Continuous Quantum Hypothesis Testing

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Quantum Probability Theory

- Wave:

- Probability (Born's rule $P(x) = |\langle x|\psi\rangle|^2$):

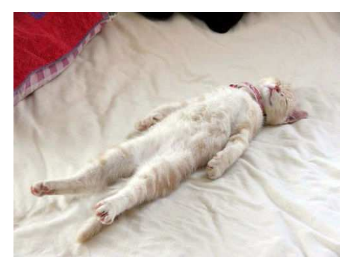
I.2 ON THE QUANTUM MECHANICS OF COLLISIONS

[Preliminary communication]†

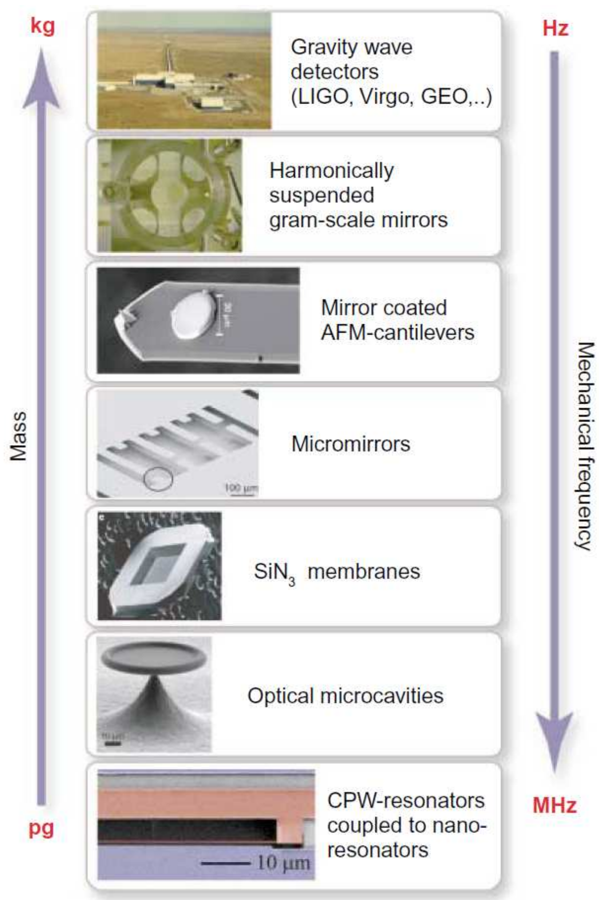
MAX BORN

* Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n_\tau m}$.

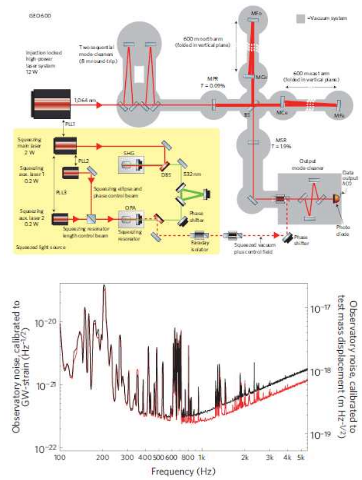
- More than classical probability:

$\Big|$  $\Big\rangle + e^{i\theta}\Big|$  $\Big\rangle$

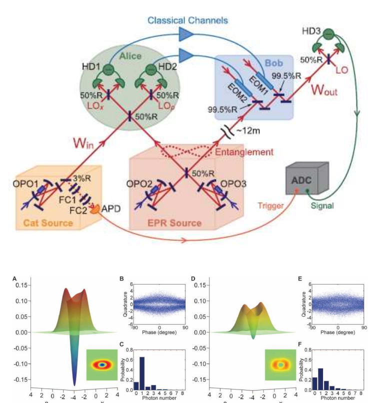
Quantum Probability Experiments



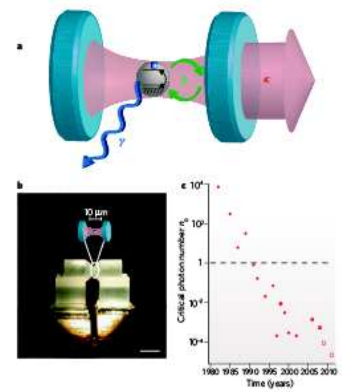
Kippenberg and Vahala, Science **321**, 1172 (2008), and references therein.



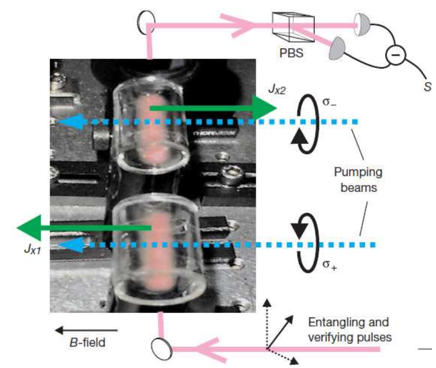
LIGO, Nature Phys. **7**, 962 (2011).



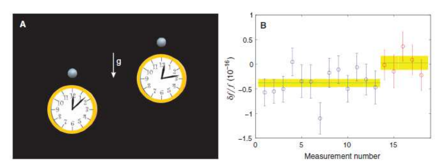
Lee *et al.*, Science **332**, 330 (2011).



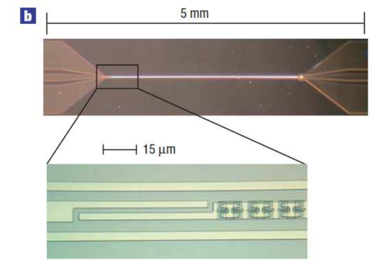
Kimble, Nature **453**, 1023 (2008).



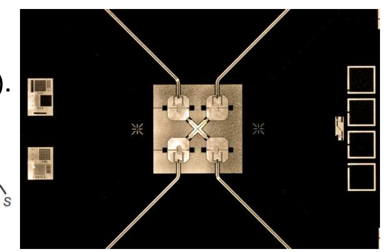
Julsgaard, Kozhekin, and Polzik, Nature **413**, 400 (2001).



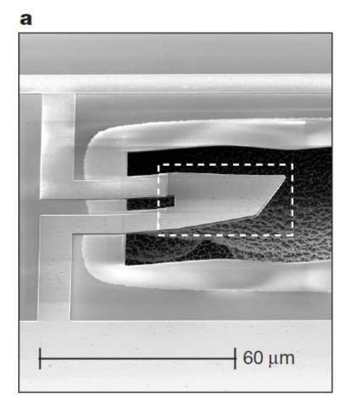
Chou *et al.*, Science **329**, 1630 (2010).



Castellanos-Beltran *et al.*, Nature Phys. **4**, 928 (2008).

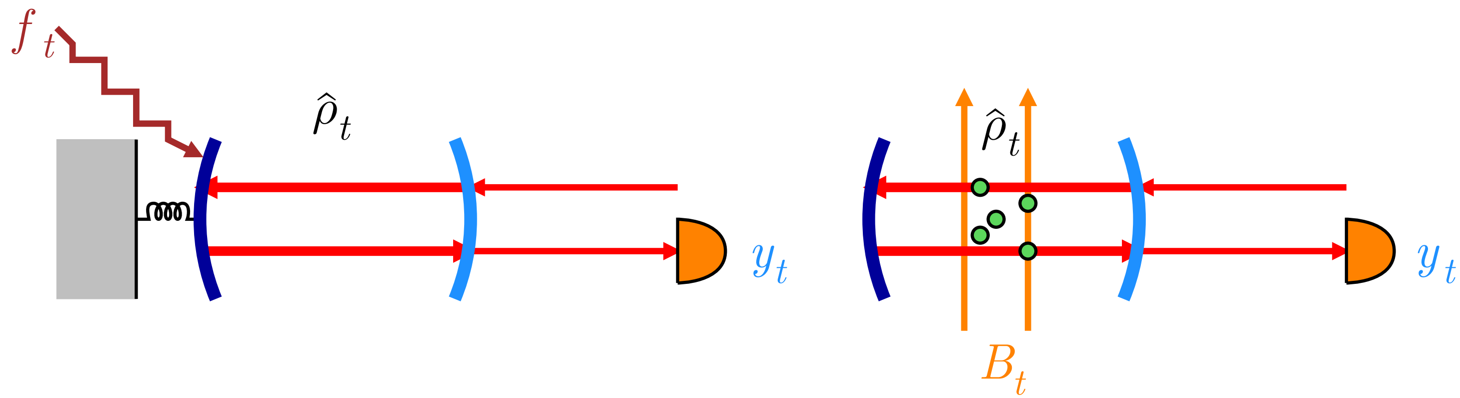


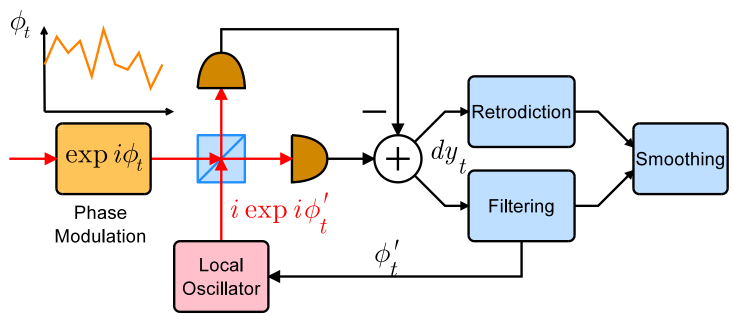
Neeley et al., Nature **467**, 570 (2010)



O'Connell *et al.*, Nature **464**, 697 (2010).

Quantum Sensing/Metrology

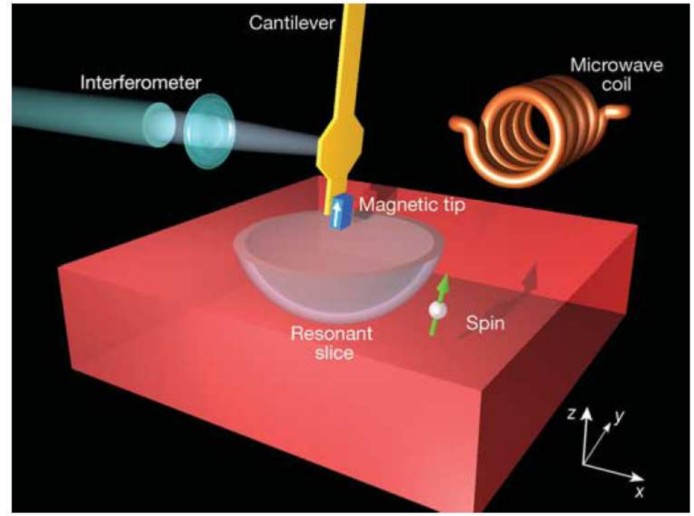




- Estimation: Optimize data processing
- Control: Optimize experiment
- Fundamental Limits: What is the ultimate sensitivity allowed by quantum mechanics?
- Examples: optical interferometry, optical imaging, optomechanical force sensing (gravitational-wave detection), atomic magnetometry, electrometer, etc.

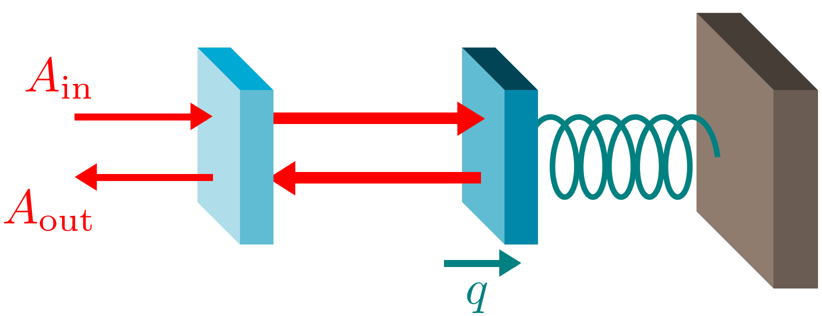
Quantum Optomechanical Force Sensing

LIGO, Hanford

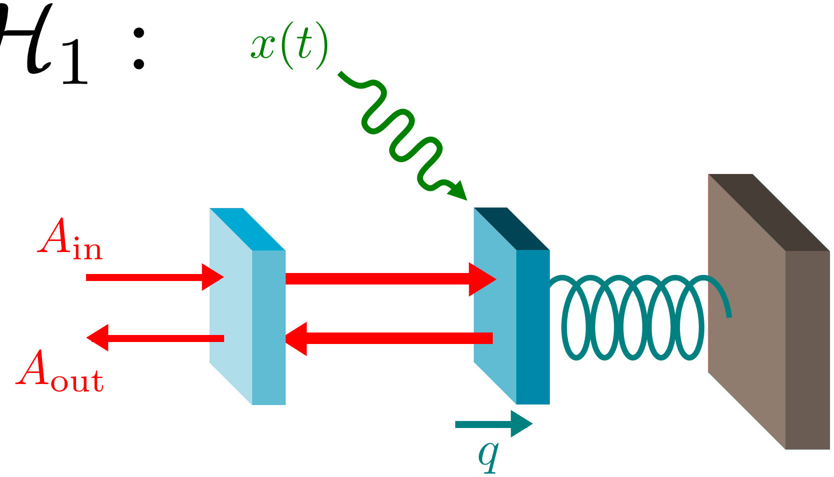


Rugar *et al.*, Nature **430**, 329 (2004).

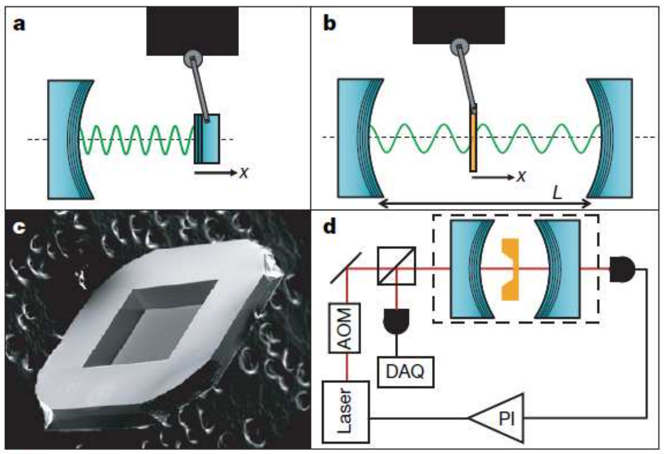
$\mathcal{H}_0$:



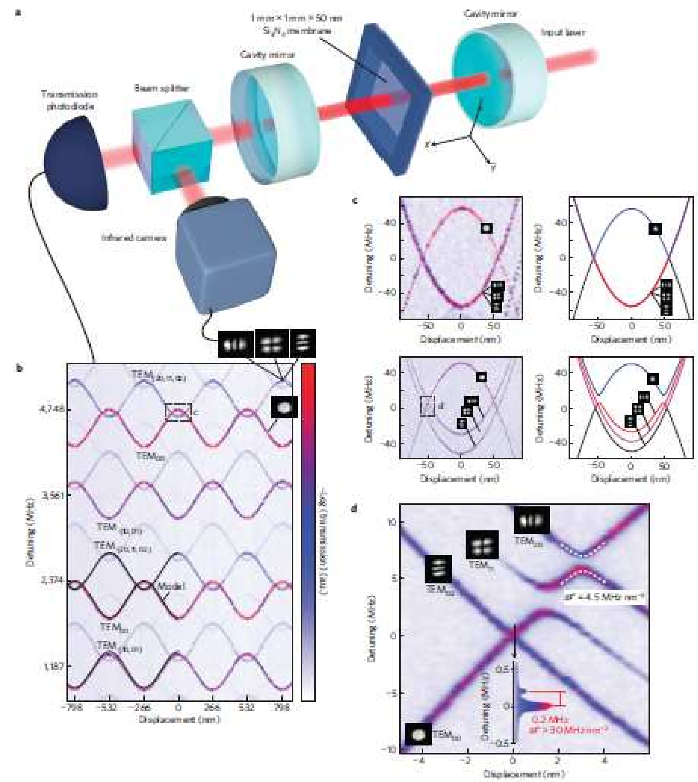
$\mathcal{H}_1$:



Energy Quantization



Thompson *et al.*, Nature **452**, 72 (2008).



Sankey *et al.*, Nature Phys. **6**, 707 (2010).

- Continuous noisy measurement of mechanical oscillator energy
- Is the energy classical (continuous) or quantum (discrete)?

Statistical Binary Hypothesis Testing

- $Y \in \Upsilon$ is an observation.
- Y is noisy: $\Pr(Y|\mathcal{H}_0)$ and $\Pr(Y|\mathcal{H}_1)$
- Given Y, $\Pr(Y|\mathcal{H}_0)$, and $\Pr(Y|\mathcal{H}_1)$, we want to decide which hypothesis is true.
- Decision rule: divide Υ into two regions Υ_0 and Υ_1:
  - If $Y \in \Upsilon_0$, we decide $\mathcal{H}_0$ is true.
  - If $Y \in \Upsilon_1$ we decide $\mathcal{H}_1$ is true.
- Type-I error probability (miss probaiblity):

$$P_{01}(\Upsilon_0, \Upsilon_1) = \sum_{Y \in \Upsilon_0} \Pr(Y|\mathcal{H}_1) \tag{1}$$

- Type-II error probability (false-alarm probaiblity):

$$P_{10}(\Upsilon_0, \Upsilon_1) = \sum_{Y \in \Upsilon_1} \Pr(Y|\mathcal{H}_0) \tag{2}$$

- How to choose Υ_0 and Υ_1 in order to minimize errors?

Likelihood-Ratio Test

- Define likelihood ratio:

$$\Lambda \equiv \frac{\Pr(Y|\mathcal{H}_1)}{\Pr(Y|\mathcal{H}_0)} \tag{3}$$

- Likelihood-ratio test given a threshold γ:
  - If $\Lambda \geq \gamma$ decide $\mathcal{H}_1$ is true.
  - If $\Lambda < \gamma$ decide $\mathcal{H}_0$ is true.
- Neyman-Pearson criterion:
  - Constrain $P_{10} \leq \alpha$ and minimize P_{01}
  - set γ such that $\Pr(\Lambda \geq \gamma) = \alpha$
- Bayes criterion:
  - minimize $P_e = P_0 P_{10} + P_1 P_{01}$
  - set $\gamma = P_0/P_1$

Quantum Probability

- Sequential measurements of a quantum system:

$$\Pr(Y|\mathcal{H}_j) = \operatorname{tr}\left[\mathcal{J}_j(y_M, t_M)\mathcal{K}_j(t_M)\ldots\mathcal{J}_j(y_M, t_1)\mathcal{K}_j(t_1)\rho_j(t_0)\right], \tag{4}$$

- $\mathcal{K}$ and $\mathcal{J}$ are completely-positive maps. In terms of Kraus operators:

$$\mathcal{K}\rho \equiv \sum_z K(z)\rho K^\dagger(z), \qquad \mathcal{J}(y)\rho \equiv \sum_z J(y,z)\rho J^\dagger(y,z). \tag{5}$$

- infinitesimal CP map (Lindblad):

$$\mathcal{K}\rho = \rho + \delta t\mathcal{L}\rho + o(\delta t). \tag{6}$$

- For weak measurements with Gaussian noise,

$$\mathcal{J}(\delta y)\rho = \tilde{P}(\delta y)\left[\rho + \frac{\delta y}{2R}\left(c\rho + \rho c^\dagger\right) + \frac{\delta t}{8Q}\left(2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c\right) + o(\delta t)\right], \tag{7}$$

$$\tilde{P}(\delta y) = \mathcal{N}(0, R\delta t). \tag{8}$$

- H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

Likelihood Ratio for Continuous Measurements

- Suppose $\tilde{P}(\delta y)$ is the same in both hypothesis. Then it can be shown that

$$\Lambda = \frac{\operatorname{tr} f_1(T)}{\operatorname{tr} f_0(T)}, \tag{9}$$

where f_1 and f_0 obey the quantum Duncan-Mortensen-Zakai (DMZ) equation:

$$df_j = dt\mathcal{L}_j f_j + \frac{dy}{2R}\left(c_j f_j + f_j c_j^\dagger\right) + \frac{dt}{8Q_j}\left(2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j\right). \tag{10}$$

- some stochastic calculus:

$$d \operatorname{tr} f_j = \operatorname{tr} df_j = \frac{dy}{2R}\operatorname{tr}\left(c_j f_j + f_j c_j^\dagger\right) = \frac{dy}{R}\frac{\operatorname{tr}\left(c_j f_j + f_j c_j^\dagger\right)}{2\operatorname{tr} f_j}\operatorname{tr} f_j, \tag{11}$$

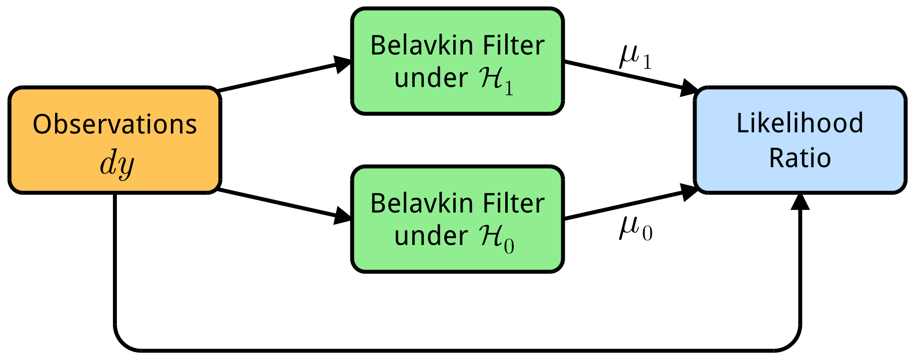
$$\ln \operatorname{tr} f_j(t) = \int_{t_0}^{T}\frac{dy}{R}\mu_j - \int_{t_0}^{T}\frac{dt}{2R}\mu_j^2, \quad \mu_j \equiv \frac{1}{\operatorname{tr} f_j}\operatorname{tr}\left(\frac{c_j + c_j^\dagger}{2} f_j\right). \tag{12}$$

Likelihood-Ratio via Quantum Filtering

- Final result:

$$\Lambda(T) = \exp\left[\int_{t_0}^{T} \frac{dy}{R}(\mu_1 - \mu_0) - \int_{t_0}^{T} \frac{dt}{2R}\left(\mu_1^2 - \mu_0^2\right)\right]. \tag{13}$$

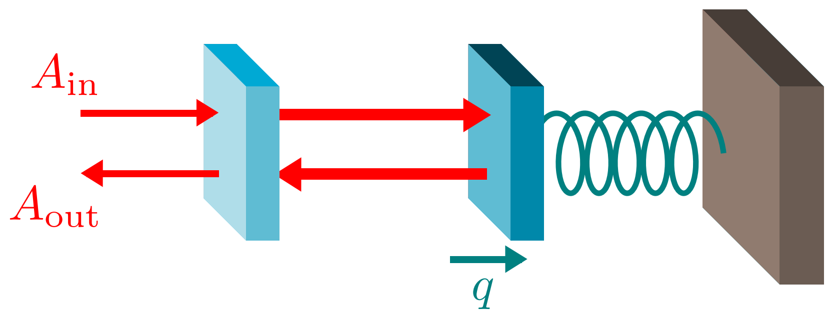
- μ_j is the expected value of $(c_j + c_j^\dagger)/2$ given the observation record, assuming $\mathcal{H}_j$ is true, can be calculated by quantum filters.



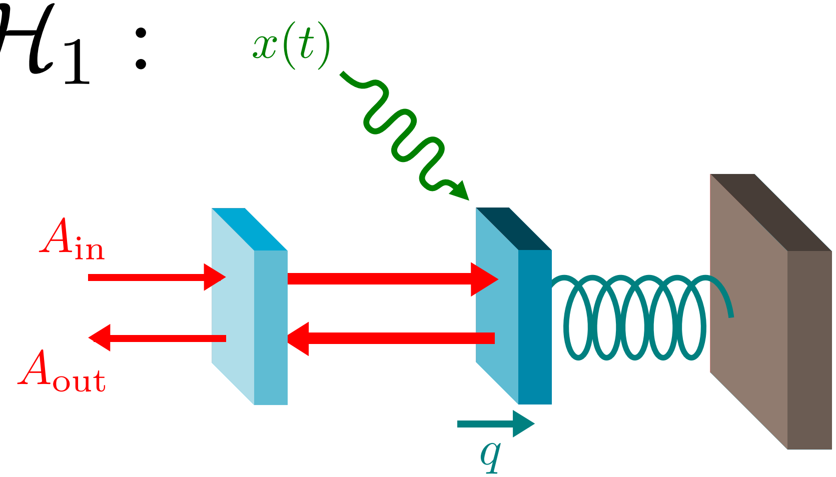
- Similar formula exists for continuous measurements with Poisson noise.
- M. Tsang, Phys. Rev. Lett. **108**, 170502 (2012) [Editors' Suggestion].
- Quantum generalizations of the Duncan-Kailath estimator-correlator formula [Duncan Inf. Control **13**, 62 (1968); Kailath IEEE TIT **15**, 350 (1969)] and Snyder's formula [Snyder, IEEE TIT **18**, 91 (1972)].

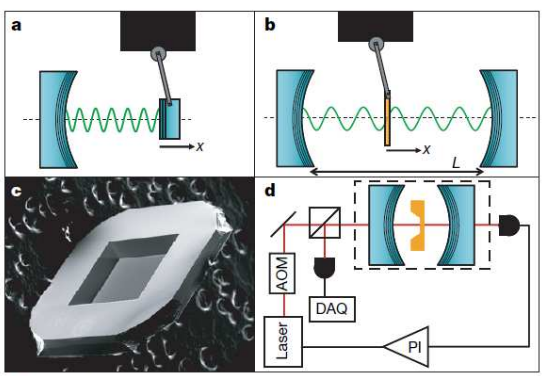
Applications

$\mathcal{H}_0$:

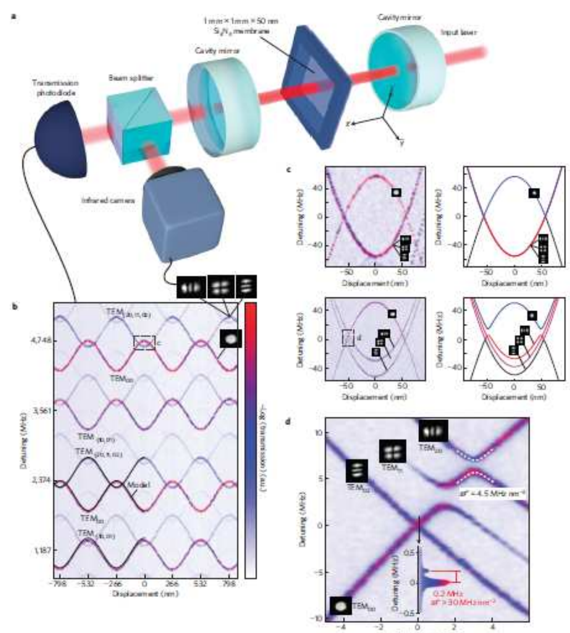


$\mathcal{H}_1$:





Thompson *et al.*, Nature **452**, 72 (2008).



Sankey *et al.*, Nature Phys. **6**, 707 (2010).

Error Probabilities

- For a likelihood-ratio test,
  - Error probabilities:

$$P_{10} = \Pr(\Lambda \geq \gamma|\mathcal{H}_0), \qquad P_{01} = \Pr(\Lambda < \gamma|\mathcal{H}_1). \tag{14}$$

    Very hard to calculate, but can be bounded using Chernoff bounds:

$$P_{10} \leq \inf_{0\leq s\leq 1} \mathbb{E}\left[\Lambda^s|\mathcal{H}_0\right]\gamma^{-s}, \qquad P_{01} \leq \inf_{0\leq s\leq 1} \mathbb{E}\left[\Lambda^s|\mathcal{H}_0\right]\gamma^{1-s}. \tag{15}$$

  - Bayesian posterior probabilities:

$$\Pr(\mathcal{H}_1|Y) = \frac{P_1\Lambda}{P_1\Lambda + P_0}, \qquad \Pr(\mathcal{H}_0|Y) = \frac{P_0}{P_1\Lambda + P_0}. \tag{16}$$

Quantum Hypothesis Testing

- C. W. Helstrom, Quantum Detection and Estimation Theory, (Academic Press, New York, 1976).
- Given two density operators ρ_0 and ρ_1,

$$\Pr(Y|\mathcal{H}_0) = \operatorname{tr} E(Y)\rho_0, \qquad \Pr(Y|\mathcal{H}_1) = \operatorname{tr} E(Y)\rho_1, \tag{17}$$

  what is the POVM that minimizes the error probability?

- Minimum error probability:

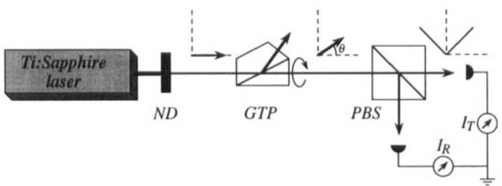
$$\min_{E(Y)} P_e = \frac{1}{2}\left(1 - ||P_1\rho_1 - P_0\rho_0||_1\right), \qquad ||A||_1 \equiv \operatorname{tr}\sqrt{AA^\dagger}, \tag{18}$$

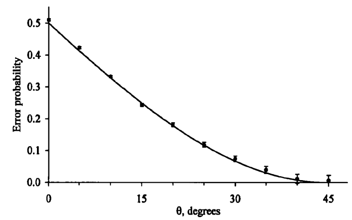
$$\min_{E(Y)} P_e \geq \frac{1}{2}\left(1 - \sqrt{1 - 4P_0P_1F}\right), \qquad F \equiv \left(\operatorname{tr}\sqrt{\sqrt{\rho_0}\rho_1\sqrt{\rho_0}}\right)^2. \tag{19}$$

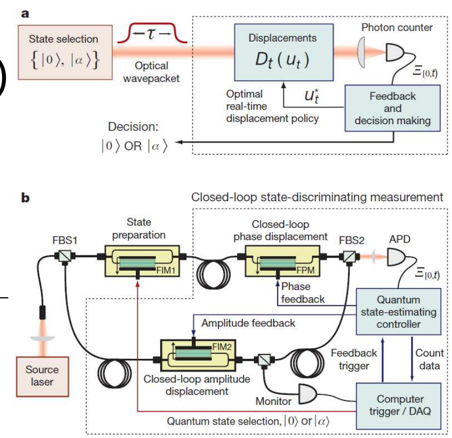
Quantum Optics

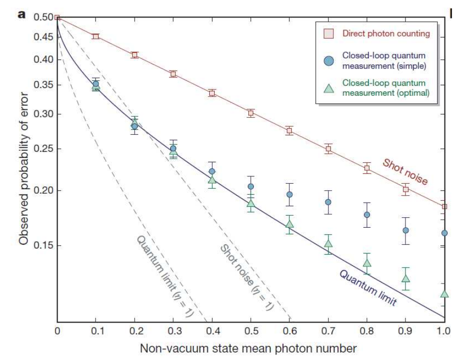
Cook *et al.*, Nature **446** 774, (2007).

Barnett and Riis, J. Mod. Opt. **44**, 1061 (1997)



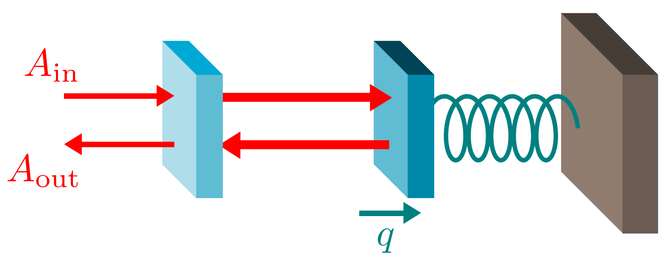




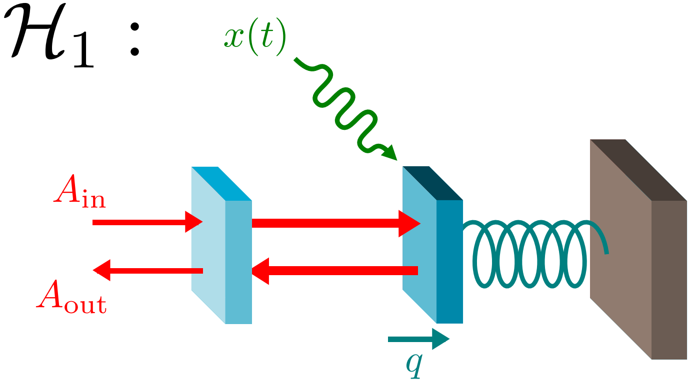


- How to apply this lower bound to waveform detection?

$\mathcal{H}_0$:

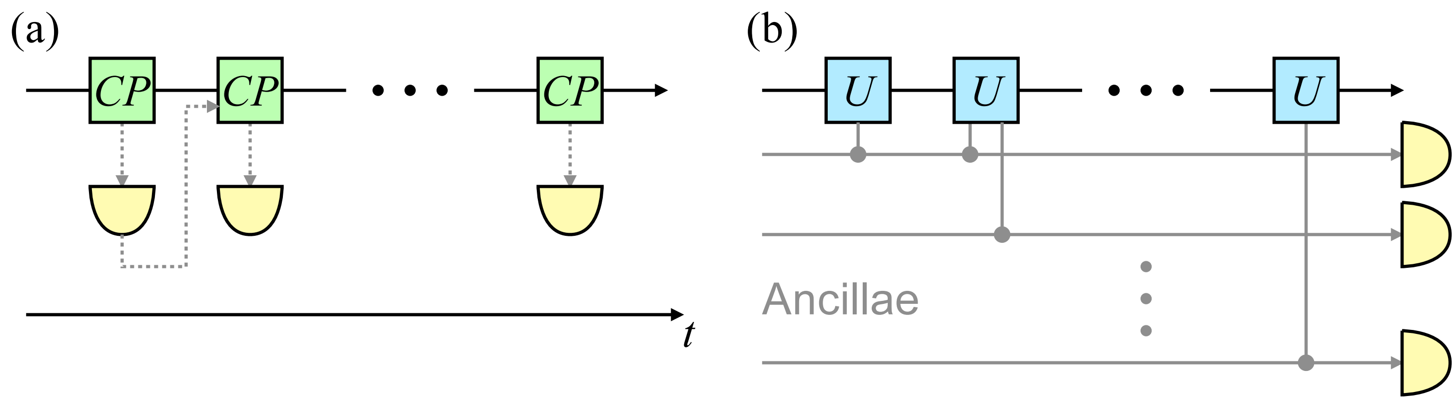


$\mathcal{H}_1$:



Quantum Information Theory to the Rescue

- Discretize time, and take continuous limit at the end
- Larger Hilbert space, Principle of Deferred Measurements:



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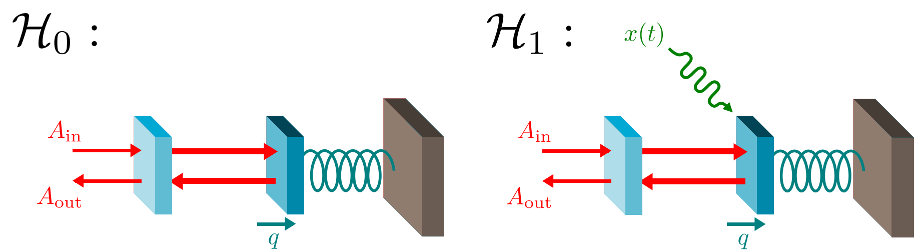
$$\rho_0 = U_0|\psi\rangle\langle\psi|U_0^\dagger, \qquad \rho_1 = \mathbb{E}_X U_1|\psi\rangle\langle\psi|U_1^\dagger, \tag{20}$$

$$U_0 = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int dt H_0(t)\right], \qquad U_1 = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int dt H_1(x(t),t)\right]. \tag{21}$$

- K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
- M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

Quantum Waveform Detection



- Lower bound in terms of fidelity:

$$P_e \geq \frac{1}{2}\left(1 - \sqrt{1 - 4P_0P_1F}\right), \tag{22}$$

$$F = \mathbb{E}_X \left|\langle\psi|U_0^\dagger U_1|\psi\rangle\right|^2 = \mathbb{E}_X \left|\langle\psi|\mathcal{T}\exp\int dt \Delta H_0(t)|\psi\rangle\right|^2, \tag{23}$$

$$\Delta H_0(t) \equiv U_0^\dagger(t,t_0)\left[H_1(t) - H_0(t)\right]U_0(t,t_0). \tag{24}$$

- Suppose $H_1 - H_0 = -qx(t)$, $U_0^\dagger q U_0$ is linear with respect to initial positions/momenta, and $|\psi\rangle$ is Gaussian (Wigner). Then

$$F = \mathbb{E}_X \exp\left[-\frac{1}{\hbar^2}\int dt \int dt' x(t) \langle : \Delta q_0(t)\Delta q_0(t') : \rangle x(t')\right]. \tag{25}$$

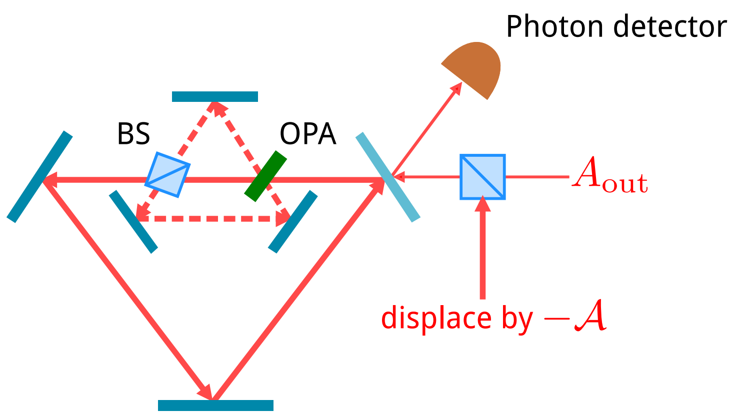
- M. Tsang and R. Nair, Phys. Rev. A **86**, 042115 (2012).

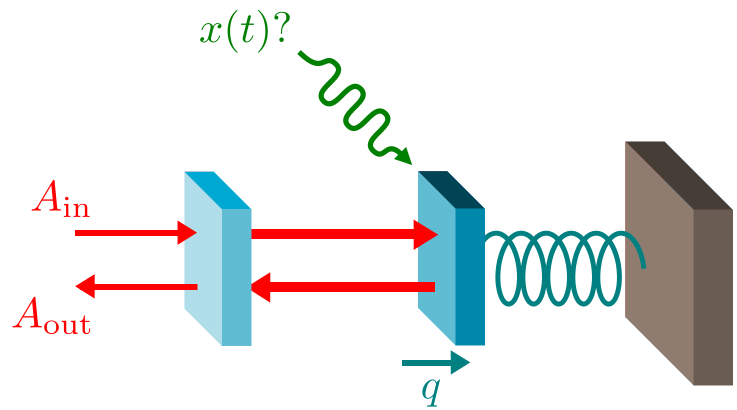
Quantum Receiver Design

- Homodyne detection of A_{out} is sub-optimal.
- Kennedy receiver is near-optimal in error exponent:

$$\lim_{T\to\infty} -\frac{1}{T}\ln P_e^{(\text{Kennedy})} = \lim_{T\to\infty} -\frac{1}{T}\ln P_e^{(\text{quantum})} = \lim_{T\to\infty} -\frac{1}{T}\ln F \tag{26}$$

for coherent state, even if $x(t)$ is stochastic.





- Require Quantum Noise Cancellation [M. Tsang and C. M. Caves, Phys. Rev. Lett. **105**, 123601 (2010)], a coherent feedforward control technique, if measurement backaction noise is significant.
- M. Tsang and R. Nair, Phys. Rev. A **86**, 042115 (2012).
- Can coherent feedback control further reduce the error probability?

Parameter Estimation

- Given y and likelihood function $P(y|x)$, estimate x.
- Let estimate be $\tilde{x}(y)$.
- Mean-square error:

$$\mathbb{E}(\delta x^2) \equiv \int dy \left[\tilde{x}(y) - x\right]^2 P(y|x). \tag{27}$$

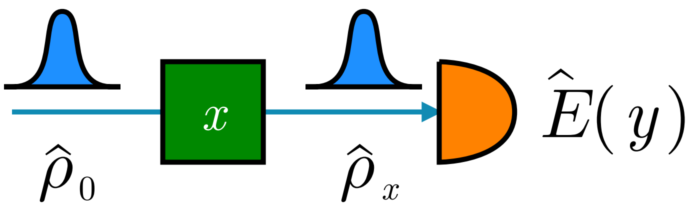
- For unbiased estimates,

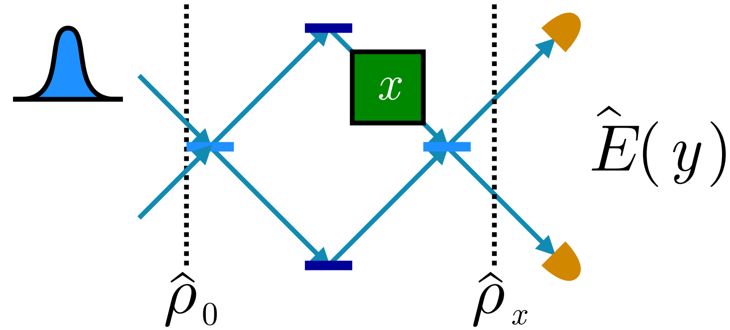
$$x = \int dy \tilde{x}(y) P(y|x), \tag{28}$$

Cramér-Rao bound:

$$\mathbb{E}\left(\delta x^2\right) \geq J^{-1}, \qquad J \equiv \int dy P(y|x) \left[\frac{\partial \ln P(y|x)}{\partial x}\right]^2. \tag{29}$$

Quantum Parameter Estimation





- Quantum:

$$P(y|x) = \mathrm{tr}\left[E(y)\rho_x\right]. \tag{30}$$

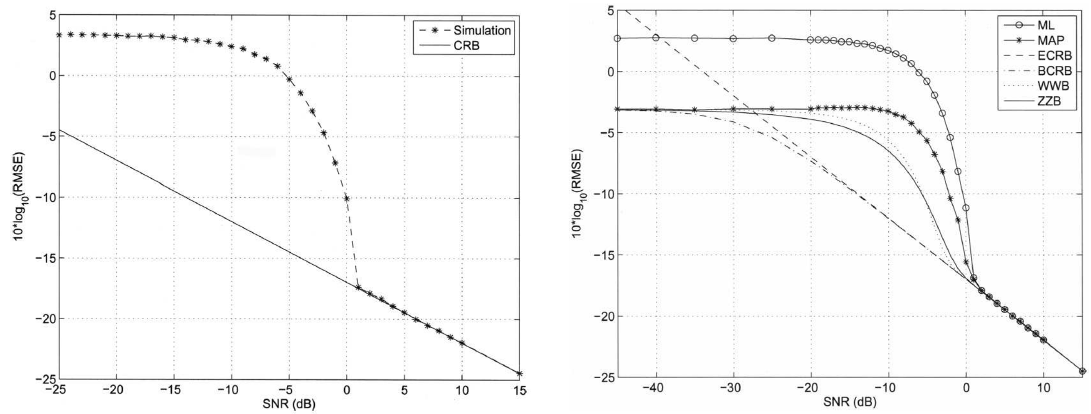
- Quantum Cramér-Rao bound (QCRB) (valid for any POVM):

$$\mathbb{E}\left(\delta x^2\right) \geq \left\langle \Delta h^2 \right\rangle^{-1}, \quad \delta x \equiv x - \tilde{x}(y), \quad \frac{\partial \rho_x}{\partial x} = \frac{1}{2}\left(h\rho_x + \rho_x h^\dagger\right). \tag{31}$$

- C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004); Nature Photon. **5**, 222 (2011).

Beyond Cramér-Rao Bounds

- If $P(y|x)$ is not a Gaussian ($Y \sim \mathcal{N}(Cx, \Sigma)$), CRB often grossly underestimate the achievable estimation error.



Van Trees and Bell, Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking (Wiley, Hoboken, 2007)

- Ziv-Zakai bound:

$$\mathbb{E}(\delta x^2) \geq \frac{1}{2}\int_0^\infty d\tau \tau \int_{-\infty}^\infty dx 2 \min\left[P_X(x), P_X(x+\tau)\right] P_e(x, x+\tau), \tag{32}$$

$P_X(x)$ is prior, $P_e(x, x+\tau)$ is the error probability of a binary hypothesis testing problem with $P(y|\mathcal{H}_0) = P(y|x)$, $P(y|\mathcal{H}_1) = P(y|x+\tau)$, and $P_0 = P_1 = 1/2$.

Quantum Ziv-Zakai Bounds

- If $P(y|x) = \mathrm{tr}\,[E(y)\rho_x]$, for any POVM $E(y)$,

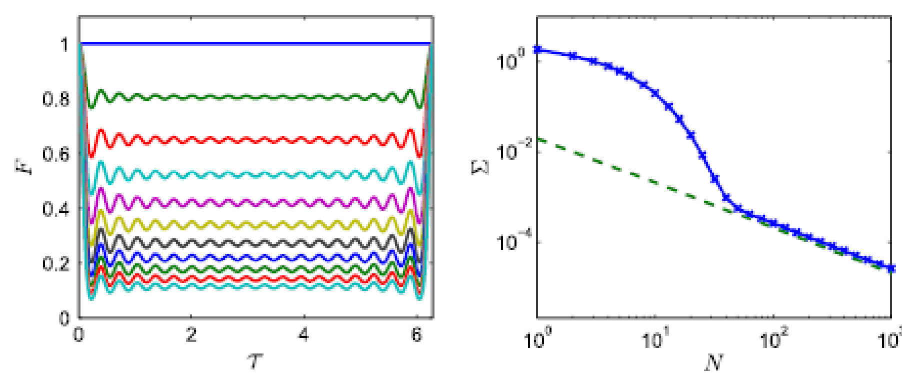
$$P_e(x, x+\tau) \geq \frac{1}{2}\left[1 - \frac{1}{2}||\rho_x - \rho_{x+\tau}||_1\right] \geq \frac{1}{2}\left[1 - \sqrt{1 - F(\rho_x, \rho_{x+\tau})}\right]. \tag{33}$$

  We immediately obtain quantum Ziv-Zakai bounds on $\mathbb{E}(\delta x^2)$.

- Can be used to prove Heisenberg limit:

$$\rho_x = \exp(-iHx)\rho\exp(iHx), \qquad \mathbb{E}(\delta x^2) \geq \frac{C}{(\langle H\rangle - E_0)^2}. \tag{34}$$

  (compare with QCRB $\mathbb{E}(\delta x^2) \geq 1/4\,\langle \Delta H^2\rangle$)



- M. Tsang, Phys. Rev. Lett. **108**, 230401 (2012).

Summary

- ## Waveform Detection
  - Estimation: M. Tsang, PRL **108**, 170502 (2012).
  - Fundamental Limits/Control: M. Tsang and R. Nair, PRA **86**, 042115 (2012).
- ## Parameter Estimation Beyond CRB
  - Quantum Ziv-Zakai Bound: M. Tsang, PRL **108**, 230401 (2012).
  - Rate distortion: R. Nair, arXiv:1204.3761.
- ## Waveform Estimation
  - Fundamental Limits: M. Tsang, H. M. Wiseman, and C. M. Caves, PRL **106**, 090401 (2011).
  - Estimation: M. Tsang, PRL **102**, 250403 (2009); PRA **80**, 033840 (2009); **81**, 013824 (2010).
  - Control: M. Tsang and C. M. Caves, PRL **105**, 123601 (2010); PRX **2**, 031016 (2012).
- ## Imaging
  - Stellar Interferometry: M. Tsang, PRL **107**, 270402 (2011).
- ## Waveform Parameter Estimation
- ## We are hiring postdocs (annual salary SGD 76K)/PhD students!
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  - `eletmk@nus.edu.sg`
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