Theory of Quantum Sensing

Estimation, Control, and Fundamental Limits

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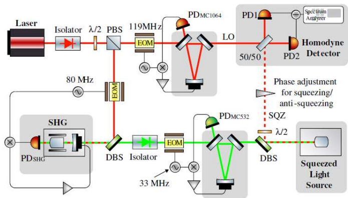
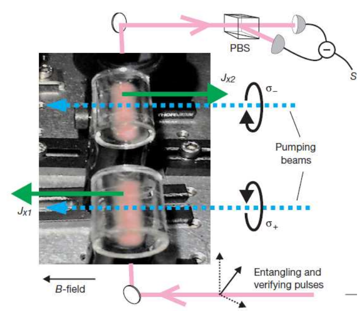
Experimental Advances



10dB squeezing, Vahlbruch *et al.*, Phys. Rev. Lett. **100**, 033602 (2008).



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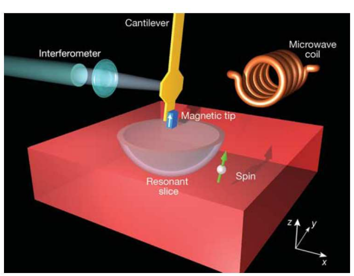
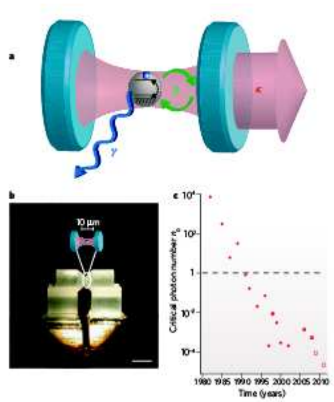
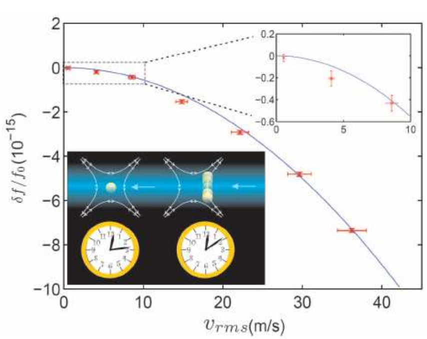
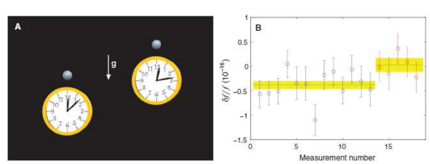


Rugar *et al.*, Nature **430**, 329 (2004).



Kimble, Nature **453**, 1023 (2008).





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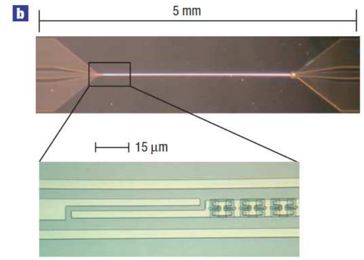

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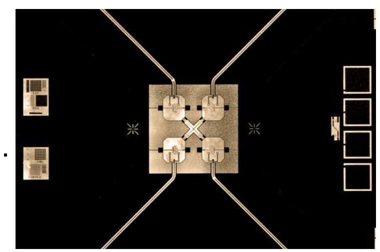
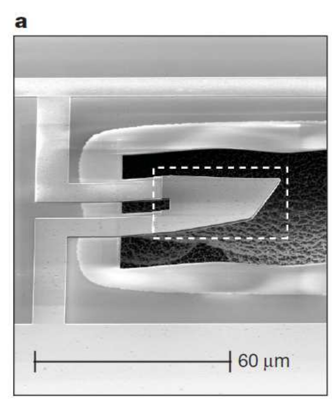


Neeley et al., Nature **467**, 570 (2010)



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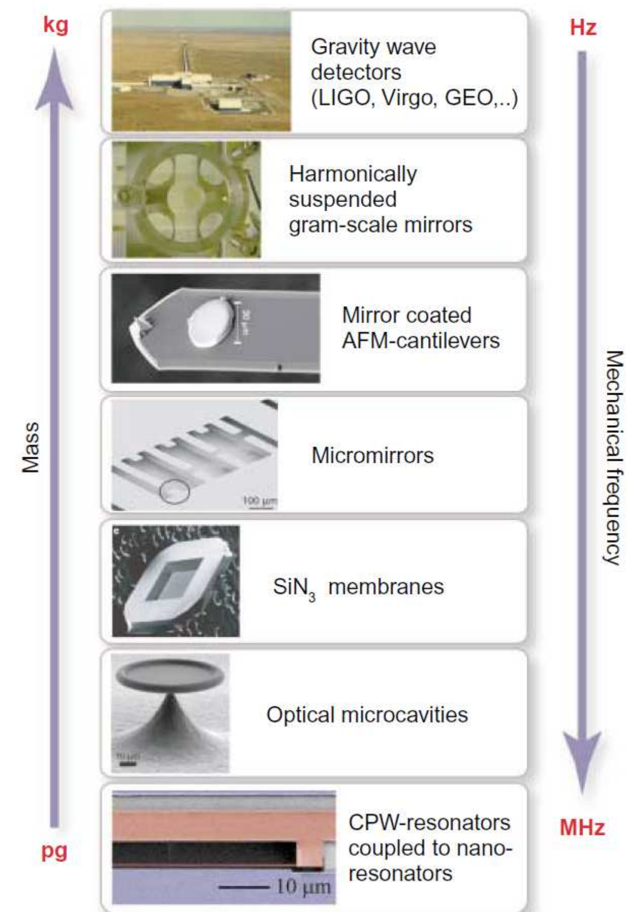
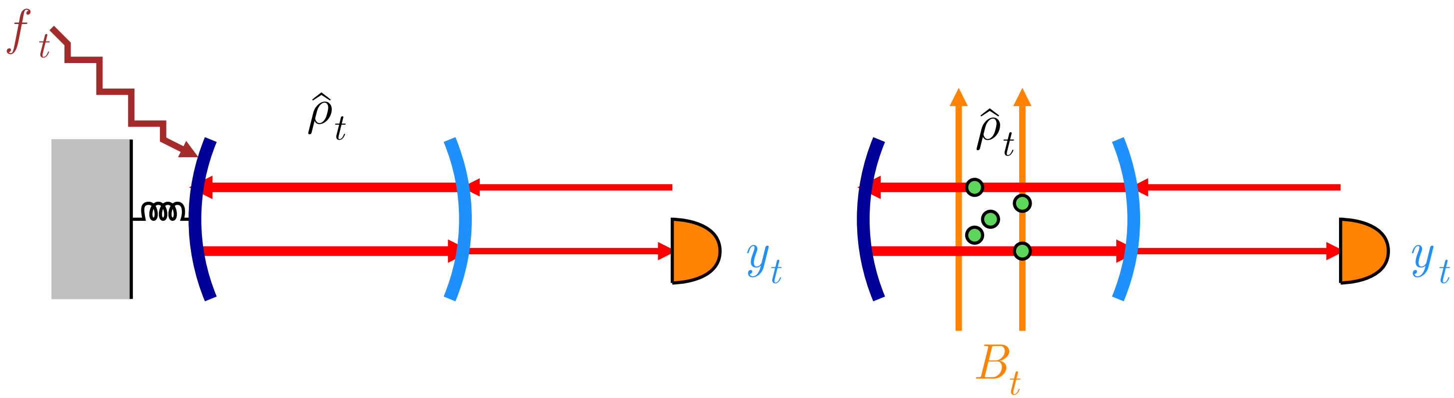
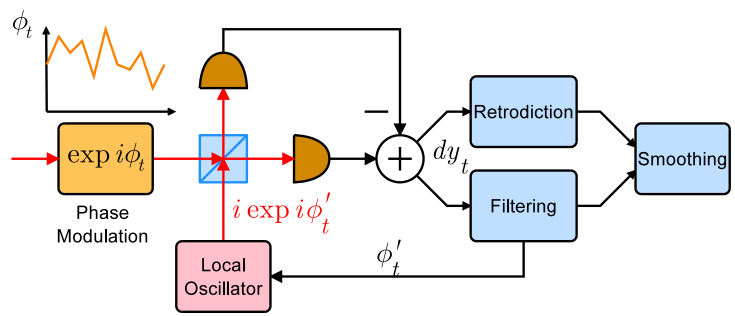


Kippenberg and Vahala, Science **321**, 1172 (2008), and references therein.

Basic Problem

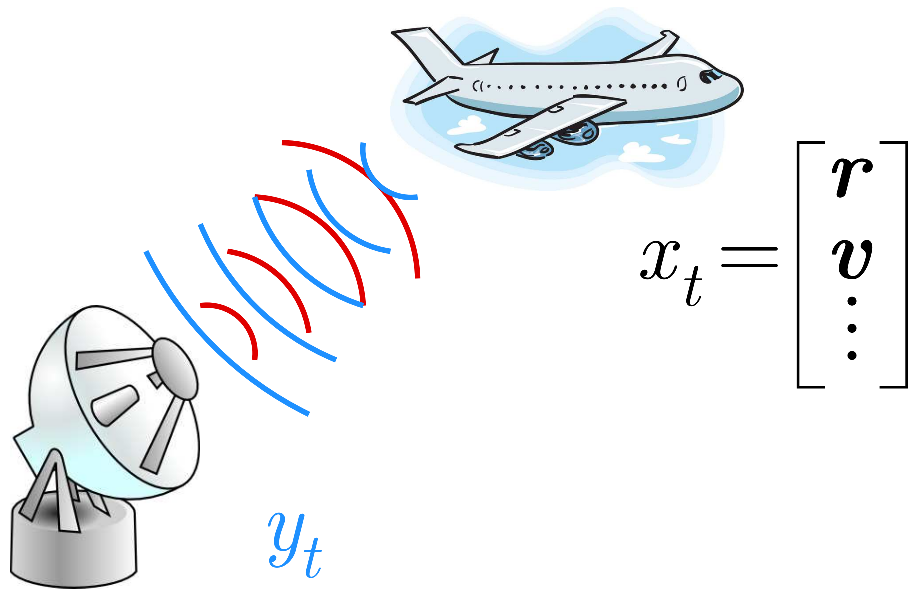




- Estimation: Optimize Data Processing
- Noise Control: Optimize Experiment
- Fundamental Limit: What is the best one can do?
- Engineering perspective
- Examples: optical phase estimation, optomechanical force sensing

Classical Bayesian Estimation: Tracking an Aircraft

- or submarine, nanoparticle, cancer cell, mosquito, terrorist, criminal, . . .

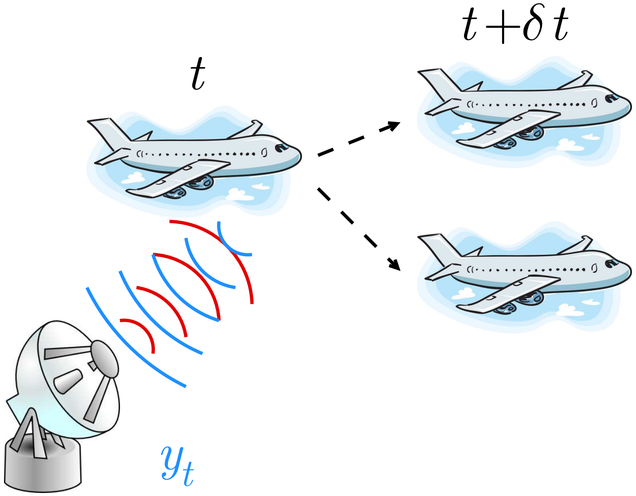


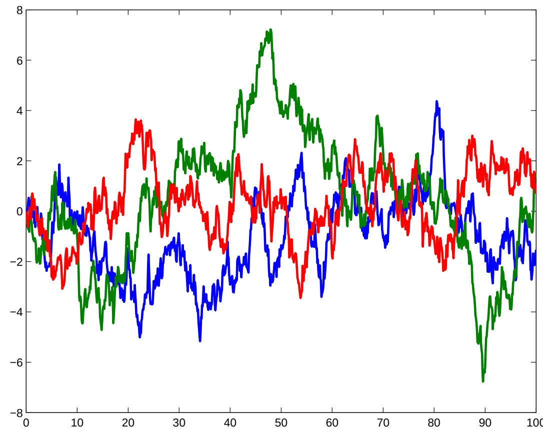
- Use Bayes theorem:

$$P(x_t|y_t) = \frac{P(y_t|x_t)P(x_t)}{\int dx_t (\text{numerator})}, \tag{1}$$

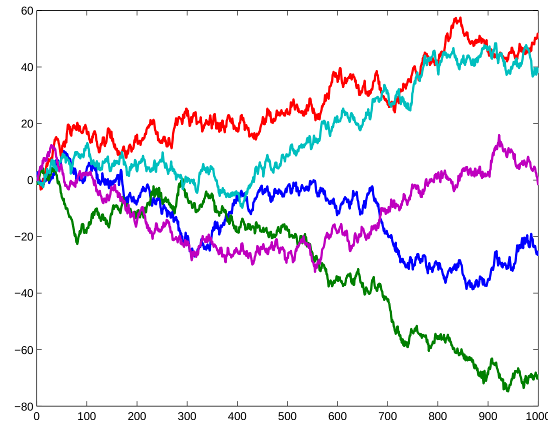
- $P(y_t|x_t)$ from observation noise, $P(x_t)$ from *a priori* information.

Prediction





3 realizations of Ornstein-Uhlenbeck process (Brownian velocity)

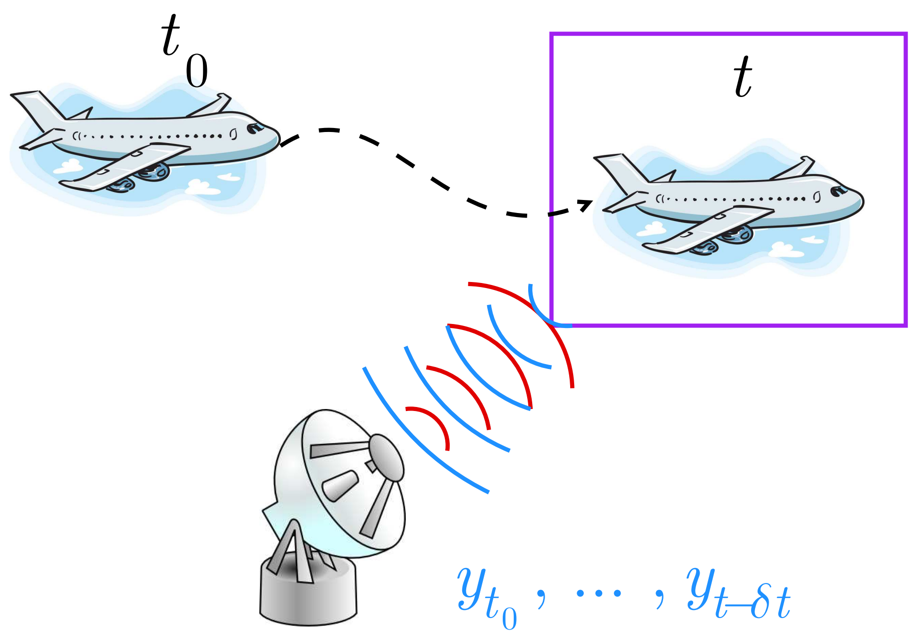


5 realizations of Wiener process (Brownian position in coarse-grained time)

- Assume x_t is a Markov process, use Chapman-Kolmogorov equation:

$$P(x_{t+\delta t}|y_t) = \int dx_t P(x_{t+\delta t}|x_t) P(x_t|y_t) \tag{2}$$

Filtering: Real-Time Estimation

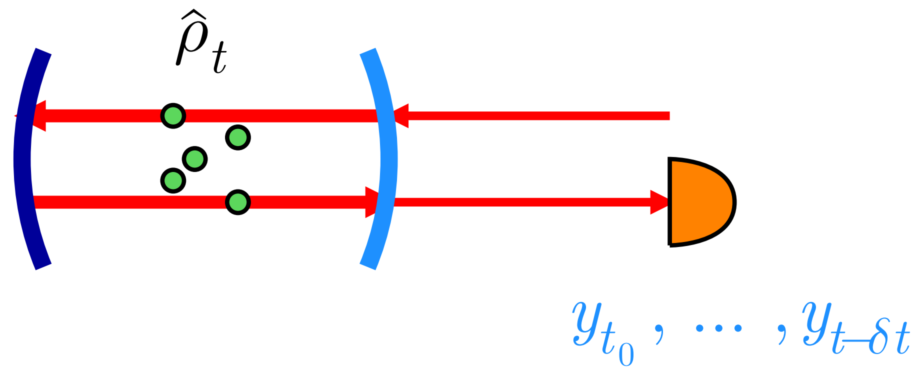


- Applying Bayes theorem and Chapman-Kolmogorov equation repeatedly

$$P(x_t | y_{t-\delta t}, \ldots, y_{t_0+\delta t}, y_{t_0}) \tag{3}$$

- Useful for control, radar, GPS, weather forecast, stock market forecast, etc.
- Wiener, Kolmogorov, Stratonovich, Kalman, Kushner, etc.
  - Bernardo and Smith, *Bayesian Theory*; Van Trees, *Detection, Estimation, and Modulation Theory*; Jazwinski, *Stochastic Processes and Filtering Theory*

Quantum Filtering



- Use "quantum Bayes theorem,"

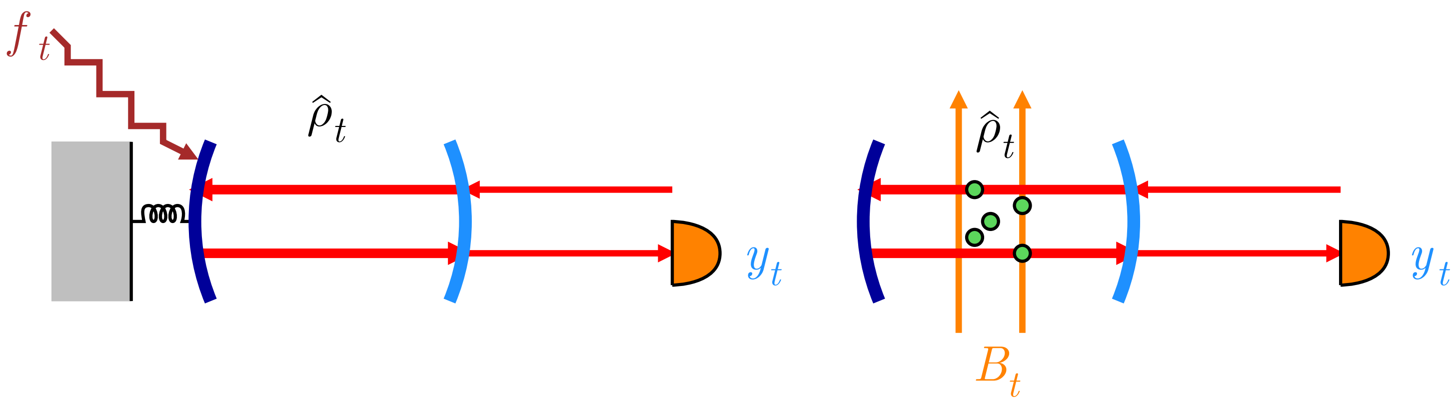
$$\hat{\rho}_t(|y_t) = \frac{\hat{M}(y_t)\hat{\rho}_t\hat{M}^\dagger(y_t)}{\mathrm{tr}(\text{numerator})} \tag{4}$$

- Use a completely positive map to evolve the system state (analogous to Chapman-Kolmogorov),

$$\hat{\rho}_{t+\delta t}(|y_t) = \sum_\mu \hat{K}_\mu \hat{\rho}_t(|y_t)\hat{K}_\mu^\dagger \tag{5}$$

- Useful for quantum state preparation, measurement-induced squeezing, etc.
- Belavkin, Barchielli, Mensky, Carmichael, Gisin, Mølmer, . . .

Hybrid Classical-Quantum Filtering



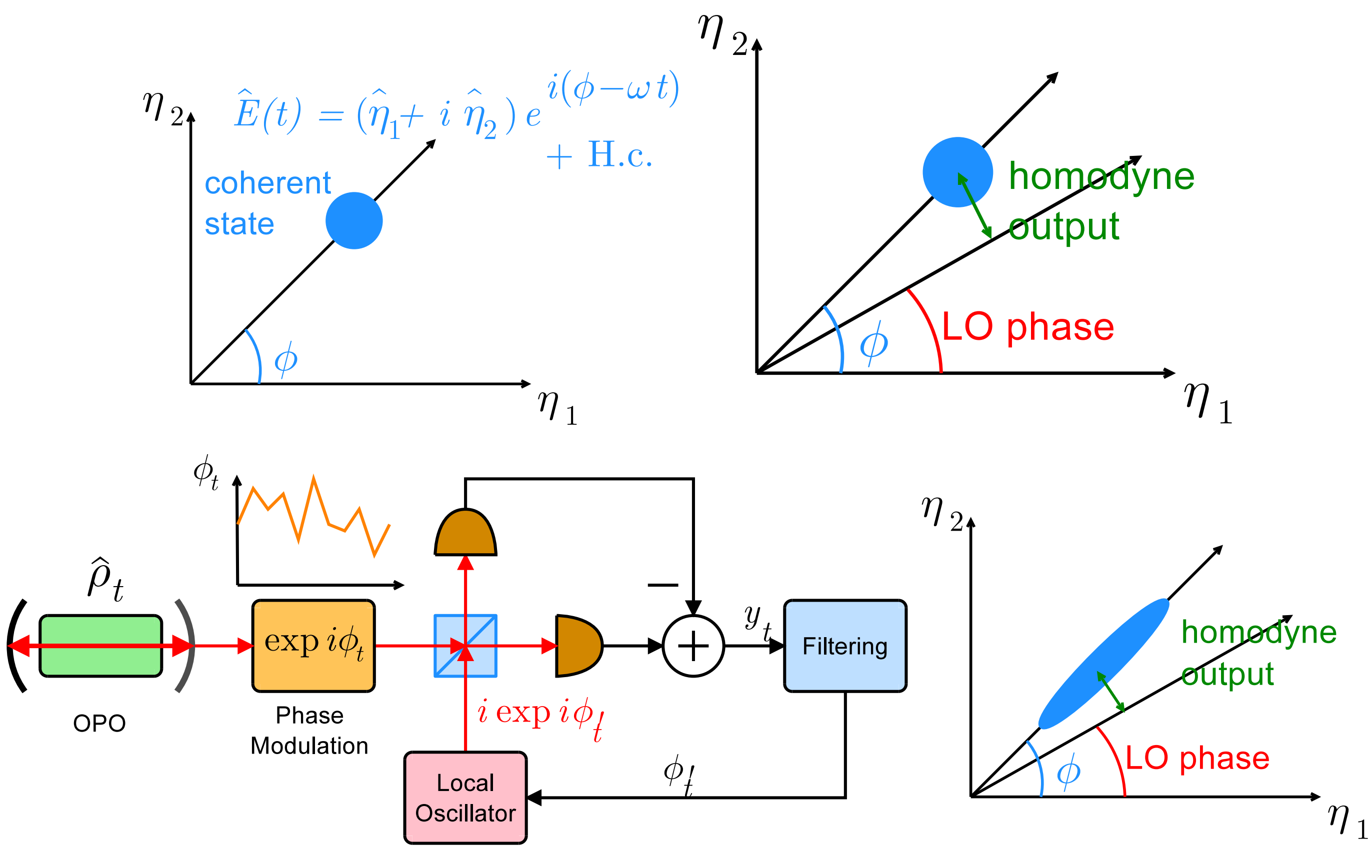
- Force, gravitational wave, magnetic field, clock signal (relative to local oscillator), etc. can all be stochastic
- Define hybrid density operator $\hat{\rho}(x_t)$, $P(x_t) = \mathrm{tr}[\hat{\rho}(x_t)]$, $\hat{\rho}_t = \int dx_t \hat{\rho}(x_t)$
- Use hybrid Bayes theorem + completely positive map:

$$\hat{\rho}(x_t|y_t) = \frac{\hat{M}(y_t|x_t)\hat{\rho}(x_t)\hat{M}^\dagger(y_t|x_t)}{\int dx_t\, \mathrm{tr}(\mathrm{numerator})}, \tag{6}$$

$$\hat{\rho}(x_{t+\delta t}|y_t) = \int dx_t \sum_\mu \hat{K}_\mu(x_{t+\delta t}|x_t)\hat{\rho}(x_t|y_t)\hat{K}_\mu^\dagger(x_{t+\delta t}|x_t) \tag{7}$$

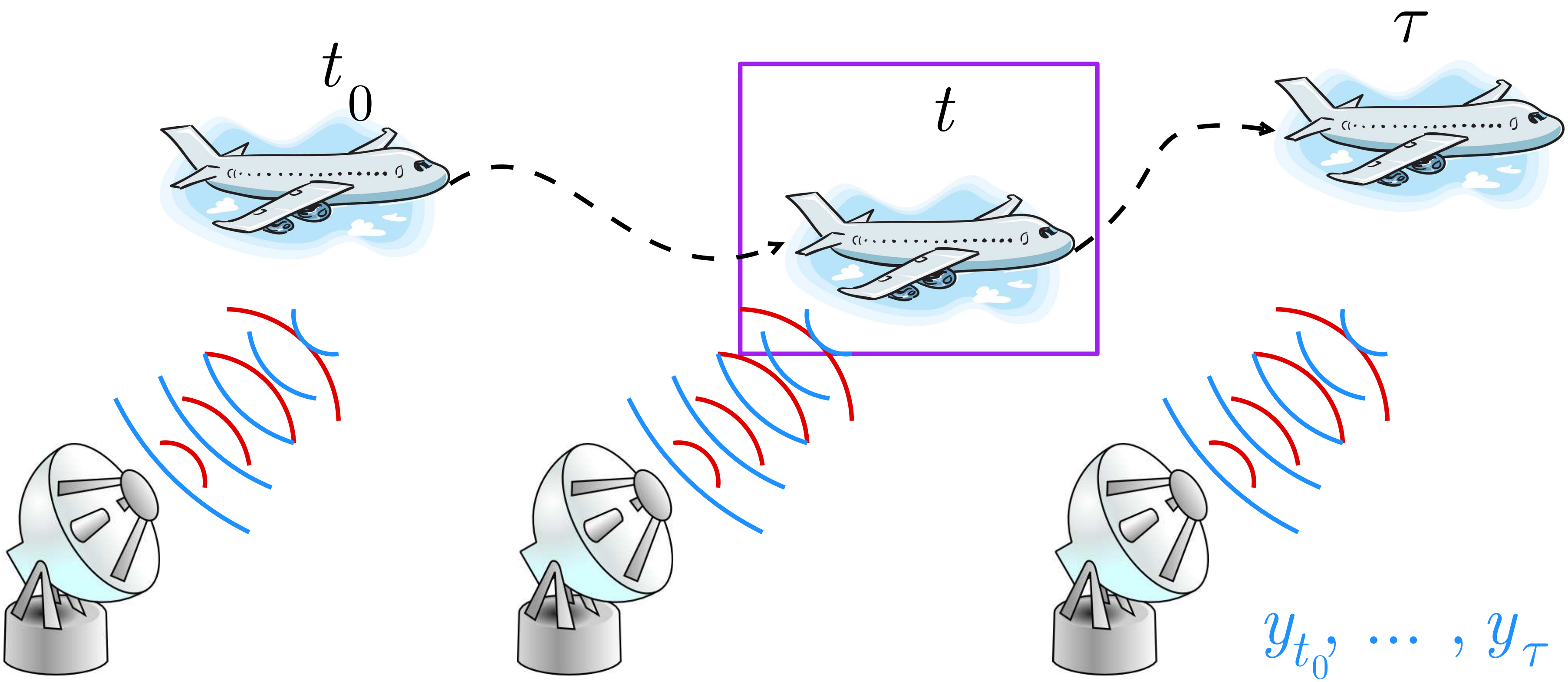
- M. Tsang, Phys. Rev. Lett. **102**, 250403 (2009); Phys. Rev. A **80**, 033840 (2009); **81**, 013824 (2010).

Adaptive Optical Phase Estimation



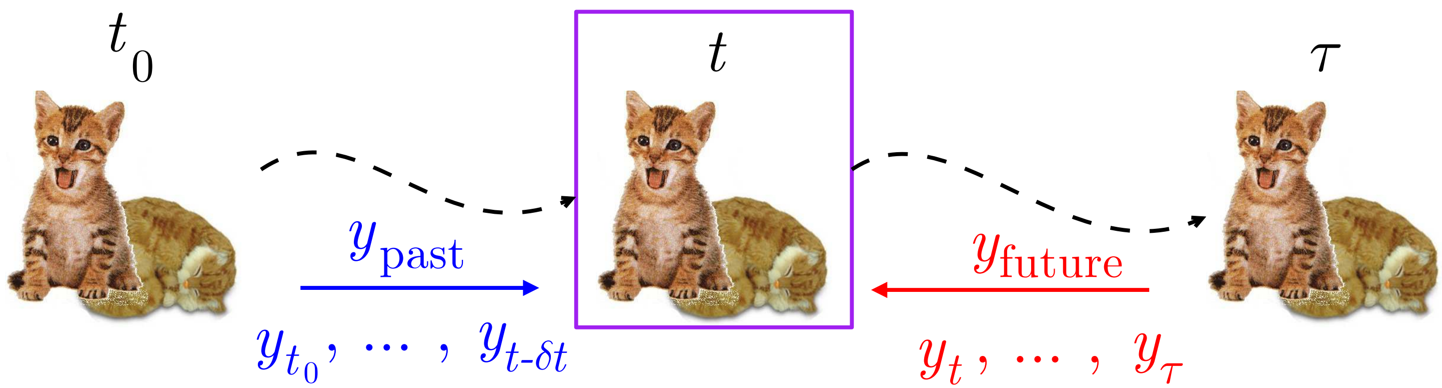
- Personick, IEEE Trans. Inform. Th. **IT-17**, 240 (1971); Yurke, McCall, and Klauder, Phys. Rev. A **33**, 4033 (1986); Wiseman, Phys. Rev. Lett. **75**, 4587 (1995); Armen, Mabuchi *et al.*, Phys. Rev. Lett. **89**, 133602 (2002); Wheatley *et al.*, Phys. Rev. Lett. **104**, 093601 (2010).
- Berry and Wiseman, Phys. Rev. A **65**, 043803 (2002); **73**, 063824 (2006).
- Tsang, Shapiro, and Lloyd, Phys. Rev. A **78**, 053820 (2008); **79**, 053843 (2009); Tsang, Phys. Rev. A **80**, 033840 (2009).

Smoothing: Estimation with Delay

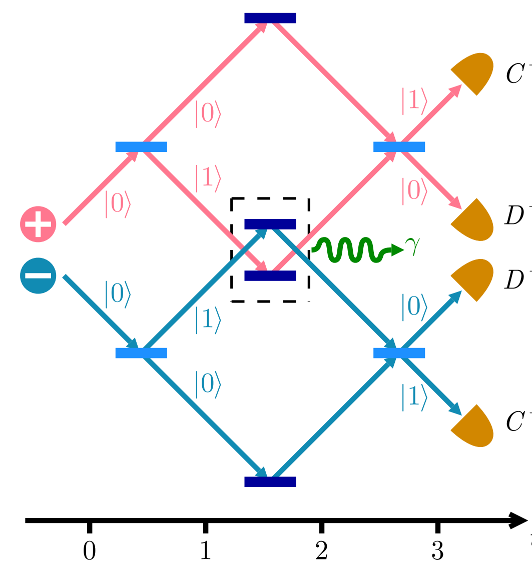


- More accurate than filtering
- Sensing, analog communication, astronomy, crime investigation, . . .
- Delay

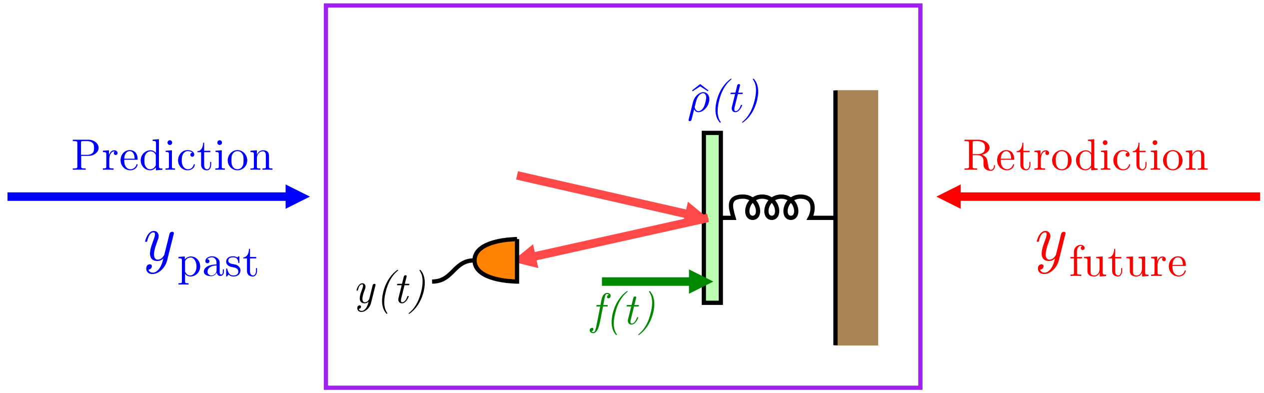
Quantum Smoothing?



- Quantum theory is a predictive theory
- Quantum state described by $|\Psi_t\rangle$ or $\hat{\rho}_t$ can only be conditioned only upon past observations and evolves only forward in time using Schrödinger equation or master equation
- Smoothing cannot be applied to quantum degrees of freedom: Belavkin, Foundation of Physics **24**, 685 (1994).
- "Weak values" by Aharonov, Vaidman, *et al.*
- Paradox: e.g. L. Hardy, Phys. Rev. Lett. **68**, 2981 (1992); Aharonov *et al.*, Phys. Lett. A **301**, 130 (2002); M. Tsang, Phys. Rev. A **81**, 013824 (2010).



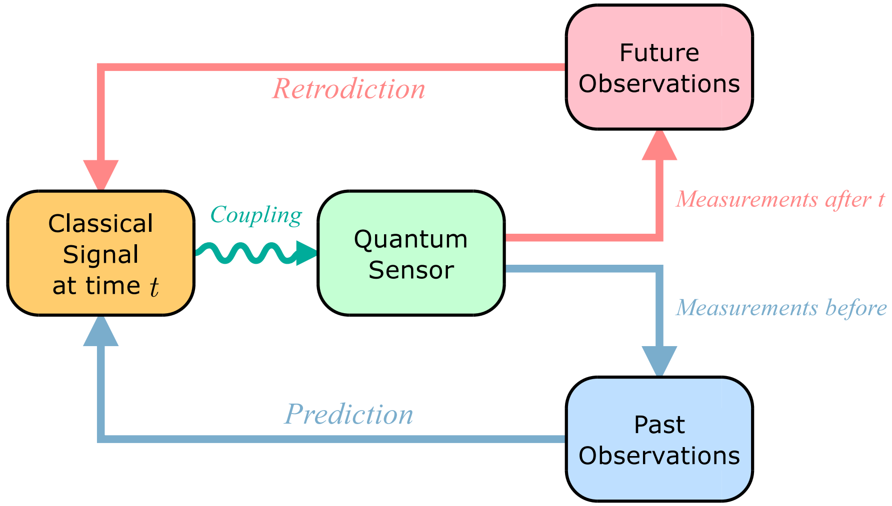
Smoothing for Quantum Sensing



- Use two operators: density operator $\hat{\rho}(x_t|Y_{\text{past}})$ and an effect operator (POVM) $\hat{E}(Y_{\text{future}}|x_t)$

$$P(x_t|Y_{\text{past}}, Y_{\text{future}}) = \frac{\operatorname{tr}\left[\hat{E}(Y_{\text{future}}|x_t)\hat{\rho}(x_t|Y_{\text{past}})\right]}{\int dx_t(\text{numerator})}. \tag{8}$$

Equations



- $\hat{f}(x, t_0) =$ *a priori* $\hat{f}$, $\hat{g}(x, T) \propto \hat{1}$.

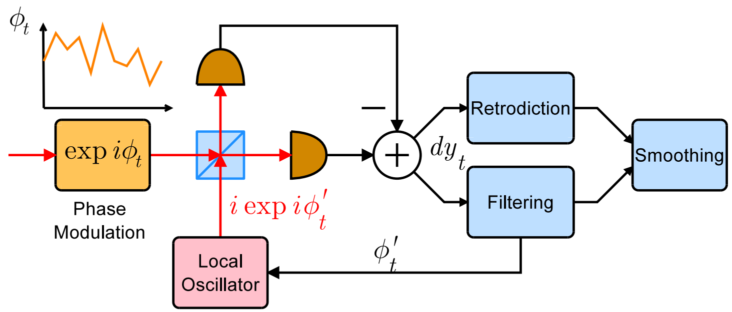
$$d\hat{f} = dt\mathcal{L}(x)\hat{f} + \frac{dt}{8}\left(2\hat{C}^T R^{-1}\hat{f}\hat{C}^\dagger - \hat{C}^{\dagger T} R^{-1}\hat{C}\hat{f} - \hat{f}\hat{C}^{\dagger T} R^{-1}\hat{C}\right) + \frac{1}{2}dy^T R^{-1}\left(\hat{C}\hat{f} + \hat{f}\hat{C}^\dagger\right)$$

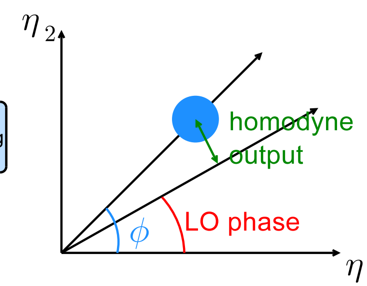
$$-d\hat{g} = dt\mathcal{L}^*(x)\hat{g} + \frac{dt}{8}\left(2\hat{C}^{\dagger T} R^{-1}\hat{g}\hat{C} - \hat{C}^{\dagger T} R^{-1}\hat{C}\hat{g} - \hat{g}\hat{C}^{\dagger T} R^{-1}\hat{C}\right) + \frac{1}{2}dy^T R^{-1}\left(\hat{C}^\dagger\hat{g} + \hat{g}\hat{C}\right)$$

$$h(x, t) = P(x_t = x|y_{\text{past}}, y_{\text{future}}) = \frac{\operatorname{tr}\left[\hat{g}(x, t)\hat{f}(x, t)\right]}{\int dx(\text{numerator})}$$

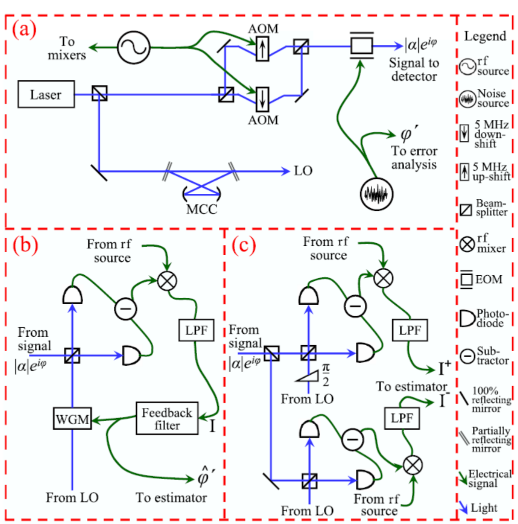
- M. Tsang, Phys. Rev. Lett. **102**, 250403 (2009); Phys. Rev. A **80**, 033840 (2009); **81**, 013824 (2010).
- Classical: Pardoux, Stochastics **6**, 193 (1982).

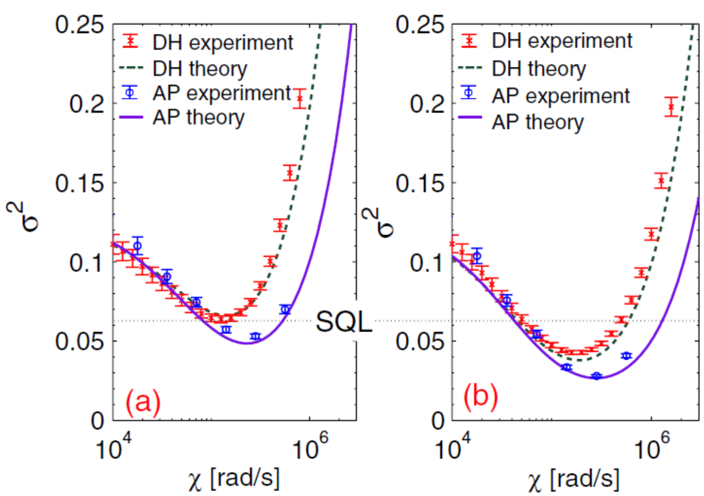
Example 1: Phase-Locked Loop with Smoother

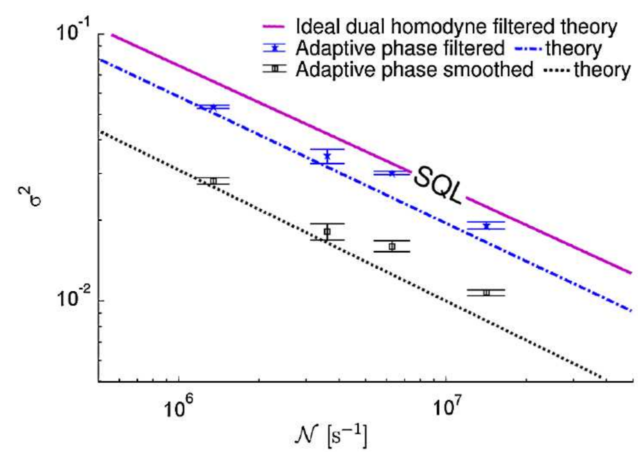




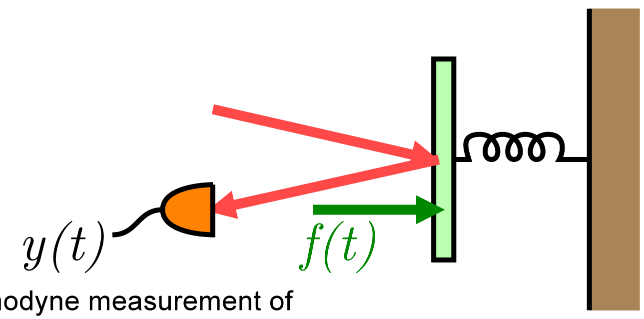
- M. Tsang, J. H. Shapiro, and S. Lloyd, Phys. Rev. A **78**, 053820 (2008); **79**, 053843 (2009).
- Wheatley *et al.*, "Adaptive Optical Phase Estimation Using Time-Symmetric Quantum Smoothing," Phys. Rev. Lett. (*Editors' Suggestion*) **104**, 093601 (2010).



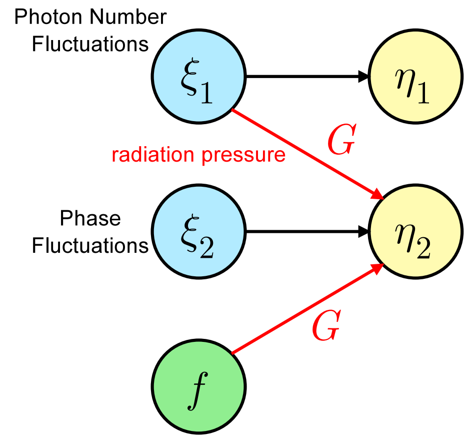


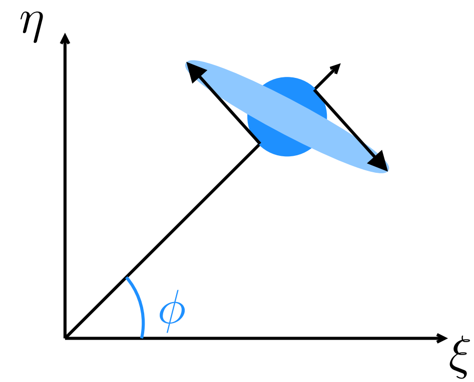


Example 2: Optomechanical Force Sensing



Adaptive homodyne measurement of optical phase





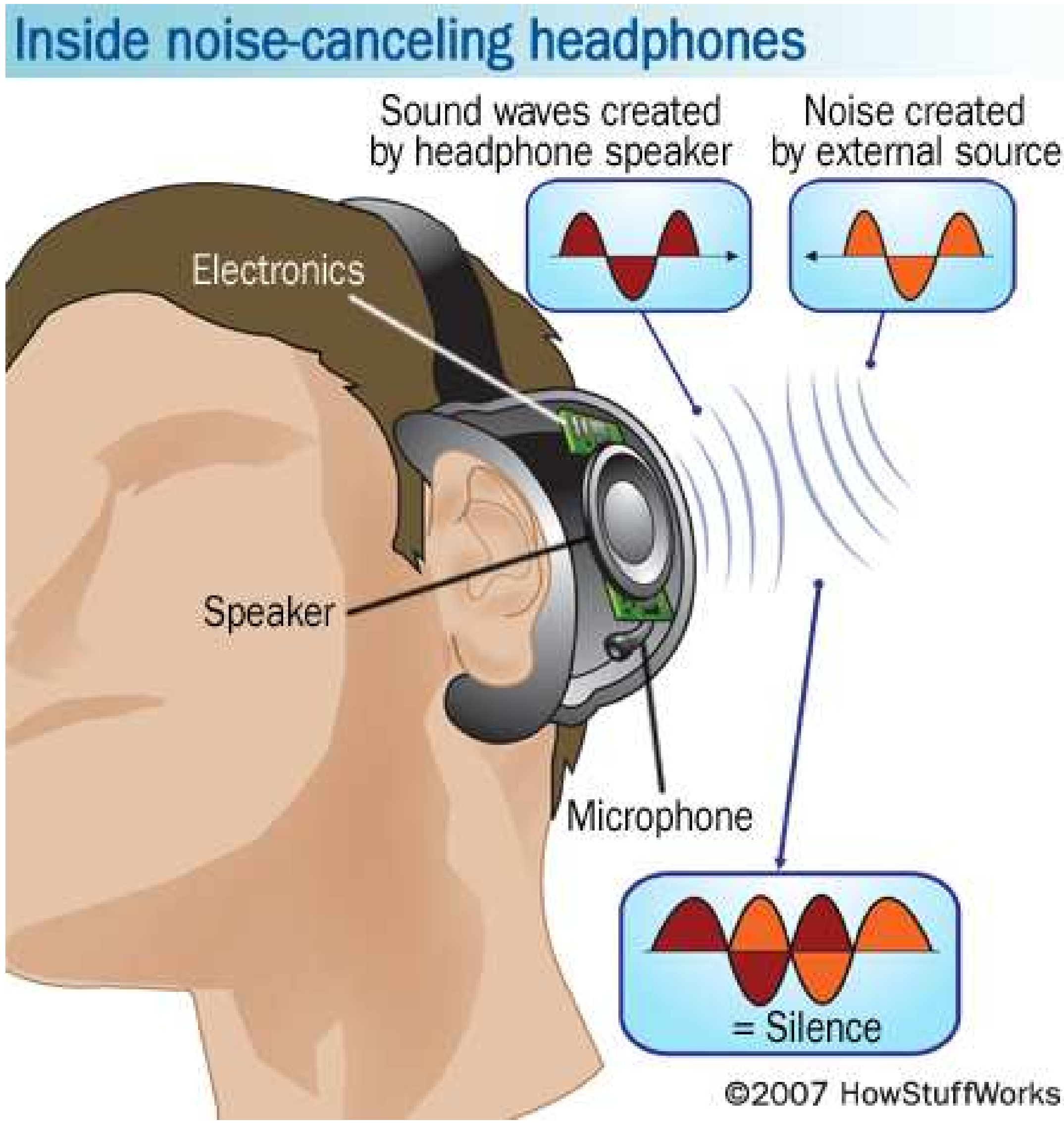
$$y(\omega) = \eta_2(\omega) = G(\omega)[f(\omega) + \text{noise}(\omega)] \tag{9}$$

$$\langle \delta f^2 \rangle_{\text{smoothing}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{1/S_{\text{noise}}(\omega) + 1/S_f(\omega)} \tag{10}$$

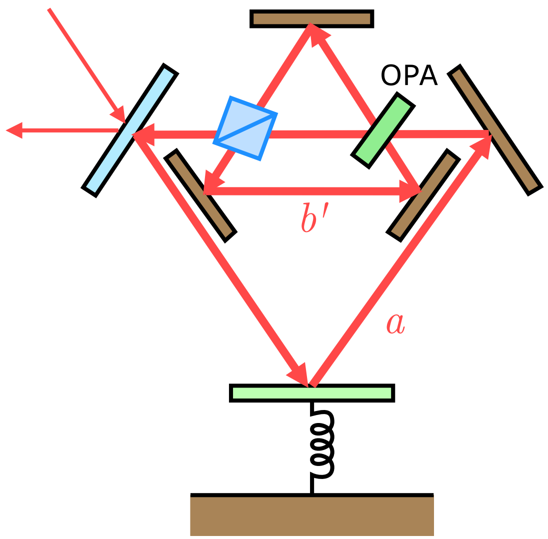
$$S_{\text{noise}}(\omega) = \frac{1}{|G(\omega)|^2} S_2(\omega) + S_1(\omega) \geq \frac{\hbar}{|G(\omega)|} = \text{SQL noise floor} \tag{11}$$

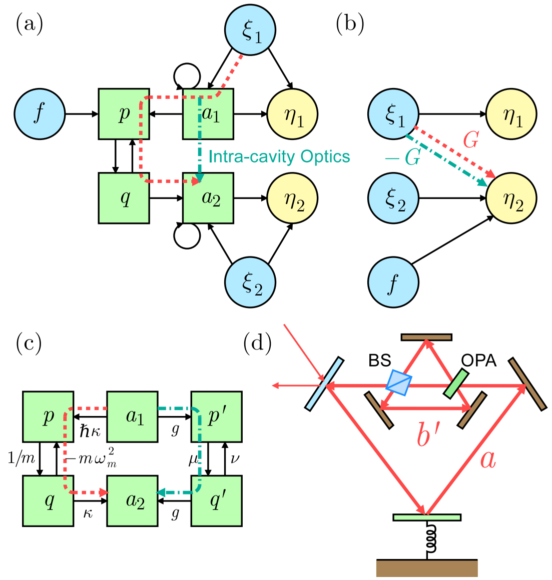
- $S_2(\omega)$: noise in optical phase quadrature $\sim 1/P$, $S_1(\omega)$ radiation-pressure fluctuations $\sim P$

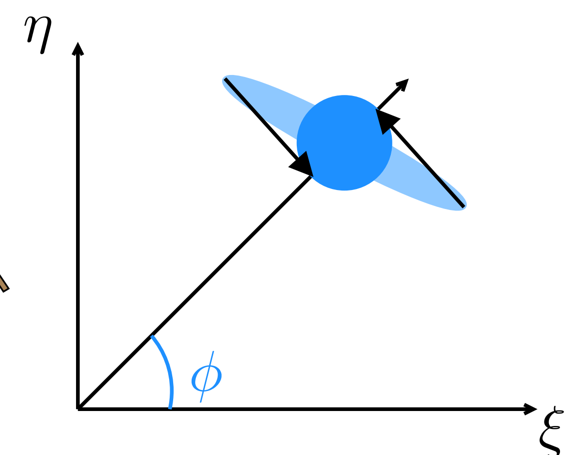
Noise Cancellation



Broadband Quantum Noise Cancellation





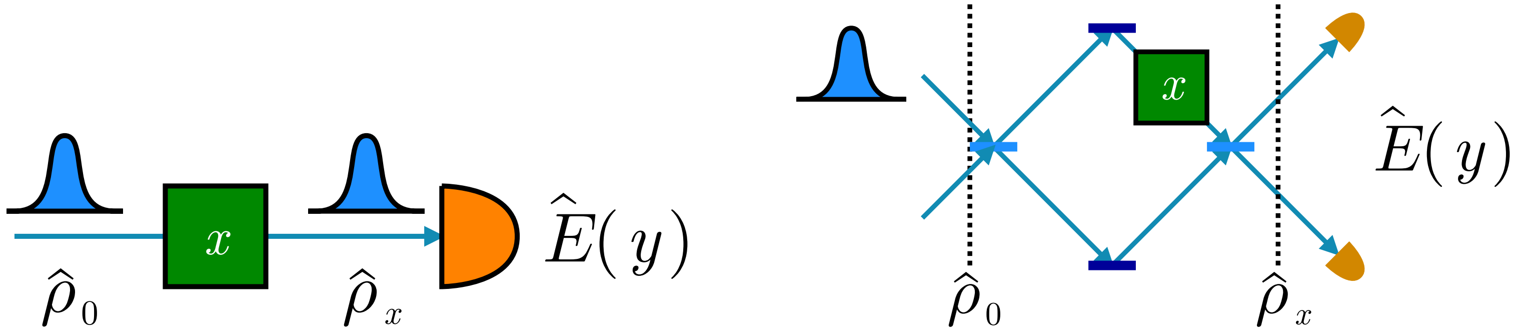


$$S_{\rm noise}(\omega) = \frac{1}{|G(\omega)|^2} S_2(\omega) + \cancel{S_1(\omega)} \tag{12}$$

$$\langle \delta f^2 \rangle_{\rm smoothing,QNC} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{|G(\omega)|^2/S_2(\omega) + 1/S_f(\omega)} \tag{13}$$

- Is this the best one can do?
- Quantum Information Theory

Quantum Cramér-Rao Bound



- Quantum Cramér-Rao bound (QCRB):

$$\langle \delta x^2 \rangle \left\langle \Delta \hat{h}^2 \right\rangle \geq \frac{\hbar^2}{4}, \tag{14}$$

$$\text{e.g. } \langle \delta \phi^2 \rangle \left\langle \Delta \hat{N}^2 \right\rangle \geq \frac{1}{4}. \tag{15}$$

- C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976); V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004); arXiv:1102.2318 (2011).
- ignores dynamics, continuous measurements, measurement back-action

Dynamical Quantum Cramér-Rao Bound

$$\boxed{\langle \delta x_t^2 \rangle \geq F^{-1}(t,t)} \tag{16}$$

- $F^{-1}(t,t')$ is the inverse of Fisher information matrix $F(t,t')$, defined as $\int dt' F(t,t') F^{-1}(t',\tau) = \delta(t-\tau)$.
- Fisher information function has two components: $F(t,t') = F^{(Q)}(t,t') + F^{(C)}(t,t')$.
- $F^{(Q)}$ is quantum:

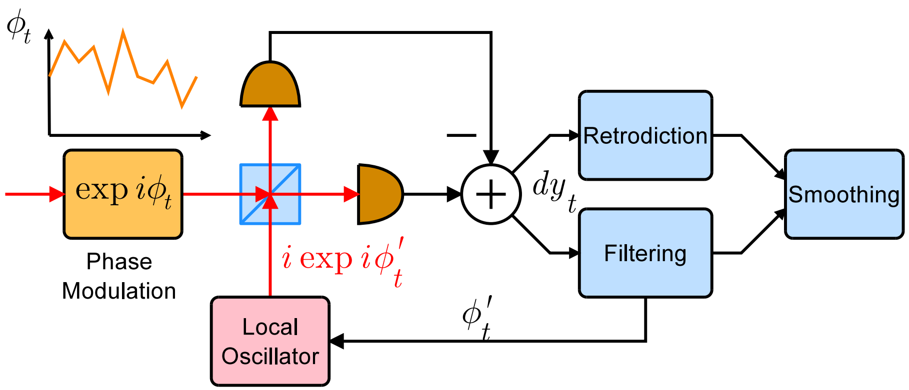
$$F^{(Q)}(t,t') = \frac{4}{\hbar^2}\left\langle \Delta \hat{h}(t) \Delta \hat{h}(t') \right\rangle, \quad \hat{h}(t) \equiv \int_{t_0}^{t_J} d\tau \hat{U}^\dagger(\tau,t_0) \frac{\delta \hat{H}[x(t),\tau]}{\delta x(t)} \hat{U}(\tau,t_0).$$

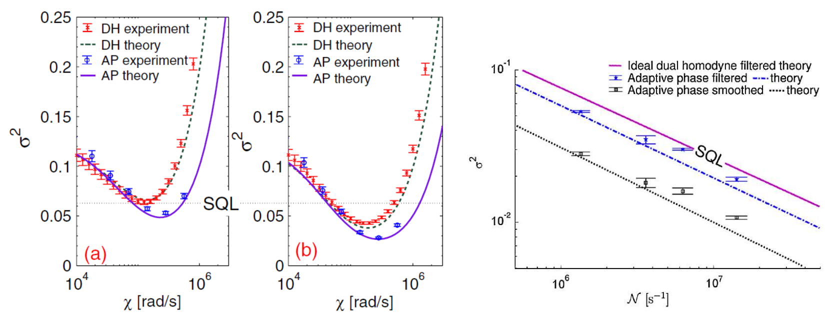
- $F^{(C)}$ incorporates *a priori* waveform information

$$F^{(C)}(t,t') = \int Dx P[x] \frac{\delta \ln P[x]}{\delta x(t)} \frac{\delta \ln P[x]}{\delta x(t')}. \tag{17}$$

- M. Tsang, H. M. Wiseman, and C. M. Caves, "Fundamental Quantum Limit to Waveform Estimation," to appear in Phys. Rev. Lett. [e-print arXiv:1006.5407].

Example 1: Optical Phase Estimation



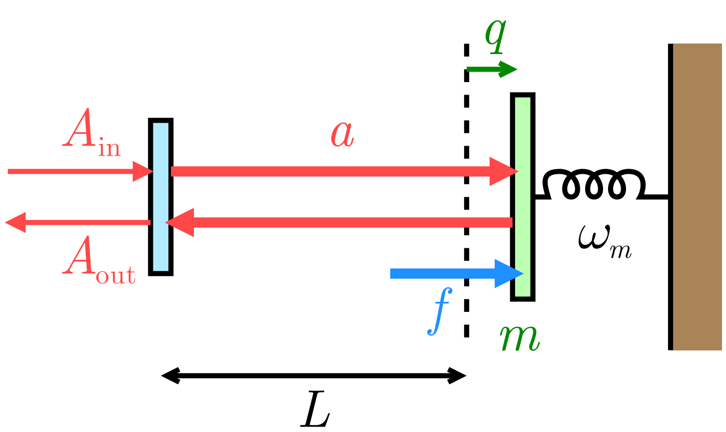


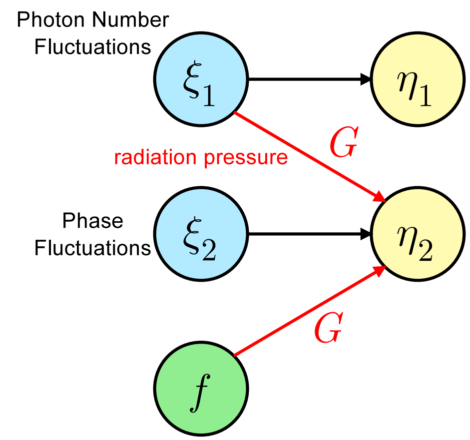
$$\langle \delta\phi_t^2\rangle_{\mathrm{QCRB}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{4S_{\Delta\hat{I}}(\omega) + 1/S_\phi(\omega)}, \quad \text{e.g. } S^{\mathrm{coh}}_{\Delta\hat{I}}(\omega) = \frac{\bar{P}}{\hbar\omega_0}, \quad S^{\mathrm{OU}}_\phi(\omega) = \frac{\kappa}{\omega^2+\epsilon^2}$$

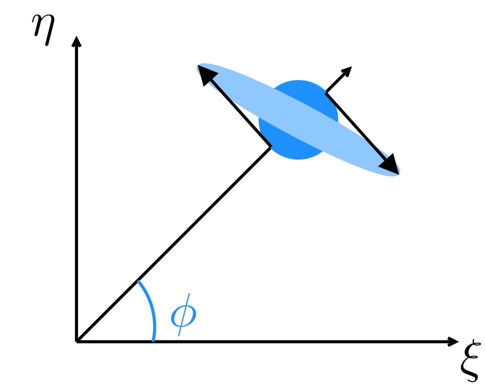
$$\langle \delta\phi_t^2\rangle_{\mathrm{smoothing}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{1/S_{\mathrm{noise}}(\omega) + 1/S_\phi(\omega)}, \quad \boxed{S_{\mathrm{noise}}(\omega) \ge \frac{1}{4S_{\Delta\hat{I}}(\omega)} = \frac{\hbar\omega_0}{4\bar{P}}}$$

- Wheatley *et al.*, Phys. Rev. Lett. (*Editors' Suggestion*) **104**, 093601 (2010): very close to QCRB for coherent state

Example 2: Optomechanical Force Sensor







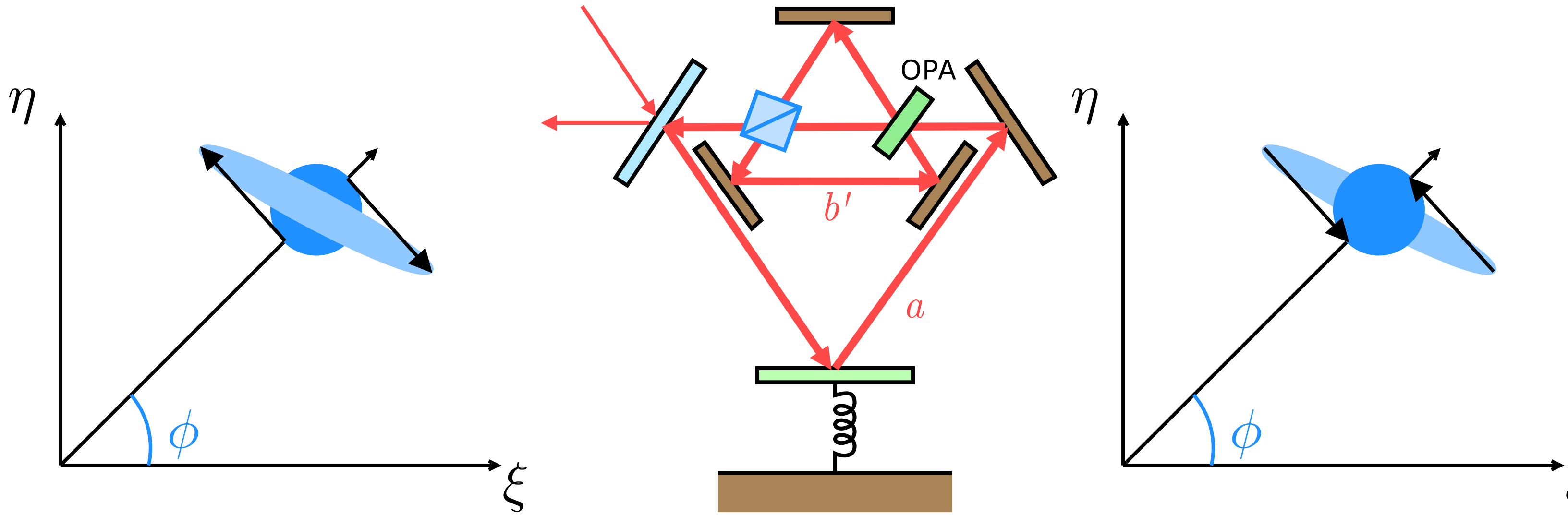
- QCRB:

$$\langle \delta f^2 \rangle_{\text{QCRB}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(4/\hbar^2) S_{\Delta \hat{q}}(\omega) + 1/S_f(\omega)} \tag{18}$$

- Apply smoothing to $y(\omega) = \eta_2(\omega) = G(\omega)[f(\omega) + \text{noise}(\omega)]$, SQL noise floor: $S_{\text{noise}}(\omega) = S_2(\omega)/|G(\omega)|^2 + S_1(\omega) \geq \hbar/|G(\omega)|$.

$$\langle \delta f^2 \rangle_{\text{smoothing}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{1/S_{\text{noise}}(\omega) + 1/S_f(\omega)} > \langle \delta f^2 \rangle_{\text{QCRB}} \,.$$

QCRB-Optimal Force Sensing

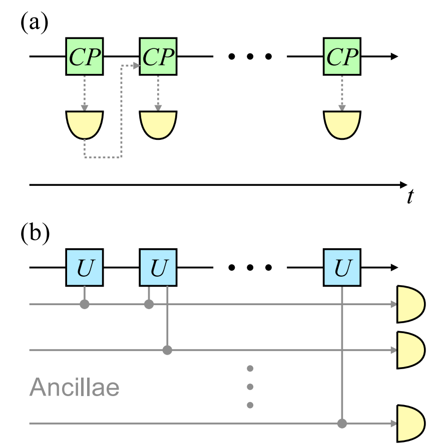


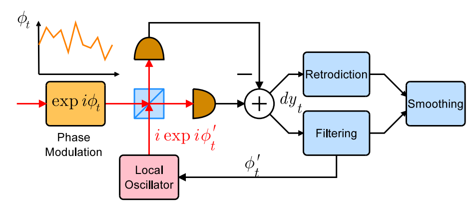
$$S_{\rm noise}(\omega) = \frac{1}{|G(\omega)|^2} S_2(\omega) + \cancel{S_1(\omega)} = \frac{\hbar^2}{4S_{\Delta\hat{q}}(\omega)}, \tag{19}$$

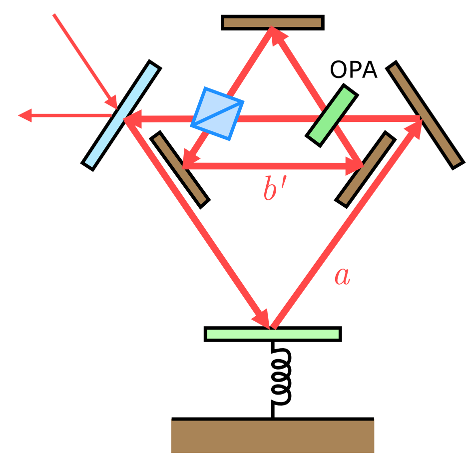
$$\langle \delta f^2 \rangle_{\rm smoothing,QNC} = \langle \delta f^2 \rangle_{\rm QCRB} \tag{20}$$

Summary

- **Estimation**: Quantum Smoothing
  - M. Tsang, J. H. Shapiro, and S. Lloyd, Phys. Rev. A **78**, 053820 (2008); **79**, 053843 (2009); M. Tsang, Phys. Rev. Lett. **102**, 250403 (2009); Phys. Rev. A **80**, 033840 (2009); **81**, 013824 (2010).
- **Control**: Quantum Noise Cancellation
  - M. Tsang and C. M. Caves, Phys. Rev. Lett. **105**, 123601 (2010).
- **Fundamental Limit**: Quantum Cramér-Rao Bound
  - M. Tsang, H. M. Wiseman, and C. M. Caves, “Fundamental Quantum Limit to Waveform Estimation,” to appear in Phys. Rev. Lett. [e-print arXiv:1006.5407].
- **Future Work**
  - Generalizations
  - Applications: Optical phase estimation, force sensing, magnetometry, atomic clocks, . . .
  - Collaborations: Elanor Huntington at ADFA@UNSW on quantum optics experiments, George Barbastathis at MIT/SMART on imaging experiments, Howard Wiseman at Griffith University on quantum theory, . . .
- `http://mankei.tsang.googlepages.com/`
- `mankei@unm.edu`







Nonlinear Filtering Equations

- Given

$$dx_t = A(x_t,t)dt + B(x_t,t)dW_t, \qquad dy_t = C(x_t,t)dt + d\eta_t, \tag{21}$$

$$\langle dW_t dW_t^T \rangle = Qdt, \qquad \langle d\eta_t d\eta_t^T \rangle = Rdt, \tag{22}$$

- Kushner-Stratonovich:

$$dP(x,t) = dt\left[-\frac{\partial}{\partial x_\mu}A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu}(BQB^T)_{\mu\nu}\right]P(x,t)$$
$$+ \left(dy_t - dt\,\langle C\rangle\right)^T R^{-1}\left(C - \langle C\rangle\right)P(x,t), \tag{23}$$

$$\langle C\rangle \equiv \int dx C(x,t)P(x,t). \tag{24}$$

- Duncan-Mortensen-Zakai:

$$df(x,t) = dt\left[-\frac{\partial}{\partial x_\mu}A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu}(BQB^T)_{\mu\nu}\right]f(x,t)$$
$$+ dy_t^T R^{-1} C f(x,t), \quad P(x,t) = \frac{f(x,t)}{\int dx f(x,t)}. \tag{25}$$

Linear Filtering

- Given

$$\dot{x}_t = Ax_t + B\xi, \qquad y_t = Cx_t + z_t, \tag{26}$$

- Kalman filtering: conditional mean $\tilde{x}$ and covariance Σ obey

$$\dot{\tilde{x}} = A\tilde{x} + \Gamma\left(y - C\tilde{x}\right), \qquad \Gamma = \Sigma C^T R^{-1}, \tag{27}$$

$$\dot{\Sigma} = A\Sigma + \Sigma A^T - \Sigma C^T R^{-1} C\Sigma + BQB^T. \tag{28}$$

Simon, *Optimal State Estimation*

- Wiener filtering: given $y(\omega) = x(\omega) + z(\omega)$, $S_x(\omega)$, and $S_z(\omega)$, design causal filter $H(\omega)$:

$$x'(\omega) = H(\omega)y(\omega). \qquad \Sigma = R\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\ln\left[1 + \frac{S_x(\omega)}{R}\right] \mathrm{if} S_z(\omega) = R. \tag{29}$$

Van Trees, *Detection, Estimation, and Modulation Theory*

Belavkin Equations

- Given $d\hat{y}_t = \hat{C}_t dt + d\hat{\eta}_t$, $\left\langle d\hat{\eta}_t d\hat{\eta}_t^T \right\rangle = R dt$,
- Belavkin equation (analogous to Kushner-Stratonovich):

$$d\hat{\rho}(t) = dt\mathcal{L}\hat{\rho}(t) + \frac{dt}{8}\left[2\hat{C}^T R^{-1}\hat{\rho}(t)\hat{C}^\dagger - \hat{C}^{\dagger T} R^{-1}\hat{C}\hat{\rho}(t) - \hat{\rho}(t)\hat{C}^{\dagger T} R^{-1}\hat{C}\right]$$
$$+ \frac{1}{2}\left(dy_t - \frac{dt}{2}\left\langle \hat{C} + \hat{C}^\dagger \right\rangle\right)^T R^{-1}\left[\left(\hat{C} - \langle \hat{C}\rangle\right)\hat{\rho}(t) + \mathrm{H.c.}\right], \quad \langle \hat{C}\rangle \equiv \mathrm{tr}[\hat{C}\hat{\rho}(t)]. \tag{30}$$

- Linear version (analogous to Duncan-Mortensen-Zakai):

$$d\hat{f}(t) = dt\mathcal{L}\hat{f}(x,t) + \frac{dt}{8}\left[2\hat{C}^T R^{-1}\hat{f}(t)\hat{C}^\dagger - \hat{C}^{\dagger T} R^{-1}\hat{C}\hat{f}(t) - \hat{f}(t)\hat{C}^{\dagger T} R^{-1}\hat{C}\right]$$
$$+ \frac{1}{2} dy_t^T R^{-1}\left[\hat{C}\hat{f}(t) + \mathrm{H.c.}\right], \quad \hat{\rho}(t) = \frac{\hat{f}(t)}{\mathrm{tr}[\hat{f}(t)]}. \tag{31}$$

- Belavkin, Barchielli, Mensky, Carmichael, Gisin, Mølmer, . . .
- Milburn, Caves, Wiseman, Mabuchi, van Handel, Bouten, James, Nurdin, Petersen, . . .
  - Carmichael, *Open Systems Approach to Quantum Optics*
  - Gardiner and Zoller, *Quantum Noise*
  - Wiseman and Milburn, *Quantum Measurement and Control*

Hybrid Filtering Equations

- Hybrid Belavkin:

$$
\begin{aligned}
d\hat{\rho}(x,t) &= dt\left[\mathcal{L}(x) - \frac{\partial}{\partial x_\mu}A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu}(BQB^T)_{\mu\nu}\right]\hat{\rho}(x,t) \\
&\quad + \frac{dt}{8}\left[2\hat{C}^T R^{-1}\hat{\rho}(x,t)\hat{C}^\dagger - \hat{C}^{\dagger T}R^{-1}\hat{C}\hat{\rho}(x,t) - \hat{\rho}(x,t)\hat{C}^{\dagger T}R^{-1}\hat{C}\right] \\
&\quad + \frac{1}{2}\left(dy_t - \frac{dt}{2}\left\langle \hat{C} + \hat{C}^\dagger\right\rangle\right)^T R^{-1}\left[\left(\hat{C} - \langle\hat{C}\rangle\right)\hat{\rho}(x,t) + \mathrm{H.c.}\right]
\end{aligned}
\tag{32}
$$

- Linear version:

$$
\begin{aligned}
d\hat{f}(x,t) &= dt\left[\mathcal{L}(x) - \frac{\partial}{\partial x_\mu}A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu}(BQB^T)_{\mu\nu}\right]\hat{f}(x,t) \\
&\quad + \frac{dt}{8}\left[2\hat{C}^T R^{-1}\hat{f}(x,t)\hat{C}^\dagger - \hat{C}^{\dagger T}R^{-1}\hat{C}\hat{f}(x,t) - \hat{f}(x,t)\hat{C}^{\dagger T}R^{-1}\hat{C}\right] \\
&\quad + \frac{1}{2}dy_t^T R^{-1}\left[\hat{C}\hat{f}(x,t) + \mathrm{H.c.}\right], \quad \hat{\rho}(x,t) = \frac{\hat{f}(x,t)}{\int dx\,\mathrm{tr}[\hat{f}(x,t)]}.
\end{aligned}
\tag{33}
$$

- Tsang, Phys. Rev. Lett. **102**, 250403 (2009); Phys. Rev. A **80**, 033840 (2009); **81**, 013824 (2010).

Phase-Space Smoothing

- Wigner distributions:

$$\mathrm{tr}[\hat{g}(x,t)\hat{f}(x,t)] \propto \int dqdp\, g(q,p,x,t) f(q,p,x,t) \tag{34}$$

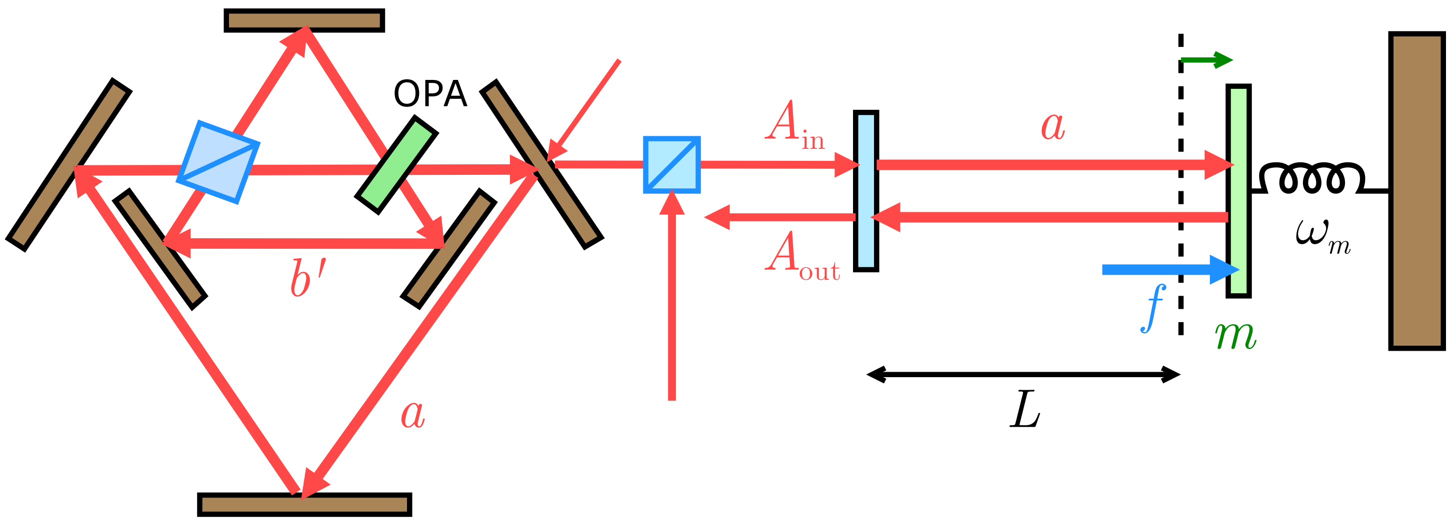
$$h(x,t) = \frac{\int dqdp\, g(q,p,x,t) f(q,p,x,t)}{\int dx\, (\mathrm{numerator})} \tag{35}$$

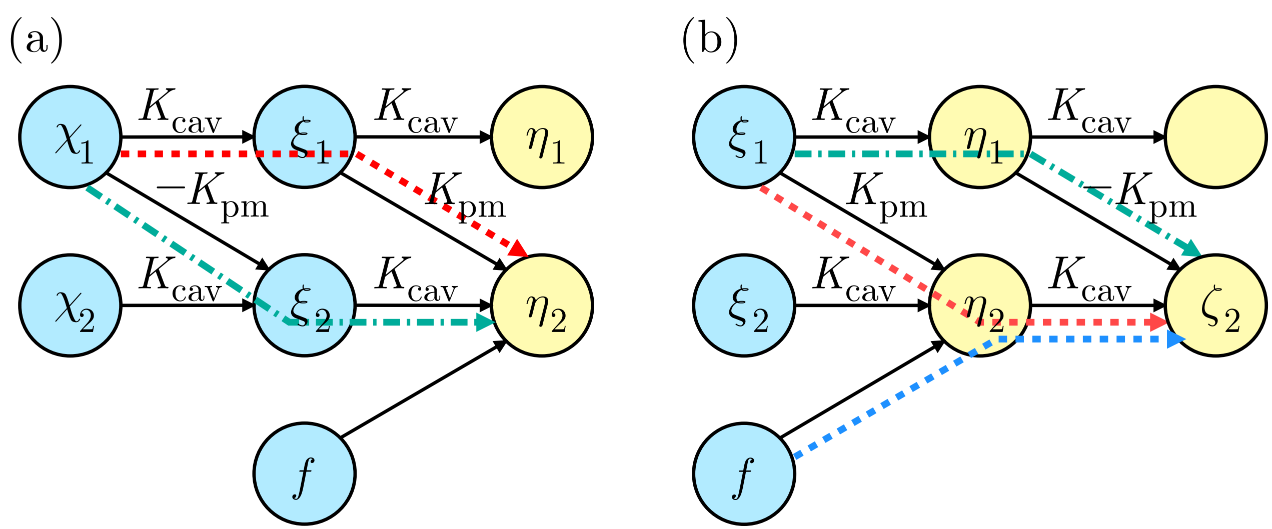
- If $f(q,p,x,t)$ and $g(q,p,x,t)$ are non-negative, equivalent to classical smoothing:

$$h(q,p,x,t) = \frac{g(q,p,x,t) f(q,p,x,t)}{\int dxdqdp\, (\mathrm{numerator})}, \qquad h(x,t) = \int dqdp\, h(q,p,x,t) \tag{36}$$

- If $f(q,p,x,t)$ and $g(q,p,x,t)$ are Gaussian, equivalent to linear smoothing, use two Kalman filters (Mayne-Fraser-Potter smoother)
- For stationary statistics, use Wiener.

Input/Output Noise Cancellation



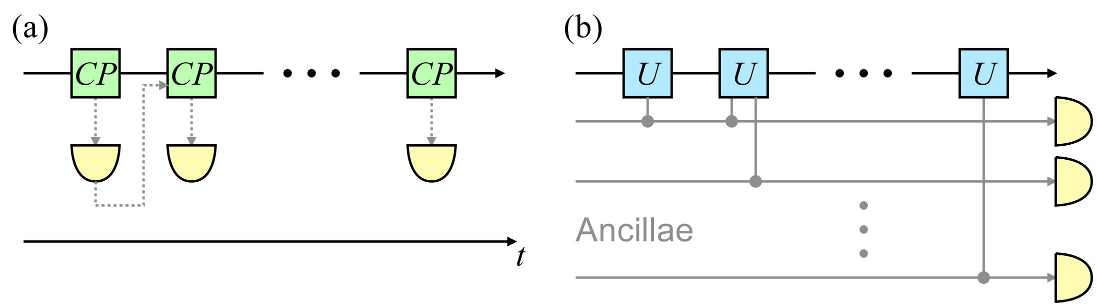


Quantum Information Theory to the Rescue

- Discretize time: finite number of measurement results and parameters
- Measurements and dynamics described by a sequence of completely positive (CP) maps:

$$P[y|x] = \mathrm{tr}\left[\mathcal{K}(y_N|x_N)\ldots\mathcal{K}(y_1|x_1)\hat{\rho}_0\right] \tag{37}$$

- Equivalent quantum circuit model (Kraus representation theorem, principle of deferred measurements):

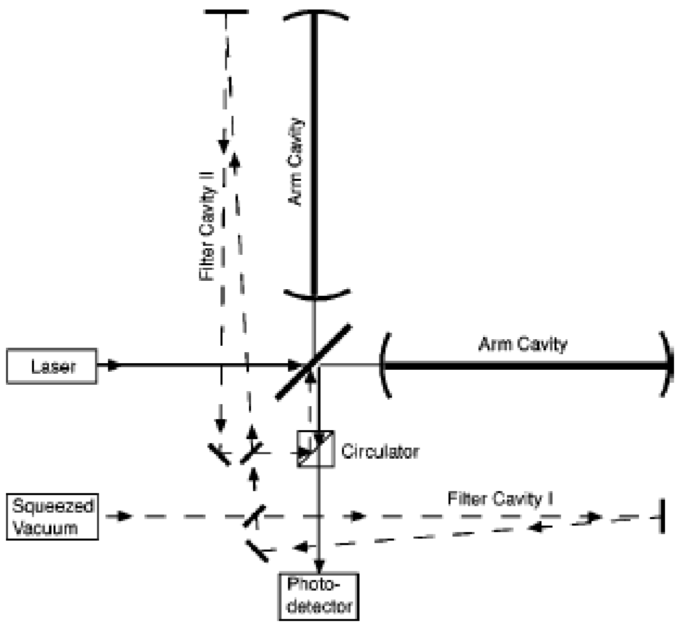


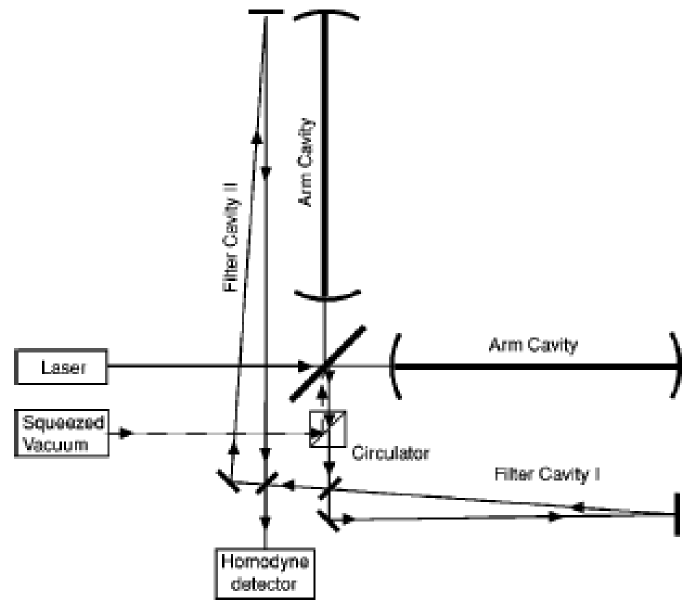
$$P[y|x] = \mathrm{tr}\left[\hat{E}(y)e^{-i\hat{H}(x_N)\delta t}\ldots e^{-i\hat{H}(x_1)\delta t}\hat{\rho}_0 e^{i\hat{H}(x_1)\delta t}\ldots e^{i\hat{H}(x_N)\delta t}\right] \tag{38}$$

- K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
- M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

Kimble Filters

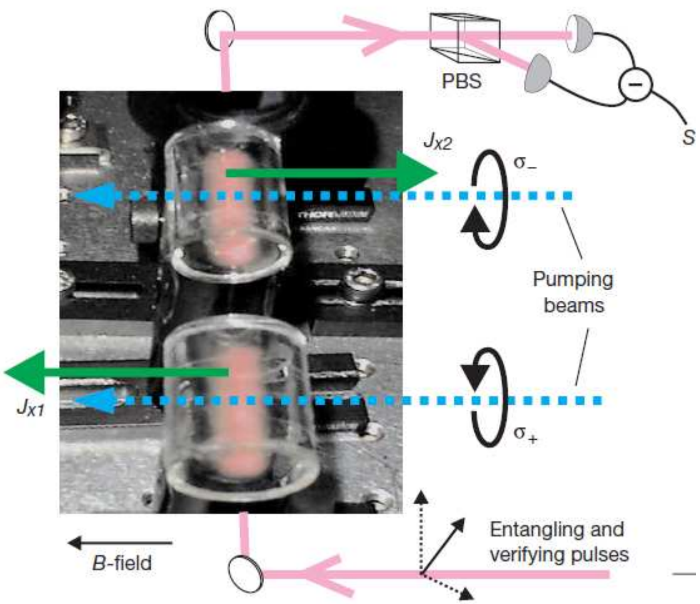
- W. G. Unruh, in *Quantum Optics, Experimental Gravitation, and Measurement Theory*, edited by P. Meystre and M. O. Scully (Plenum, New York, 1982), p. 647; Kimble *et al.*, PRD **65**, 022002 (2001):

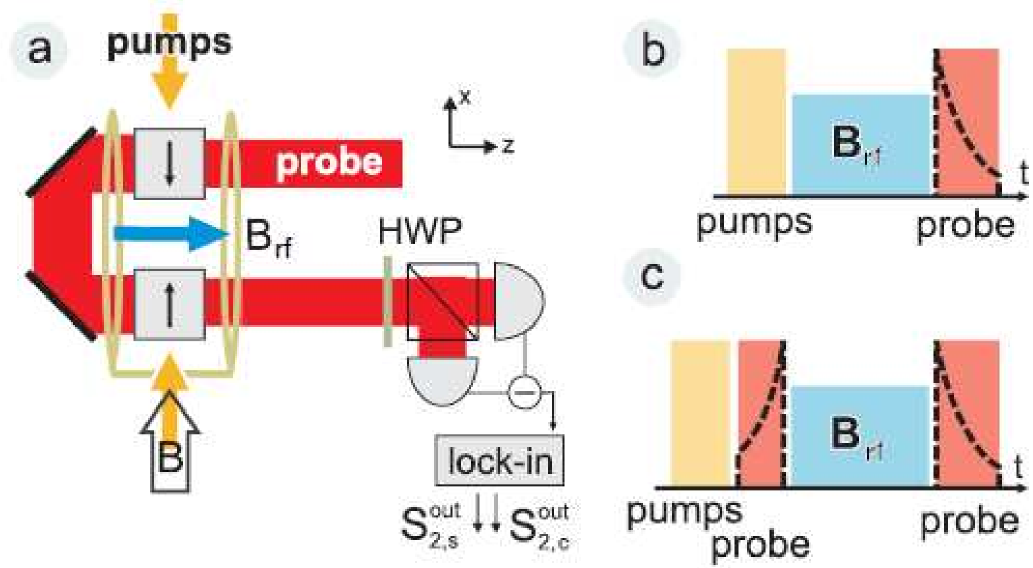




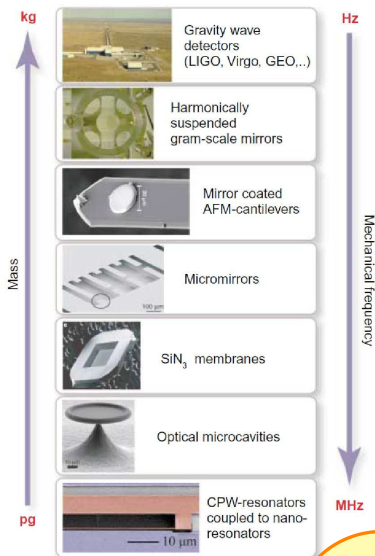
Magnetometry

- Smoothing for magnetometry with linear Gaussian approximation: Petersen and Mølmer, Phys. Rev. A **74**, 043802 (2006).
- Quantum Noise Cancellation:
  - Julsgaard, Kozhekin, and Polzik, “Experimental long-lived entanglement of two macroscopic objects,” Nature **413**, 400 (2001).
  - Wasilewski *et al.*, “Quantum Noise Limited and Entanglement-Assisted Magnetometry,” Phys. Rev. Lett. **104**, 133601 (2010).

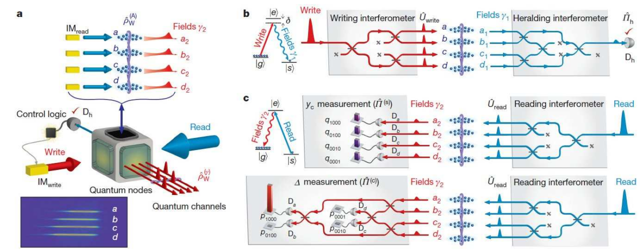




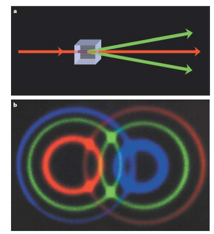
Coupling Optics, Atoms, Mechanics, and Microwaves

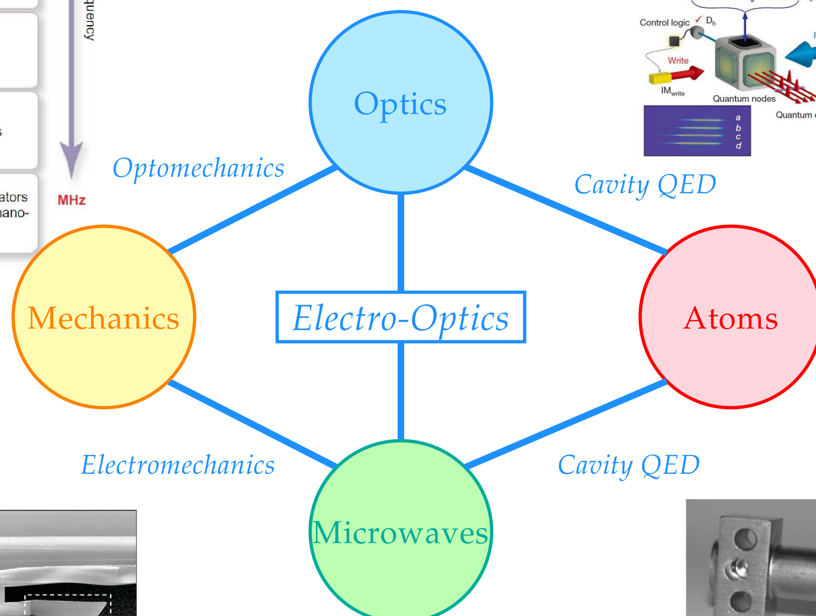


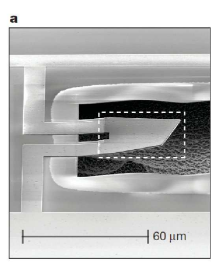
Kippenberg and Vahala,
Science 321, 1172 (2008)



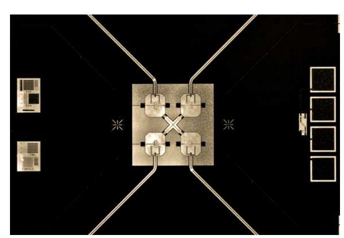
Choi, Goban, Papp, van Enk, and Kimble,
Nature 468, 412 (2010)



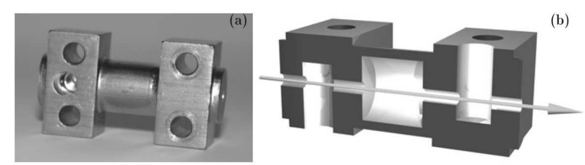




O'Connell et al.,
Nature 464, 697 (2010)

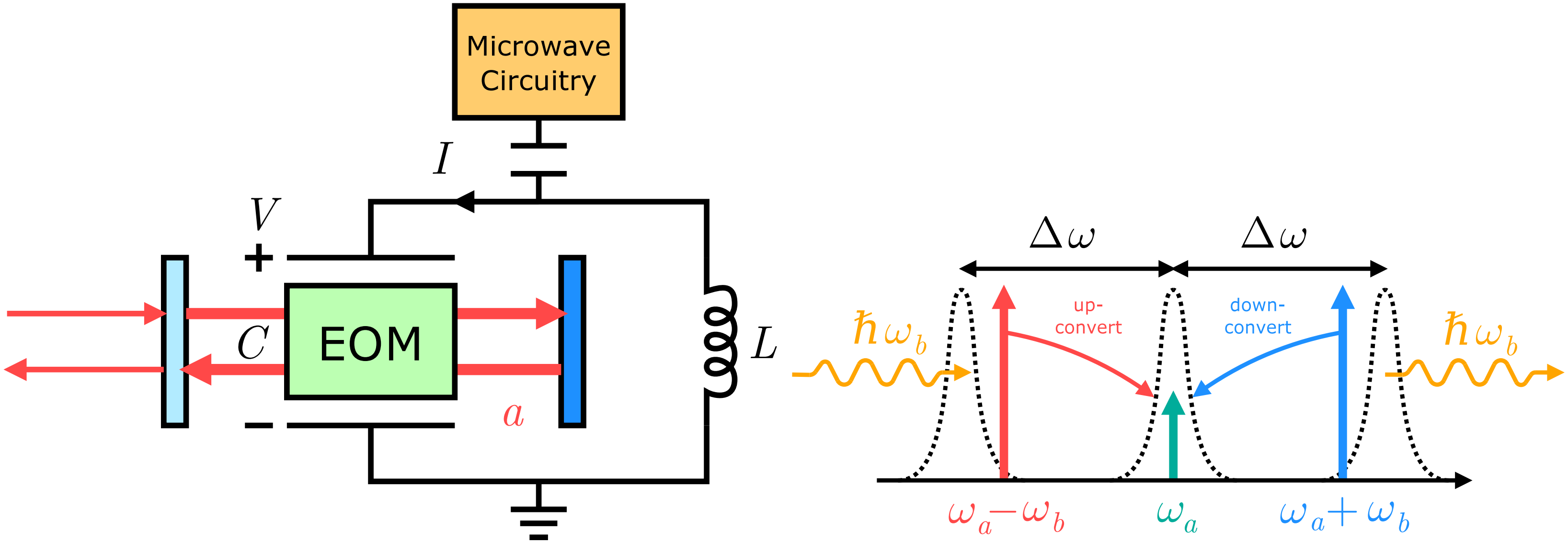


Neeley et al., Nature 467, 570 (2010)



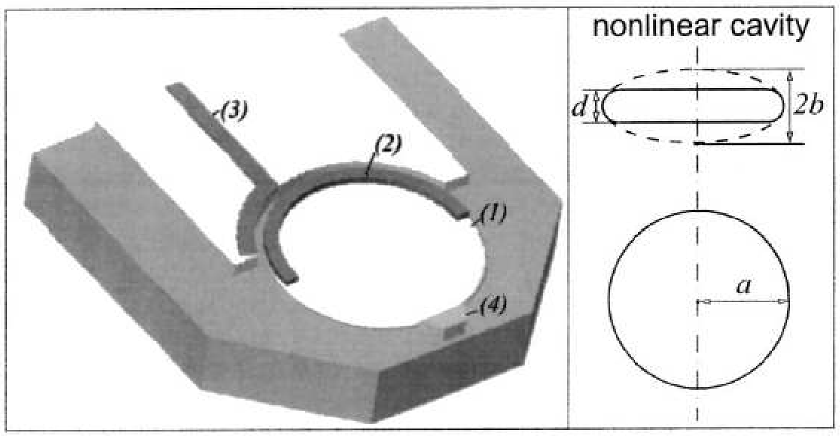
Haroche and Raimond,
Exploring the Quantum

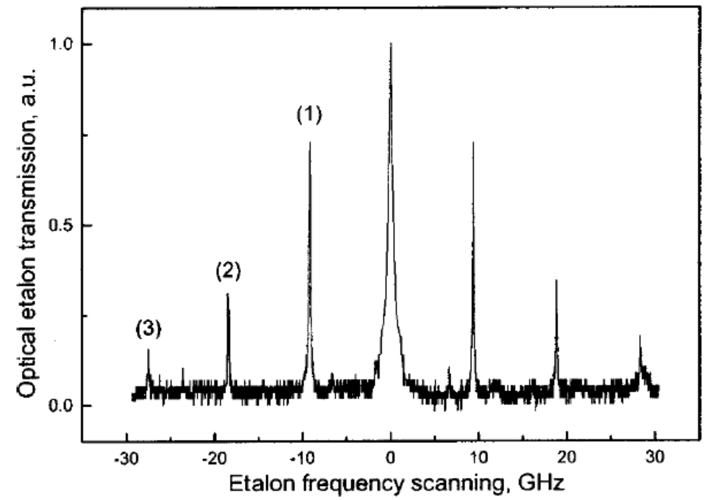
Cavity Quantum Electro-Optics



- Electro-Optic Modulator (EOM): second-order nonlinear optical effect in non-centrosymmetric material (e.g. lithium niobate), off-resonant (coherent and ultrafast, same effect used in OPA), widely used in optics
- interaction Hamiltonian: $\hat{H}_I = \hbar\omega(\hat{V})\hat{a}^\dagger\hat{a}$, $\omega(\hat{V}) = \omega_0 + \omega_1\hat{V}$, same as optomechanics ($\hat{V} \sim \hat{q}$)
- Optical pump at red sideband: coherent electro-optic up conversion, laser cooling of microwave mode, . . . Pump at blue sideband: electro-optic parametric amplification, . . .
- M. Tsang, "Cavity Quantum Electro-Optics," Phys. Rev. A **81**, 063837 (2010).

Experimental Prospect





- Ilchenko *et al.* (JPL), JOSAB **20**, 333 (2003): $LiNbO_3$ whispering-gallery mode optical resonator + half-wave microstrip resonator

$$\omega_b \sim 2\pi \times 9\,\mathrm{GHz}, \quad \lambda_a \sim 1550\,\mathrm{nm}, \quad G = \frac{g^2 N_{\mathrm{pump}}}{\gamma_a \gamma_b}, \quad \boxed{g \sim 2\pi \times 20\,\mathrm{Hz}} \tag{39}$$

$$\gamma_b \sim 2\pi \times 90\,\mathrm{MHz}, \quad Q_b \sim 100, \quad \gamma_a \sim 2\pi \times 40\,\mathrm{MHz}, \quad Q_a \sim 5 \times 10^6 \tag{40}$$

- Optical pump at $2\,\mathrm{mW}$, $G \sim 2 \times 10^{-5}$.
- If one can shrink the device to optical wavelength scale, $g \sim 5\,\mathrm{kHz}$, if one uses superconducting microwave resonator, γ_b can be reduced by $10^4 - 10^6$.