Reversing Time and Space in Classical and Quantum Optics

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Overview

- Nonlinear Optics
  - Reverse Propagation
  - Spectral Phase Conjugation

- Quantum Optics
  - Time-Entangled Photons
  - Quantum Imaging using Path-Entangled Photons

- Nano-Imaging
  - Evanescent-Wave Amplification and Conversion

- Quantum Sensing
  - Bayesian Quantum Estimation
  - Time-Symmetric Smoothing
Ultrashort Optical Pulse Propagation in Fiber

\[
\frac{\partial A}{\partial z} = -\frac{\alpha}{2} A - \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6} \frac{\partial^3 A}{\partial T^3} + i\gamma \left[ |A|^2 A + \frac{i}{\omega_0} \frac{\partial}{\partial T} (|A|^2 A) - T_R A \frac{\partial |A|^2}{\partial T} \right]
\]

- Agrawal, *Nonlinear Fiber Optics*
- Phase Conjugation, Solitons, Evolution Algorithm, etc.
Reverse Propagation

\[ \frac{\partial A}{\partial z} = -\frac{\alpha}{2} A - \frac{i\beta_2}{2} \frac{\partial^2 A}{\partial T^2} + \frac{\beta_3}{6} \frac{\partial^3 A}{\partial T^3} + i\gamma \left[ |A|^2 A + \frac{i}{\omega_0} \frac{\partial}{\partial T} (|A|^2 A) - T_R A \frac{\partial |A|^2}{\partial T} \right] \]

Fig. 2. Reverse propagation of an experimental output pulse. The experimental output pulse shape is plotted at \( z = -9 \) m and numerically propagates in reverse from \( z = 0 \) m to \( z = -10 \) m.

Fig. 3. Comparison of the input obtained from reverse propagation and the actual experimental input.

Temporal Phase Conjugation


can only compensate for even-order dispersion and Kerr effect

\[ 2\omega_0 \]

\[ \omega_0 \]

\[ \chi^{(2)} \]

\[ A_s(t) \]

\[ A_i(t) \sim A_s^*(t) \]

\[ a_s(\omega) \]

\[ a_i(\omega) \sim a_s^*(2\omega_0 - \omega) \]
Spectral Phase Conjugation


Compensate for dispersion of all orders as well as Kerr effect

$A_s(t)$

$A_i(t) \sim A_s^*(t_0-t)$

$a_s(\omega)$

$a_i(\omega) \sim a_s^*(\omega)$

Kuzucu et al., CLEO 2009 Postdeadline paper CPDB3
How to Perform SPC


Spontaneous Parametric Down Conversion

\[ \hat{H} \sim \chi \hat{A}_p \hat{A}_s^\dagger \hat{A}_i^\dagger + \text{H.c.}, \quad |\Psi\rangle \sim |0\rangle + \eta \hat{A}_p \hat{A}_s^\dagger \hat{A}_i^\dagger |0\rangle \quad (1) \]
Spontaneous SPC

Coincident Frequency Entanglement

SPC via Extended Phase Matching

\[ A_s(L, \tau) = A_s(0, \tau) \sec G + iA_i^*(0, -\tau) \tan G \]  \hfill (2)

\[ A_i(L, \tau) = A_i(0, \tau) \sec G + iA_s^*(0, -\tau) \tan G \]  \hfill (3)

\[ G = \frac{v \chi}{1 - k_p'/k_s'} \int d\tau A_p(\tau) \]  \hfill (4)

predicted photon pair generation rate = \( f_R \tan^2 G \approx 3.6 \times 10^6 \text{ s}^{-1} \), matches Kuzucu et al.’s experiment (4 \times 10^6 \text{ s}^{-1})
Mirrorless Parametric Oscillator

What happens when $G = \pi/2$, $\tan^2 G = \infty$?


43% down conversion efficiency, 140 dB equivalent gain
**What’s Special about Coin. Freq. Entanglement?**

- **Quantum enhancement of timing accuracy** [Giovannetti et al., Nature 412, 417 (2001)].

\[
\begin{align*}
\frac{t_2 - t_1}{\sqrt{2}} & \quad \frac{t_1 + t_2}{\sqrt{2}} \\
\Delta t & \quad \Delta \omega
\end{align*}
\]

- Analogous to how *mutual funds* work: selecting negatively-correlated stocks reduces risk.

Multiphoton Enhancement


- $N$ independent photons:

$$\Delta T \geq \frac{1}{2\sqrt{N}\Delta \omega}$$  
(Standard Quantum Limit)  
(5)

- e.g.: $W = 100$ ps, $N = 10^{10}$, $\Delta T = 1$ fs

- $N$ negatively-time-correlated photons:

$$\Delta T \geq \frac{1}{2N\Delta \omega}$$  
(Ultimate Quantum Limit)  
(6)

- $N = 2$ is quite useless compared to $N \gg 1$

- How to create multiphoton time anti-correlation with $N \gg 1$?

- Extended Phase Matching doesn’t work when number of generated photon pairs > 1.
Adiabatic Control of Optical Solitons

Adiabatic Soliton Expansion

Normal GVD, Linear

Quantum Imaging

Boto et al., Phys. Rev. Lett. 85, 2733 (2000), resolution $\sim \lambda/N$

Near-Field Imaging


\[ n = -1 \]

\[ \varepsilon < 0 \]

Continuous Waveform Estimation

In real-world applications, the signal $x(t)$ to be estimated is usually a random process.

Nonclassical photons need to be created continuously for continuous estimation.

Bayesian Estimation Theory
Tracking an Aircraft

or submarine, terrorist, criminal, mosquito, cancer cell, nanoparticle, ...

Use Bayes theorem:

\[
P(x_t|\delta y_t) = \frac{P(\delta y_t|x_t)P(x_t)}{\int dx_t} \tag{7}
\]

\(P(\delta y_t|x_t)\) from observation noise, \(P(x_t)\) from *a priori* information.
Prediction

Assume $x_t$ is a Markov process, use the Chapman-Kolmogorov equation:

$$P(x_{t+\delta t} | \delta y_t) = \int dx_t P(x_{t+\delta t} | x_t) P(x_t | \delta y_t)$$

(8)
Filtering: Real-Time Estimation

Applying Bayes theorem and Chapman-Kolmogorov equation repeatedly, we can obtain

\[ P(x_t | \delta y_{t-\delta t}, \ldots, \delta y_{t_0+\delta t}, \delta y_{t_0}) \]  

Useful for control, weather and finance forecast, etc.

Wiener, Stratonovich, Kalman, Kushner, etc.
Quantum Filtering

Use "quantum Bayes theorem,"

$$\hat{\rho}_t(\mid\delta y_t) = \frac{\hat{M}(\delta y_t)\hat{\rho}_t\hat{M}^\dagger(\delta y_t)}{\text{tr}(\text{numerator})}$$  \hspace{1cm} (10)

Use a completely positive map to evolve the system state (analogous to Chapman-Kolmogorov),

$$\hat{\rho}_{t+\delta t}(\mid\delta y_t) = \sum_{\mu} \hat{K}_\mu \hat{\rho}_t(\mid\delta y_t)\hat{K}_\mu^\dagger$$  \hspace{1cm} (11)

Belavkin, Barchielli, Carmichael, Caves, Milburn, Wiseman, Mabuchi, etc.

Useful for cavity QED, quantum optics, etc.
Define hybrid density operator $\hat{\rho}(x_t), P(x_t) = \text{tr}[\hat{\rho}(x_t)], \hat{\rho}_t = \int dx_t \hat{\rho}(x_t)$.

Use generalized quantum Bayes theorem and positive map + Chapman-Kolmogorov:

$$\hat{\rho}(x_t|\delta y_t) = \frac{\hat{M}(\delta y_t|x_t)\hat{\rho}(x_t)\hat{M}^\dagger(\delta y_t|x_t)}{\int dx_t \text{tr}(\text{numerator})},$$

$$\hat{\rho}(x_{t+\delta t}|\delta y_t) = \int dx_t P(x_{t+\delta t}|x_t) \sum_\mu \hat{K}_\mu(x_t)\hat{\rho}(x_t|\delta y_t)\hat{K}_\mu^\dagger(x_t)$$

Hybrid Filtering Equations

Hybrid Belavkin equation (analogous to Kushner-Stratonovich):

$$d\hat{\rho}(x, t) = dt \left[ L_0 + L(x) - \frac{\partial}{\partial x_\mu} A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu} B_{\mu\nu} \right] \hat{\rho}(x, t)$$

$$+ \frac{dt}{8} \left[ 2\hat{C}^T R^{-1} \hat{\rho}(x, t) \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{\rho}(x, t) - \hat{\rho}(x, t) \hat{C}^\dagger T R^{-1} \hat{C} \right]$$

$$+ \frac{1}{2} \left( dy_t - \frac{dt}{2} \left< \hat{C} + \hat{C}^\dagger \right> \right)^T R^{-1} \left[ \left( \hat{C} - \left< \hat{C} \right> \right) \hat{\rho}(x, t) + \text{H.c.} \right]$$

(14)

Linear version (analogous to Duncan-Mortensen-Zakai):

$$d\hat{f}(x, t) = dt \left[ L_0 + L(x) - \frac{\partial}{\partial x_\mu} A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu} B_{\mu\nu} \right] \hat{f}(x, t)$$

$$+ \frac{dt}{8} \left[ 2\hat{C}^T R^{-1} \hat{f}(x, t) \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{\rho}(x, t) - \hat{f}(x, t) \hat{C}^\dagger T R^{-1} \hat{C} \right]$$

$$+ \frac{1}{2} dy_t^T R^{-1} \left[ \hat{C} \hat{f}(x, t) + \text{H.c.} \right], \quad \hat{\rho}(x, t) = \frac{\hat{f}(x, t)}{\int dx \text{tr}[\hat{f}(x, t)]}.$$  

(15)
Phase-Locked Loop Design


Smoothing: Estimation with Delay

- More accurate than filtering
- Sensing, analog communication, astronomy, crime investigation, ...
- Delay
Conventional quantum theory is a **predictive** theory

Quantum state described by $|\Psi_t\rangle$ or $\hat{\rho}_t$ can only be conditioned only upon past observations

“Weak values” by Aharonov, Vaidman, *et al.*

“Quantum retrodiction” by Barnett, Pegg, Yanagisawa, etc.
Use two operators to describe system: density operator $\hat{\rho}(x_t|\delta y_{\text{past}})$ and a retrodictive likelihood operator $\hat{E}(\delta y_{\text{future}}|x_t)$

$$P(x_t|\delta y_{\text{past}}, \delta y_{\text{future}}) = \frac{\text{tr} \left[ \hat{E}(\delta y_{\text{future}}|x_t) \hat{\rho}(x_t|\delta y_{\text{past}}) \right]}{\int dx \text{(numerator)}}$$  \hspace{1cm} (16)
Solve the predictive equation from $t_0$ to $t$ using \textit{a priori} $\hat{f}$ as initial condition, and solve the retrodictive equation from $T$ to $t$ using $\hat{g}(x, T) \propto 1$ as the final condition.

\[
d\hat{f} = dt\mathcal{L}(x)\hat{f} + \frac{dt}{8} \left( 2\hat{C}_R^{-1}\hat{f}\hat{C}^\dagger - \hat{C}_R^\dagger\hat{C}_R^{-1}\hat{f} - \hat{C}^\dagger\hat{f}\hat{C}_R^{-1}\hat{C} \right) + \frac{1}{2} dy_T T R^{-1} \left( \hat{C}\hat{f} + \hat{f}\hat{C}^\dagger \right)
\]

\[
-d\hat{g} = dt\mathcal{L}^*(x)\hat{g} + \frac{dt}{8} \left( 2\hat{C}^\dagger R^{-1}\hat{g}\hat{C} - \hat{C}^\dagger R^{-1}\hat{g} - \hat{g}\hat{C}^\dagger R^{-1}\hat{C} \right) + \frac{1}{2} dy_T T R^{-1} \left( \hat{C}^\dagger\hat{g} + \hat{g}\hat{C} \right)
\]

\[
h(x, t) = P(x_t = x|\delta y_{\text{past}}, \delta y_{\text{future}}) = \frac{\text{tr} \left[ \hat{g}(x, t)\hat{f}(x, t) \right]}{\int dx (\text{numerator})}
\]


Phase-Space Smoothing

Convert to Wigner distributions:

\[ \text{tr}[\hat{g}(x, t)\hat{f}(x, t)] \propto \int dqdp \ g(q, p, x, t)f(q, p, x, t) \]  

(17)

\[ h(x, t) = \frac{\int dqdp \ g(q, p, x, t)f(q, p, x, t)}{\int dx \ (\text{numerator})} \]  

(18)

If \( f(q, p, x, t) \) and \( g(q, p, x, t) \) are non-negative, equivalent to classical smoothing:

\[ h(q, p, x, t) = \frac{g(q, p, x, t)f(q, p, x, t)}{\int dx dq dp \ (\text{numerator})}, \quad h(x, t) = \int dqdp \ h(q, p, x, t) \]  

(19)

\( q \) and \( p \) can be regarded as classical, with \( h(q, p, x, t) \) the classical smoothing probability distribution.

If \( f(q, p, x, t) \) and \( g(q, p, x, t) \) are Gaussian, equivalent to linear smoothing, use two Kalman filters (Mayne-Fraser-Potter smoother)
Phase-Locked Loop: Post-Loop Smoother

Force Sensor

Assume force is an Ornstein-Uhlenbeck process:

\[ dx_t = -ax_t dt + bdW_t \]  

Hamiltonian:

\[ \hat{H}(x_t) = \frac{\hat{p}^2}{2m} - x_t\hat{q} \]

Filtering

Filtering equation:

\[ d\hat{f}(x, t) = -\frac{i}{\hbar}dt \left[ \hat{H}(x), \hat{f}(x, t) \right] + dt \left( a \frac{\partial}{\partial x} + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} \right) \hat{f}(x, t) \]
\[ + \frac{\gamma}{8} dt \left[ 2\hat{q}\hat{f}(x, t)\hat{q} - \hat{q}^2 \hat{f}(x, t) - \hat{f}(x, t)\hat{q}^2 \right] + \frac{\gamma}{2} dy_t \left[ \hat{q}\hat{f}(x, t) + \hat{f}(x, t)\hat{q} \right], \]

In terms of \( f(q, p, x, t) \):

\[ df = dt \left( -\frac{p}{m} \frac{\partial}{\partial q} - x \frac{\partial}{\partial p} + a \frac{\partial}{\partial x} + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} + \frac{\hbar^2}{8} \gamma \frac{\partial^2}{\partial p^2} \right) f + \gamma dy_t q f. \]

Kalman filter:

\[ d\mu = A\mu dt + \Sigma C^T \gamma d\eta, \quad \frac{d\Sigma}{dt} = A \Sigma + \Sigma A^T - \Sigma C^T \gamma C \Sigma + Q, \]
\[ A = \begin{pmatrix} 0 & 1/m & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -a \end{pmatrix}, \quad C = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar^2 \gamma/4 & 0 \\ 0 & 0 & b^2 \end{pmatrix}. \]
Steady-State Filtering

- \( \Sigma_{qq}, \Sigma_{qp}, \) and \( \Sigma_{pp} \) determine conditional sensor quantum state \( \hat{\rho} \), \( \Sigma_{xx} \) is mean-square force estimation error.

- At steady state, let \( d\Sigma/dt = 0 \) (solve numerically).

- \( x_t = 0 \) limit (analytic):

\[
\Sigma_{qq} = \sqrt{\frac{\hbar}{m\gamma}}, \quad \Sigma_{qp} = \frac{\hbar}{2}, \quad \Sigma_{pp} = \frac{\hbar\sqrt{\hbar m\gamma}}{2}, \quad \Sigma_{qq}\Sigma_{pp} - \Sigma_{qp}^2 = \frac{1}{4} \tag{22}
\]

Smoothing

- Write retrodictive equation for $\hat{g}(x, t)$
- Convert $\hat{g}(x, t)$ to $g(q, p, x, t)$
- solve for mean vector $\nu$ and covariance matrix $\Xi$ of $g(q, p, x, t)$ using a retrodictive Kalman filter
- Define

$$h(q, p, x, t) = \frac{g(q, p, x, t)f(q, p, x, t)}{\int dqdpdx \ g(q, p, x, t)f(q, p, x, t)}$$  \hspace{1cm} (23)

Mean $\xi$ and covariance $\Pi$ of $h(q, p, x, t)$:

$$\xi = \begin{pmatrix} \xi_q \\ \xi_p \\ \xi_x \end{pmatrix} = \Pi \left( \Sigma^{-1} \mu + \Xi^{-1} \nu \right) \quad \Pi = \begin{pmatrix} \Pi_{qq} & \Pi_{qp} & \Pi_{qx} \\ \Pi_{pq} & \Pi_{pp} & \Pi_{px} \\ \Pi_{xq} & \Pi_{xp} & \Pi_{xx} \end{pmatrix} = (\Sigma^{-1} + \Xi^{-1})^{-1}$$

$\xi_x$ is smoothing estimate of force, $\Pi_{xx}$ is smoothing error

- but what are $\xi_q$, $\xi_p$, $\Pi_{qq}$, $\Pi_{qp}$, $\Pi_{pp}$, and $h(q, p, x, t)$ in general?
Filtering vs Smoothing at Steady State

\[ \frac{a}{\sqrt{b}} = 0.01/(\hbar m)^{1/4}, \quad 1/\beta = (\gamma/b)(\sqrt{\hbar^3/m}), \quad s_{qq} = \sqrt{m\gamma/\hbar \Sigma_{qq}}, \quad s_{qp} = \Sigma_{qp}/\hbar, \]

\[ s_{pp} = \Sigma_{pp}/\sqrt{\hbar^3 m\gamma}, \] blue: filtering, green: \( x_t = 0 \), red: smoothing
Quantum Smoothing

Can we use

\[ h(q, p, x, t) = \frac{g(q, p, x, t) f(q, p, x, t)}{\int dqdpdx \, \text{numerator}} \]  

(24)

to estimate \( q \) and \( p \)?

- when \( g \) and \( f \) are non-negative, problem becomes classical

\[ h(x, t) = \int dqph(q, p, x, t) \]  

(25)

- \( \xi_q, \xi_p \equiv \text{real part of weak values} \)
- arises from statistics of weak measurements

\( h(q, p, x, t) \) can go negative, many versions of Wigner distributions for discrete degrees of freedom
Sensing Beyond Heisenberg?

An engineer would conclude that the sensor did have sub-Heisenberg uncertainties some time in the past to enable the enhanced sensing accuracy.

Shouldn’t we be able to learn more about a system, whether classical or quantum, in retrospect?
Summary

Reverse Propagation and Spectral Phase Conjugation:

Generation of Temporally Anti-correlated Entangled Photons:

Quantum Imaging and Near-Field Imaging:

Quantum Smoothing:

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