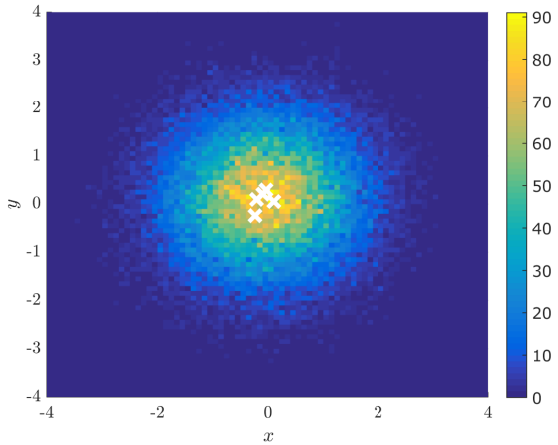


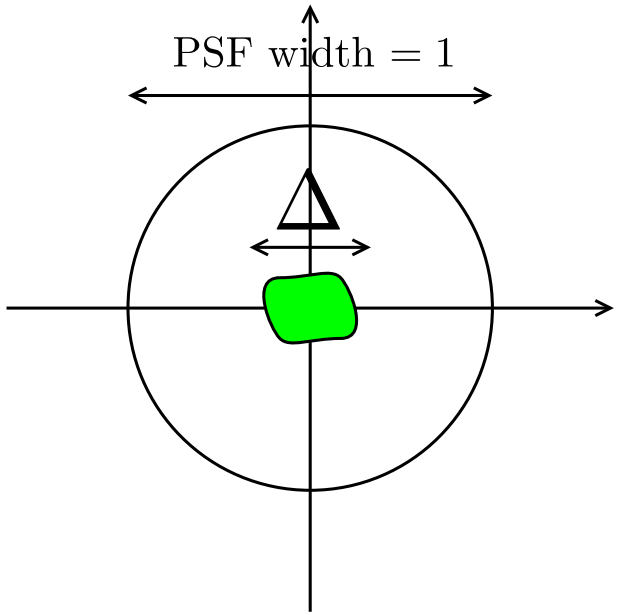
Defining the Subdiffraction Regime

Sparse (**Good Regime**, PALM, STED, compressed sensing, etc.)



Subdiffraction (Worst-Case Regime)





$$\boxed{\text{object width (relative to origin)} \equiv \Delta \ll 1.}$$

Define the parameter of interest as an object moment:

$$\boxed{\beta = \int [X^{\mu} + o(\Delta^{\mu})] F(X) dX, \quad \mu = \text{integer}}$$

Arbitrary F, scalar parameter of interest: **Semiparametric**

Performance of Generalized SPADE

- Tsang, NJP (2017); PRA (2018):

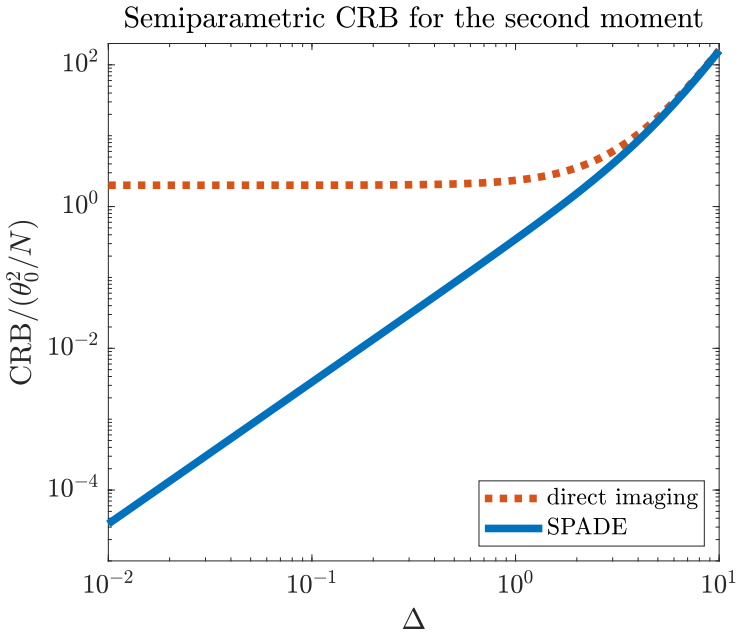
$$\boxed{\mathrm{MSE}^{(\mathrm{SPADE})} = \frac{\Theta\left(\Delta^{2\lfloor \mu/2 \rfloor}\right)}{N}.}$$

- **Good news**: compare with direct imaging (Gaussian PSF) [Tsang, PRR (2019)]:

$$\boxed{\mathrm{CRB}^{(\mathrm{direct})} = \frac{\Theta(1)}{N}.}$$

big enhancement over direct imaging when

  - $\Delta \ll 1$ (subdiffraction)
  - $\mu = \mu_X + \mu_Y \geq 2$



- **Bad news**: $\beta^2 = O(\Delta^{2\mu})$, SNR:

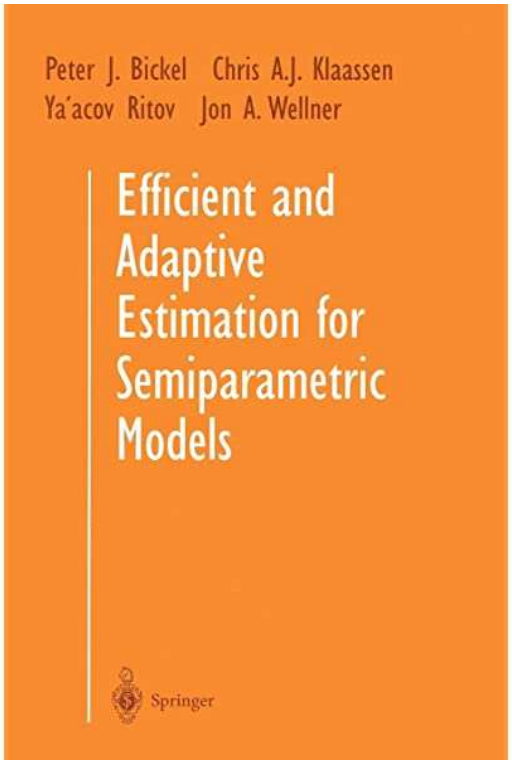
$$\boxed{\frac{\beta^2}{\mathrm{MSE}^{(\mathrm{SPADE})}} = NO(\Delta^{2\lceil \mu/2 \rceil}).}$$

Need many photons, especially for large μ.

- **Quantum limit?**

Classical Semiparametric Estimation

- CRB$^{(\text{direct})}$ is not for the faint-hearted!
- multiparameter CRB: $(\partial\beta)^\top J^{-1}\partial\beta$, J is **infinite-dimensional**
- Bickel *et al.*, *Efficient and Adaptive Estimation for Semiparametric Models*.



- Reformulate CRB using **abstract Hilbert-space theory**
- Gives **exact** CRB$^{(\text{direct})}$ for Gaussian PSF [Tsang, PRR (2019)]

Quantum Semiparametric Estimation

PHYSICAL REVIEW X **10,** 031023 (2020)

Quantum Semiparametric Estimation

Mankei Tsang,[1,2,*] Francesco Albarelli,[3,4,†] and Animesh Datta[4,‡]

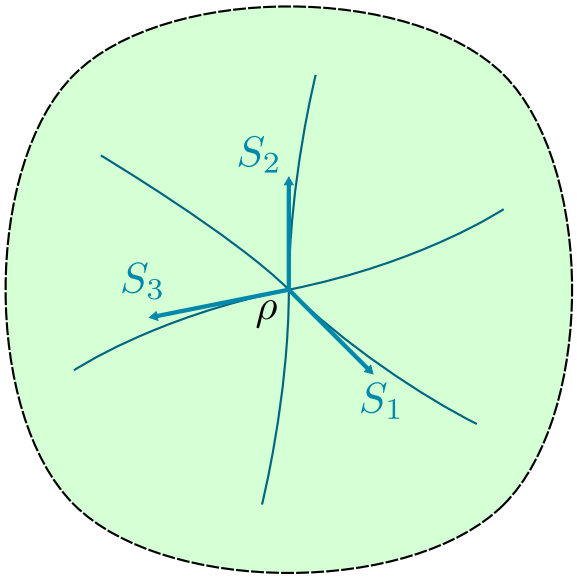
[1]*Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117583*

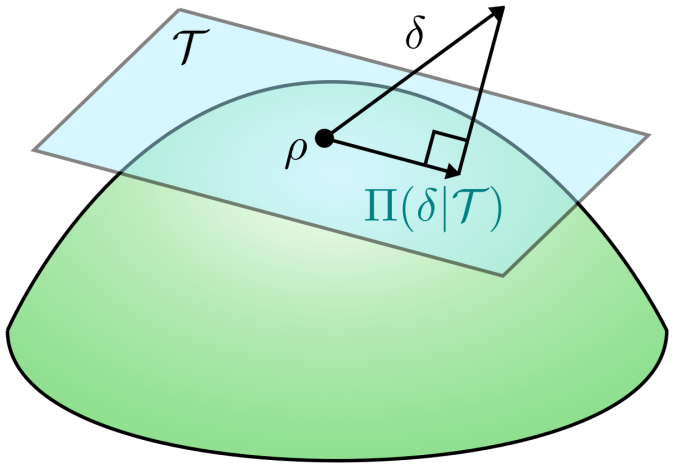
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- Tsang, Albarelli, Datta, PRX (2020).

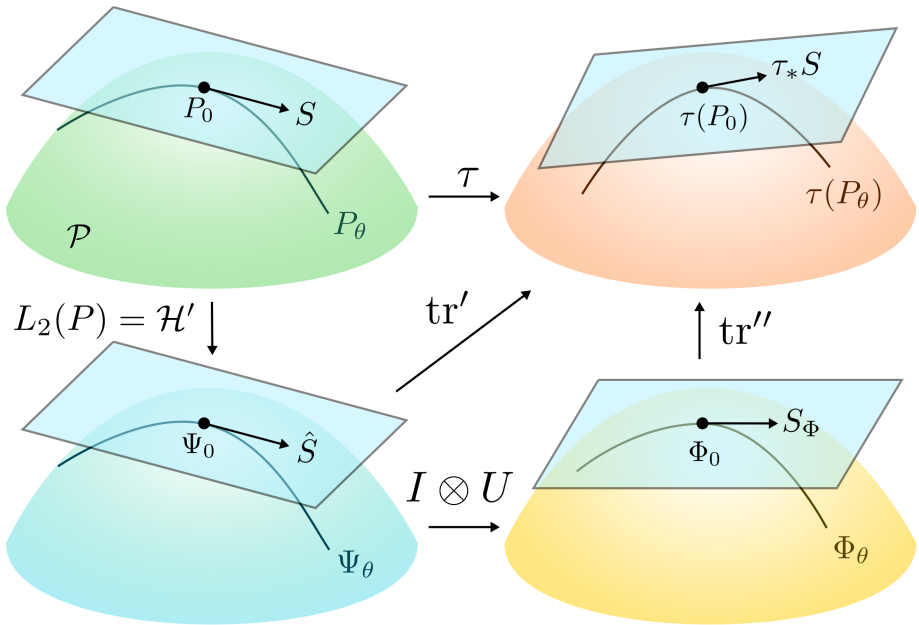
Quantum Limit on Semiparametric Imaging

- One-photon density operator: $\rho_1(F) = \int e^{-i\hat{k}X} |\psi\rangle \langle\psi| e^{i\hat{k}X} F(X)dX$, $F \in \{\text{any probability density}\}$, $\beta(F) = \int [X^\mu + o(\Delta^\mu)]F(X)dX$.
- **Quantum bound**:

$$\boxed{\text{QCRB} = \frac{\Omega(\Delta^{2\lfloor \mu/2 \rfloor})}{N},} \quad (1)$$

compare with

$$\text{MSE}^{(\text{SPADE})} = \frac{\Theta(\Delta^{2\lfloor \mu/2 \rfloor})}{N}. \quad (2)$$



Tsang, PRA (2021)

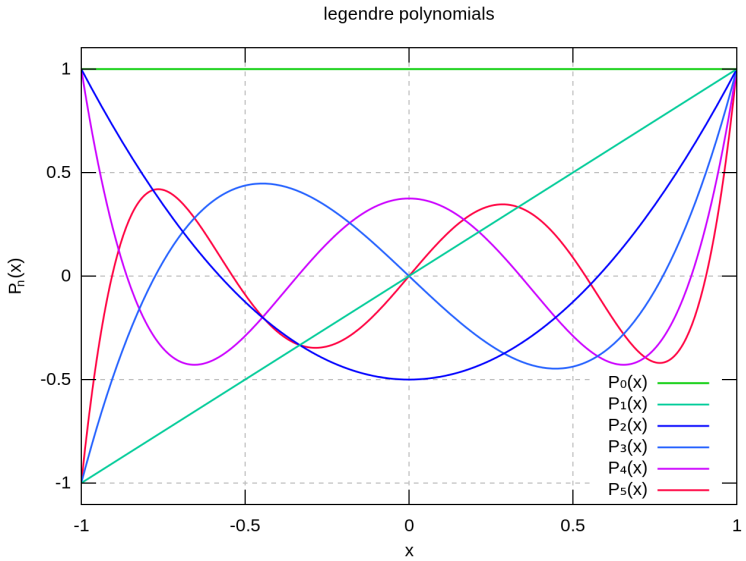
- See also Tsang, PRA (2019); Zhou & Jiang (Yale/Chicago), PRA (2019).

Superoscillation

-

$$\alpha_\mu = a_0\beta_0 + a_1\beta_1 + \cdots + a_\mu\beta_\mu = \int \underbrace{(a_0 + a_1 X + \cdots + a_\mu X^\mu)}_{\text{polynomial}} F(X)dX \quad (3)$$

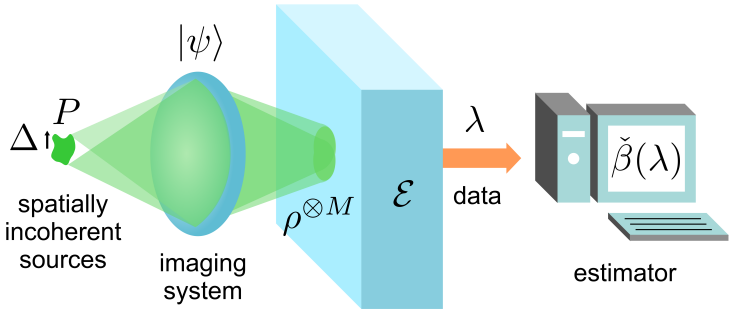
- The polynomial can be an **orthogonal polynomial** (oscillatory).

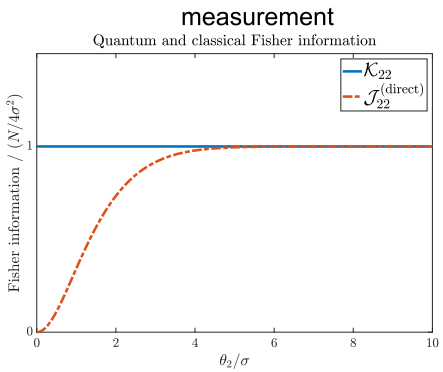


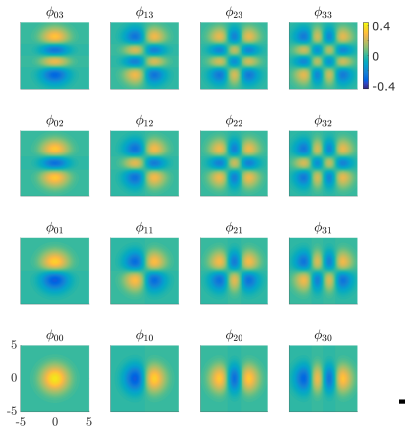
(Legendre polynomials, from wikipedia)

- α_μ is like a **Fourier coefficient**.
- Resembles **superoscillation** and **Slepian** theory
- Tsang, IEEE JSTSP (2022).

Conclusion







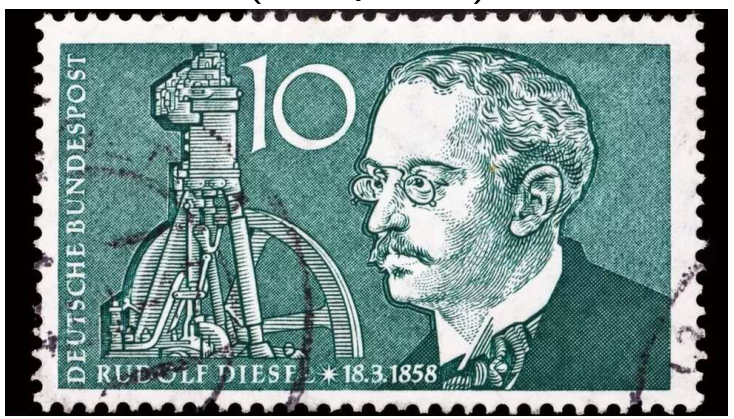
- **Quantum limits to imaging**
  - analogy: 2nd law of thermodynamics
- **Optimal measurement**: SPADE
  - analogy: Carnot engine
  - substantial improvement over direct imaging
- **Open problem**: general image reconstruction
- National Research Foundation Singapore
  - Quantum Engineering Program (QEP-P7): experimental fluorescence microscopy
- mankei@nus.edu.sg
- https://blog.nus.edu.sg/mankei/

From Carnot to Diesel

- 1824: Carnot published "*Reflections on the Motive Power of Fire*"
- 1832: asylum, died of cholera, aged 36
- 1850s: inspired Clausius, Kelvin to formulate 2nd law
- 1873–1879: As student, Diesel learned about Carnot engine, huge theoretical improvement over steam engines. Inspired to implement.
- 1892–1893: first experimental proposal (wrong)
- 1897: first successful demo
- 1898: millionaire
- 1913: disappeared (suicide or murdered)

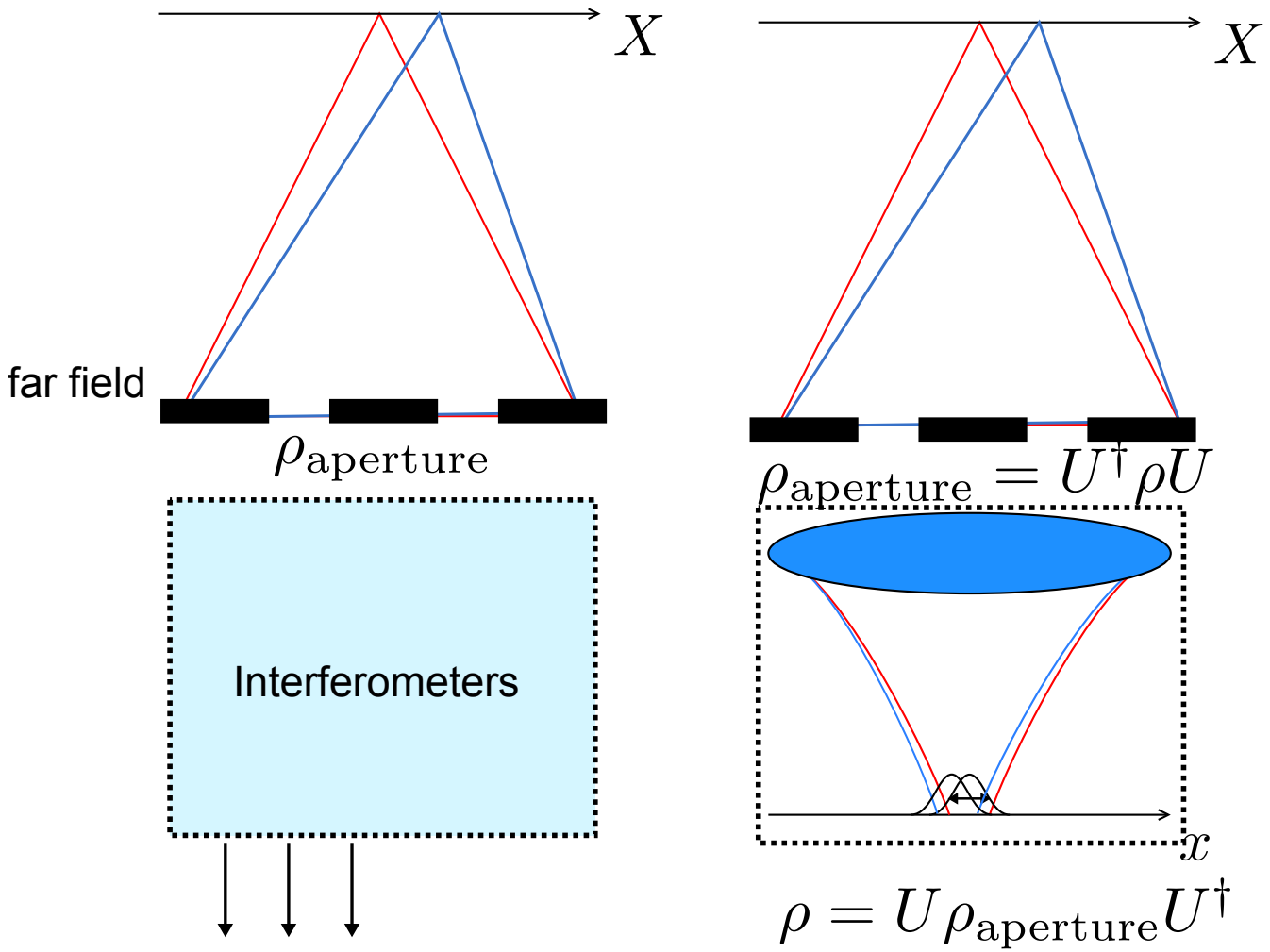


(wikipedia)



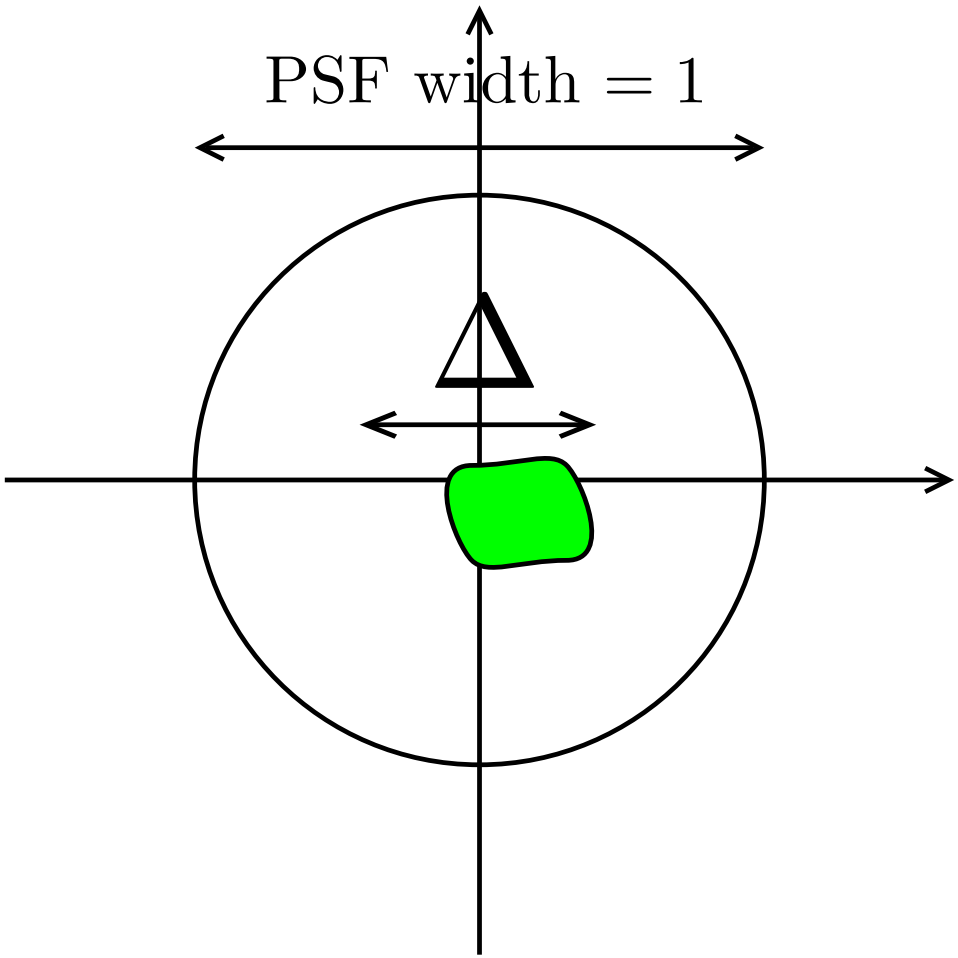
(bbc.com)

Stellar Interferometry



- Same principles apply to stellar interferometry if we **back-propagate**.
- Use the aperture function as the optical transfer function $(\psi(x) \rightarrow \Psi(k))$

Misalignment

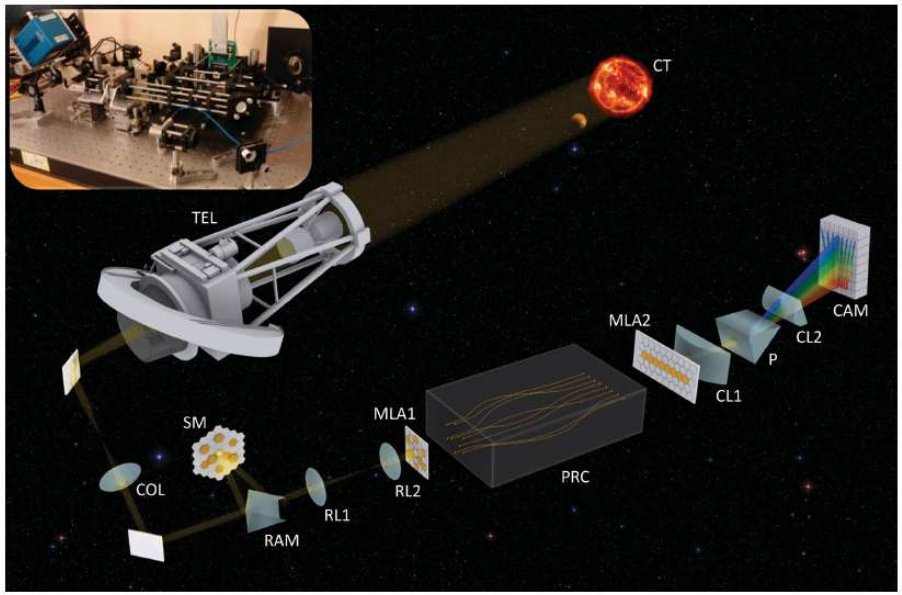


$\Delta =$ **centroid displacement** $+$ **object size** $\ll 1$

Adaptive measurements: Grace *et al.* (Arizona), JOSAA **37**, 1288 (2020).

Stellar Interferometry

- **Astrophotonics**: photonic circuits for stellar interferometry

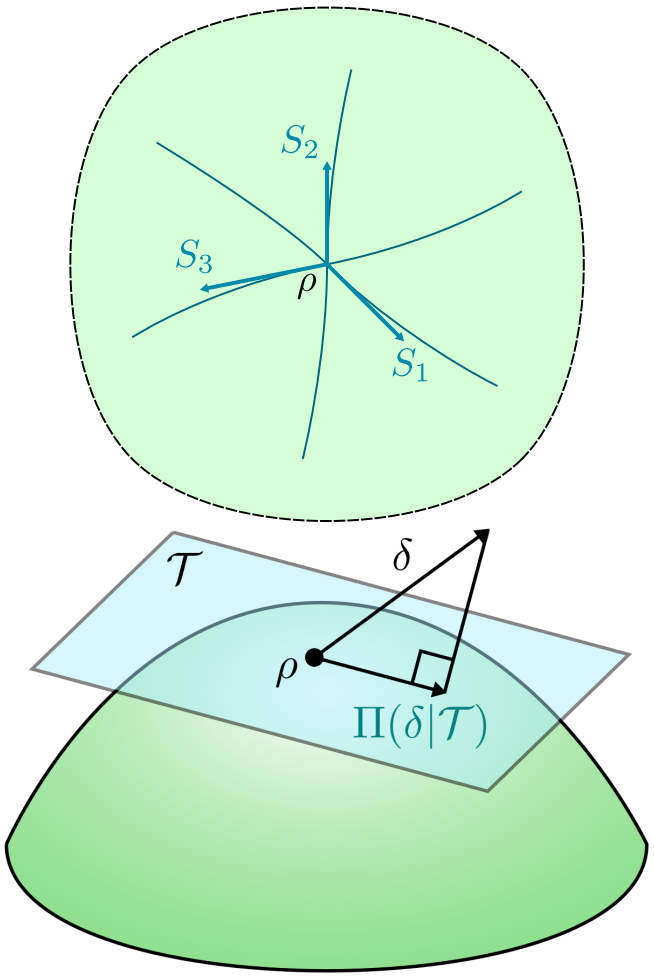


"Dragonfly," Jovanovic *et al.*, Mon. Not. R. Astron. Soc. **427**, 806 (2012)

- **Conventional wisdom**: robust against atmospheric turbulence, **can't compete with direct imaging at diffraction limit**; see, e.g.,
  - Goodman, *Statistical Optics*
  - Zmuidzinas, JOSA A (2003): *It is important to remember that the imperfect beam patterns of sparse-aperture interferometers extract a sensitivity penalty as compared with filled-aperture telescopes, even after accounting for the differences in collecting areas.*
- **Our work**:
  - **Advantage with diffraction + photon shot noise**
  - **Optimal modes adapted to PSF/aperture**
  - **Quantum limit**

Detour: Quantum Semiparametric Estimation

- Let ρ be the true density operator.
- **Hilbert space of observables** $L_2(\rho)$ with $\langle a, b\rangle \equiv \mathrm{tr}(a \circ b)\rho$ (Holevo), $a \circ b \equiv (ab + ba)/2$.
- Zero-mean subspace: $\mathcal{Z} \equiv \{h \in L_2(\rho) : \langle h, I\rangle = \mathrm{tr}\,\rho h = 0\} \subset L_2(\rho)$.
- Each 1D **parametric submodel** with scalar θ_j gives a **SLD score operator** S_j that obeys $\partial\rho/\partial\theta_j = S_j \circ \rho$.
- **Tangent space**: $\mathcal{T} \equiv \overline{\mathrm{span}}\{S_j\} \subseteq \mathcal{Z}$.
- Define parameter of interest $\beta(\theta)$, error operator $\delta \equiv \int \check{\beta}(x)E(dx) - \beta(\theta)$ (E = POVM).
- δ is an "influence operator," defined by $\partial_j\beta = \langle S_j, \delta\rangle$. δ acts like a gradient of β.

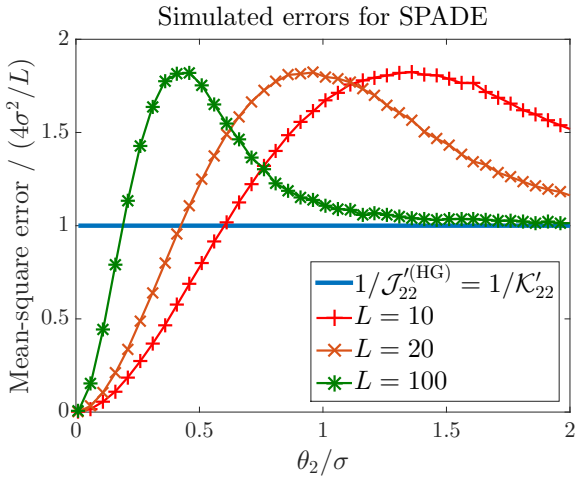


$$\boxed{\mathrm{MSE} \geq \|\delta\|^2 \geq \mathrm{QCRB} = \|\Pi(\delta|\mathcal{T})\|^2.} \quad (4)$$

- Tsang, Albarelli, Datta, PRX (2020); see also Fujiwara's thesis [Chap. 18 in Hayashi (2005)].
- If $\{S_j\}$ is a finite and linearly independent set, $\mathrm{QCRB} = (\partial\beta)^\top K^{-1}\partial\beta$, $K_{jk} \equiv \langle S_j, S_k\rangle$.

Detour: Bayesian Cramér-Rao Bounds

- CRB works for **unbiased** estimators/in a **local asymptotic** sense only. For finite sample size, CRB may be violated.
- Solution: Bayesian/minimax bounds
- Assume $\theta \in \Theta \subseteq \mathbb{R}^p$, a prior density $\pi(\theta)$, $\beta : \Theta \to \mathbb{R}$, n i.i.d. samples. Schutzenberger/Van Trees/**Gill-Levit**:



$$\langle \mathsf{MSE} \rangle \geq \mathsf{BCRB} \equiv \frac{\langle v^\top \partial\beta \rangle^2}{n\langle v^\top J v \rangle + \langle \mathsf{P} \rangle}, \quad \langle \cdots \rangle \equiv \int \cdots \pi(\theta) d^p\theta, \quad \mathsf{P} \equiv \left[\frac{1}{\pi}\partial^\top (\pi v)\right]^2.$$

(J =Fisher information matrix)

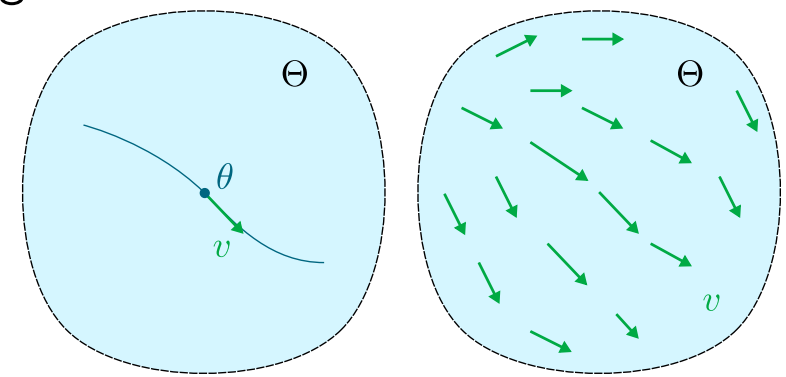
- Problems:
  1. **Reparametrization** of θ leads to a different bound.
  2. **Family of bounds**: How to choose the vector v?

Invariant Gill-Levit Bound

- Treat Θ as a **manifold**. Assume a Riemannian metric g_{ab}. Define $\epsilon \equiv \sqrt{\det g} d^p\theta$, $\rho \equiv \pi/\sqrt{\det g}$. Define covariant derivative ∇_a.
- **Invariant Gill-Levit Bound** [Tsang PRA (2020); see also Jupp JMVA (2010)]:

$$\text{BCRB} = \frac{\langle v^a \nabla_a \beta \rangle^2}{n\langle v^a J_{ab} v^b \rangle + \langle \text{P} \rangle}, \quad \langle \cdots \rangle \equiv \int \cdots \rho\epsilon, \quad \text{P} = \left[\frac{1}{\rho}\nabla_a(\rho v^a)\right]^2. \tag{5}$$

  (differential geometry notations)
- Treat v as a **vector field**. Transform v^a in a contravariant way upon reparametrization.
- v generalizes the tangent vector in CRB:



Optimal Gill-Levit Bound

- There exists a **least favorable** v that maximizes the bound. It obeys the differential equation

$$\boxed{nJ_{ab}v^b - \nabla_a \left[\frac{1}{\rho}\nabla_b \left(\rho v^b\right)\right] = \nabla_a \beta.} \tag{6}$$

- Tsang, PRA (2020)
- Example 1: assume $\beta(\theta) = u^\top \theta$, $g = I$ (flat manifold), J independent of θ, prior $\rho(\theta) = \pi(\theta)$ is Gaussian with covariance matrix Σ. Then

$$v = \left(nJ + \Sigma^{-1}\right)^{-1} u, \qquad \text{BCRB} = u^\top \left(nJ + \Sigma^{-1}\right)^{-1} u, \tag{7}$$

  coincides with Schutzenberger/Van Trees. Tight for linear Gaussian models.
- Example 2: $n \to \infty$

$$nJv \approx \nabla\beta, \qquad v \approx \frac{1}{n} J^{-1}\nabla\beta, \qquad \text{BCRB} \approx \frac{1}{n}\left\langle (\nabla\beta)^\top J^{-1} \nabla\beta \right\rangle. \tag{8}$$

  coincides with the least favorable tangent vector for CRB, consistent with local asymptotic minimax theorem.

Minimax Bound

- Minimax:

$$\sup_{\theta\in\Theta} \mathrm{MSE}(\theta) \geq \langle \mathrm{MSE} \rangle \text{ for any } \pi \text{ with support } \subseteq \Theta. \tag{9}$$

  Choose unfavorable prior to tighten bound.
- Assume $\beta = \theta$ (scalar), $u = 1$. Choose constant $v = 1$, $g = 1$ (Van Trees version).
- Trick: let $\pi(\theta) = [\psi(\theta)]^2$.

$$\mathrm{BCRB} = \frac{1}{n\langle \mathrm{J} \rangle + \langle \mathrm{P} \rangle}, \qquad n\langle \mathrm{J} \rangle + \langle \mathrm{P} \rangle = \int \left[nJ\psi^2 + \psi(-4\partial^2)\psi \right] d\theta. \tag{10}$$

  Think of $\psi(\theta)$ as a **wavefunction**, $nJ(\theta)$ as **potential energy**, $-4\partial^2$ as **kinetic energy**. Choose $\pi(\theta)$ to maximize the bound $\rightarrow$ choose $\psi(\theta)$ to minimize total energy. Best bound = $1/$**ground-state energy**.
- Tsang, JMO (2018); PRA (2020)
- Example 1: If $\min_\theta J(\theta) > 0$, then ground-state energy $= n \min_\theta J(\theta) + o(n)$, BCRB $= \Theta(n^{-1})$ (parametric rate).
- Example 2: If $J(\theta) \leq c\theta^m$, then ground-state energy $= O(n^{2/(m+2)})$, BCRB $= \Omega(n^{-2/(m+2)})$ (subparametric rate).
- Example 3: $J(\theta) \leq c\theta^2$ (quadratic potential), then ground-state energy $= O(n^{1/2})$, BCRB $= \Omega(n^{-1/2})$.

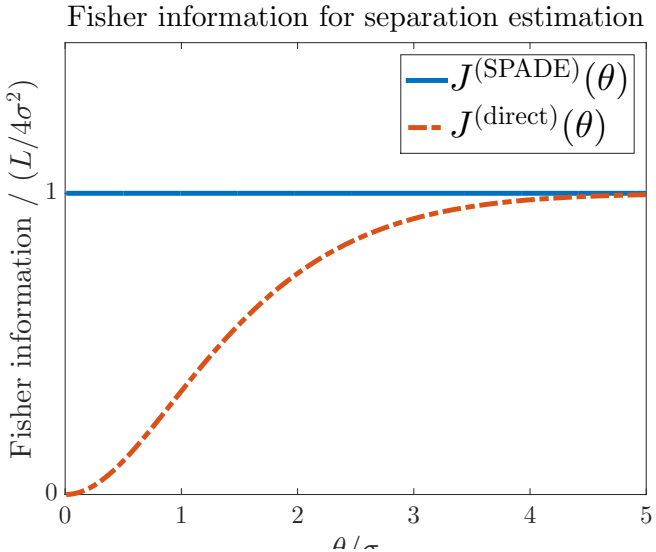
Imaging Example

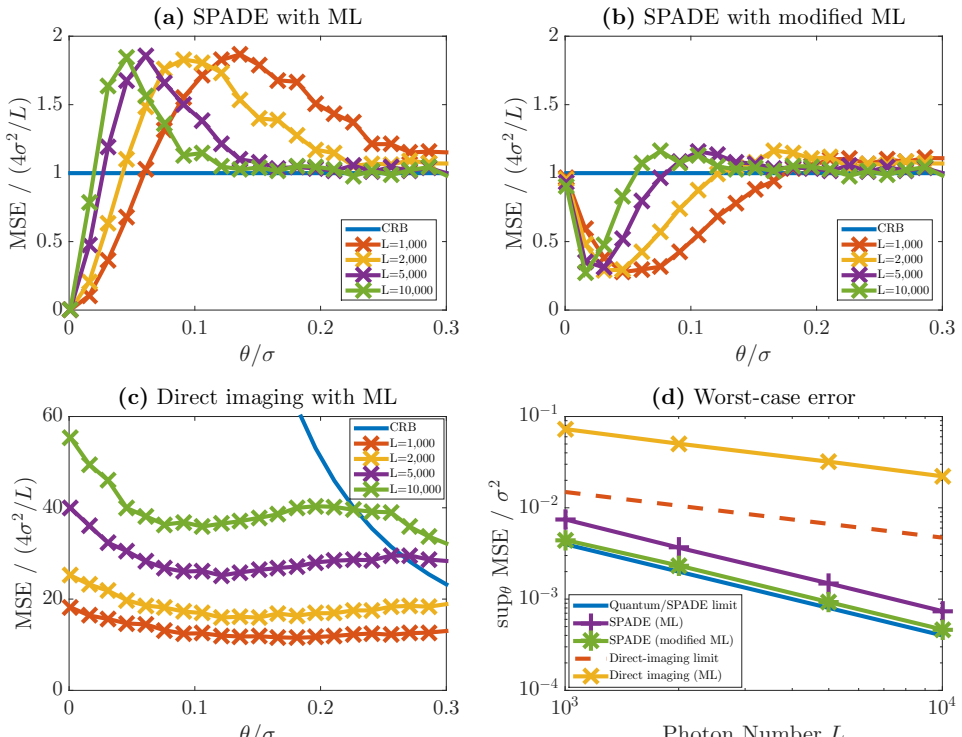
- Two-point separation estimation, direct imaging with Gaussian PSF:

$$J \leq c\theta^2,$$

$$\sup_\theta \mathrm{MSE}(\theta) = \Omega(n^{-1/2}).$$

- SPADE: $\sup_\theta \mathrm{MSE}(\theta) = \Omega(n^{-1})$.
- Tsang, JMO (2018)





Semiparametric Examples

- If parameter space = {any density operator} (nonparametric), any $h \in \mathcal{Z}$ can be a score because I can construct a parametric submodel from any h, e.g.,

$$\sigma(\theta) = \frac{[I + \tanh(\theta h/2)]\rho[I + \tanh(\theta h/2)]}{\mathrm{tr}(\text{numerator})}. \tag{11}$$

  Hence $\boxed{\mathcal{T} = \mathcal{Z},}$ tangent space is **full-dimensional**.
- Example 1: $\beta = \mathrm{tr}\, Y\rho$. $\partial_j \beta = \mathrm{tr}\, Y \partial_j \rho = \mathrm{tr}\, Y(S_j \circ \rho) = \langle Y, S_j \rangle$.
- $\Pi(Y|\mathcal{Z}) = Y - \langle Y, I\rangle = Y - \mathrm{tr}\,\rho Y = \delta$ is an influence operator.
- Efficient influence: $\Pi(\delta|\mathcal{T}) = \Pi(\delta|\mathcal{Z}) = Y - \mathrm{tr}\,\rho Y$.

$$\boxed{\text{QCRB} = \|Y - \mathrm{tr}\,\rho Y\|^2 = \mathrm{tr}\,\rho\,(Y - \mathrm{tr}\,\rho Y)^2\,.} \tag{12}$$

Measurement of Y is efficient. (See also Watanabe *et al.*, PRL (2010))

- Example 2: $\beta = \mathrm{tr}\,\rho^2$. $\partial_j \beta = 2\,\mathrm{tr}\,\rho \partial_j \rho = 2\,\mathrm{tr}\,\rho(S_j \circ \rho) = \langle S_j, 2\rho\rangle$.
- $\Pi(2\rho|\mathcal{Z}) = 2\rho - \langle 2\rho, I\rangle$ is an influence operator.
- Efficient influence: $\Pi(\delta|\mathcal{T}) = \Pi(\delta|\mathcal{Z}) = 2\rho - \langle 2\rho, I\rangle$.

$$\boxed{\text{QCRB} = 4\,\mathrm{tr}\,\rho\left(\rho - \mathrm{tr}\,\rho^2\right)^2\,.} \tag{13}$$

- **Open question**: optimal measurement?

Quantum Limit on Semiparametric Imaging

- One-photon density operator: $\rho_1(F) = \int e^{-i\hat{k}X} |\psi\rangle \langle\psi| e^{i\hat{k}X} F(X) dX$, $F \in \{\text{any probability density}\}$, $\beta(F) = \int [X^\mu + o(\Delta^\mu)] F(X) dX$.
- Exact QCRB is still intractable! ($\mathcal{T}$ is a nontrivial subspace of $\mathcal{Z}$)
- Find a looser bound using an unfavorable 1D submodel $\{\rho_1(F_\theta) : \theta \in \Theta \subseteq \mathbb{R}\}$ [Lemma 2, Tsang, PRA (2021)]:

$$\text{QCRB} = \sup_{\text{submodel} \in \{\text{1D submodels}\}} \text{QCRB}^{(\text{submodel})}.$$

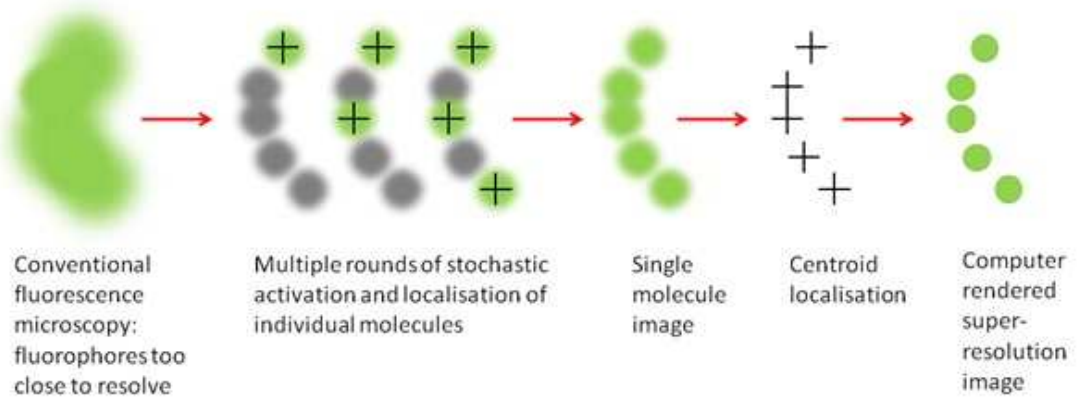
  Classical: Stein (1956).
- Still intractable! (ρ_1 is mixed and ∞-dim) Find lower bound on $\text{QCRB}^{(\text{submodel})}$ in terms of a unfavorable purification of ρ_1.
- **Quantum bound** [Tsang, PRA (2021)]:

$$\boxed{\text{QCRB}^{(\text{purified 1D submodel})} = \frac{\Omega(\Delta^{2\lfloor \mu/2 \rfloor})}{N},} \quad \text{compare with } \text{MSE}^{(\text{SPADE})} = \frac{\Theta(\Delta^{2\lfloor \mu/2 \rfloor})}{N}.$$

- See also Tsang, PRA (2019); Zhou & Jiang (Yale/Chicago), PRA (2019).

Superresolution Microscopy

- PALM, STORM, STED, etc.: make **sparse subsets** of fluorophores emit



https://cam.facilities.northwestern.edu/588-2/single-molecule-localization-microscopy/

- **avoid violating Rayleigh**
- Need **controllable** fluorophores
- **slow**, **phototoxicity**
- doesn't work for stars, passive imaging



The Nobel Prize in Chemistry 2014
Eric Betzig, Stefan W. Hell, William E. Moerner

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The Nobel Prize in Chemistry 2014

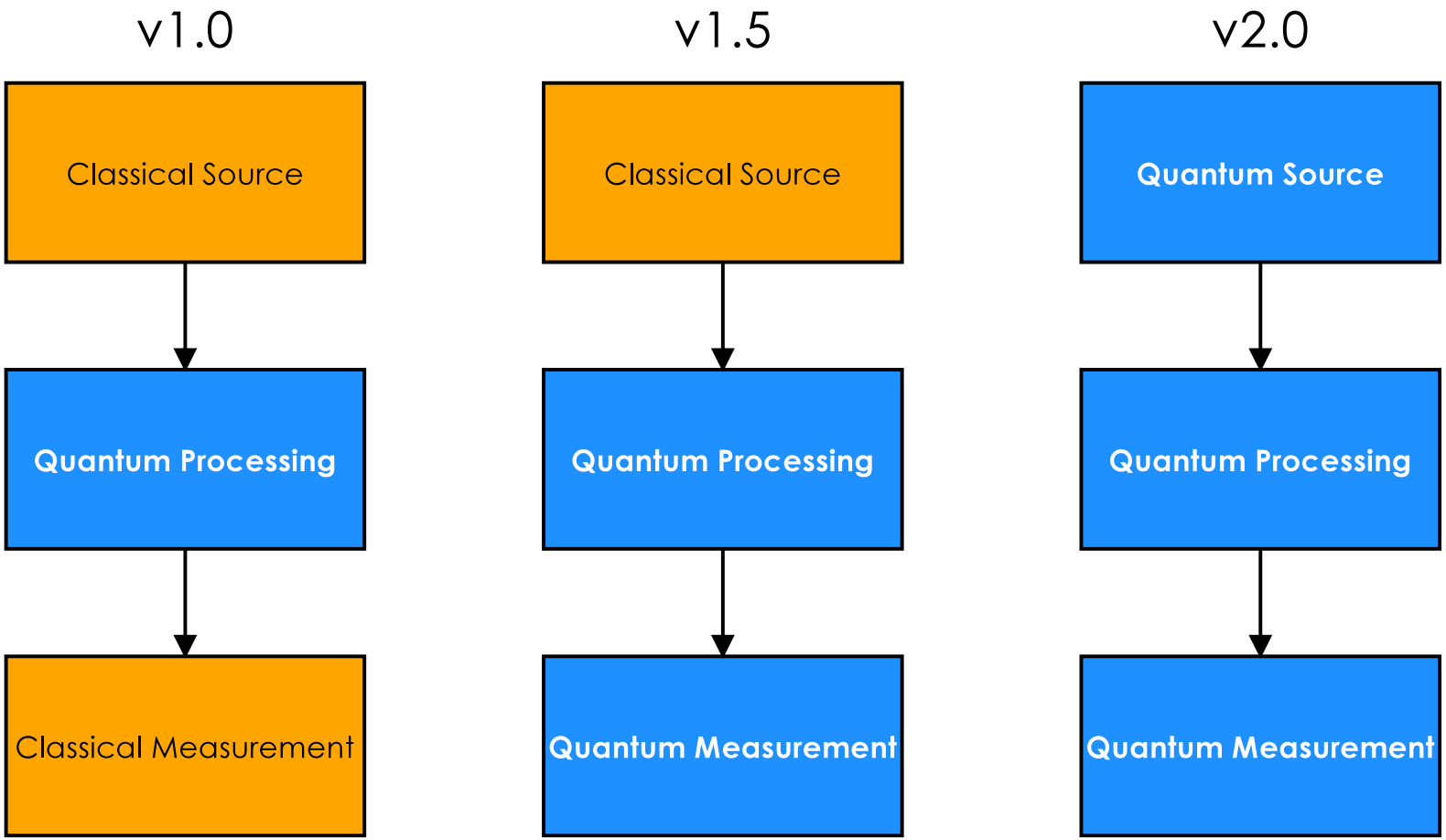
Photo: A. Mahmoud
Eric Betzig
Prize share: 1/3

Photo: A. Mahmoud
Stefan W. Hell
Prize share: 1/3

Photo: A. Mahmoud
William E. Moerner
Prize share: 1/3

The Nobel Prize in Chemistry 2014 was awarded jointly to Eric Betzig, Stefan W. Hell and William E. Moerner *"for the development of super-resolved fluorescence microscopy"*.

Quantum Technology 1.5



Solid-state
Electronics

Sensing
Imaging

Quantum-Enhanced Metrology
Quantum Computer
Quantum Internet