I. Quantum Waveform Detection Theory
II. Quantum Waveform Estimation Theory
III. Quantum Microwave Photonics *

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Quantum Probability Theory

- Wave:

- Probability (Born’s rule $P(x) = |\langle x|\psi\rangle|^2$):

  I.2 ON THE QUANTUM MECHANICS OF COLLISIONS
  [Preliminary communication]†

  MAX BORN

  * Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{nm}$.

- More than classical probability:
Quantum Probability Experiments

Kippenberg and Vahala, Science 321, 1172 (2008), and ref. therein.


**Quantum Sensing/Metrology**

- **Fundamental Limits**: What is the ultimate sensitivity allowed by quantum mechanics?
- **Control**: Optimize experiment
- **Estimation**: Optimize data processing
- **Examples**: optical interferometry, optical imaging, optomechanical force sensing (gravitational-wave detection), atomic magnetometry, gyroscopes, electrometer, etc.
Quantum Optomechanical Force Detection

LIGO, Hanford

\[ \mathcal{H}_0 : \]

\[ A_{\text{in}} \rightarrow A_{\text{out}} \]

\[ \mathcal{H}_1 : \]

\[ x(t) \]

- $Y \in \mathcal{Y}$ is an observation.
- $Y$ is noisy: $\Pr(Y|\mathcal{H}_0)$ and $\Pr(Y|\mathcal{H}_1)$
- Given $Y$, $\Pr(Y|\mathcal{H}_0)$, and $\Pr(Y|\mathcal{H}_1)$, we want to decide which hypothesis is true.
- **Decision rule:** divide $\mathcal{Y}$ into two regions $\mathcal{Y}_0$ and $\mathcal{Y}_1$:
  - If $Y \in \mathcal{Y}_0$, we decide $\mathcal{H}_0$ is true.
  - If $Y \in \mathcal{Y}_1$ we decide $\mathcal{H}_1$ is true.

- **Type-I error probability (false-alarm probability):**
  \[
P_{10}(\mathcal{Y}_0, \mathcal{Y}_1) = \sum_{Y \in \mathcal{Y}_1} \Pr(Y|\mathcal{H}_0) \tag{1}
  \]

- **Type-II error probability (miss probability):**
  \[
P_{01}(\mathcal{Y}_0, \mathcal{Y}_1) = \sum_{Y \in \mathcal{Y}_0} \Pr(Y|\mathcal{H}_1) \tag{2}
  \]

- How to choose $\mathcal{Y}_0$ and $\mathcal{Y}_1$ in order to minimize errors?
Likelihood-Ratio Test

- Define likelihood ratio:
  \[ \Lambda \equiv \frac{\Pr(Y|H_1)}{\Pr(Y|H_0)} \]  \( (3) \)

- Likelihood-ratio test given a threshold \( \gamma \):
  - If \( \Lambda \geq \gamma \) decide \( H_1 \) is true.
  - If \( \Lambda < \gamma \) decide \( H_0 \) is true.

- Neyman-Pearson criterion:
  - Constrain \( P_{10} \leq \alpha \) and minimize \( P_{01} \)
  - set \( \gamma \) such that \( P_{10} = \Pr(\Lambda \geq \gamma | H_0) = \alpha \)

- Bayes criterion (given prior probabilities \( P_0 \) and \( P_1 \)):
  - Define the cost of deciding on \( H_j \) given \( H_k \) as \( C_{jk} \) (loss function).
  - minimize average cost (Bayes risk):
    \[ C = \sum_{j,k} P_{jk} P_k C_{jk} \]  \( (4) \)
    - e.g., \( P_e = P_0 P_{10} + P_1 P_{01} \).
    - set \( \gamma = (C_{10} - C_{00}) P_0 / (C_{01} - C_{11}) P_1 \).

For a likelihood-ratio test,

- Error probabilities:

\[ P_{10} = \Pr(\Lambda \geq \gamma | \mathcal{H}_0), \quad P_{01} = \Pr(\Lambda < \gamma | \mathcal{H}_1). \]  

Very hard to calculate, but can be bounded using Chernoff bounds:

\[ P_{10} \leq \inf_{0 \leq s \leq 1} \mathbb{E} [\Lambda^s | \mathcal{H}_0] \gamma^{-s}, \quad P_{01} \leq \inf_{0 \leq s \leq 1} \mathbb{E} [\Lambda^s | \mathcal{H}_0] \gamma^{1-s}. \]  

- Lower bounds:

\[ \min_{\mathcal{Y}_{0,1}} P_e = \frac{1}{2} \left[ 1 - ||P_0 \Pr(Y|\mathcal{H}_0) - P_1 \Pr(Y|\mathcal{H}_1)||_1 \right] \geq \frac{1}{2} \left( 1 - \sqrt{1 - 4P_0 P_1 F} \right), \]  

\[ ||A(Y)||_1 \equiv \sum_Y |A(Y)|, \quad F \equiv \left[ \sum_Y \sqrt{\Pr(Y|\mathcal{H}_1) \Pr(Y|\mathcal{H}_0)} \right]^2. \]  

valid for any decision rule.

- Bayesian posterior probabilities:

\[ \Pr(\mathcal{H}_1 | Y) = \frac{P_1 \Lambda}{P_1 \Lambda + P_0}, \quad \Pr(\mathcal{H}_0 | Y) = \frac{P_0}{P_1 \Lambda + P_0}. \]
Quantum Hypothesis Testing

Define **density operator** as mixture of pure states

\[ \rho = \sum_j P_j |\psi_j\rangle \langle \psi_j|, \]  

(10)

A generalized measurement (generalized Born’s rule) is described by

\[ |\Psi_j\rangle = |\psi_j\rangle_A \otimes |\phi\rangle_B, \quad \Pr(Y) = \sum_j P_j |\langle Y|U|\Psi_j\rangle|^2 = \text{tr} [E(Y)\rho], \]  

(11)

where \( E(Y) = B \langle \phi|U^\dagger|Y\rangle \langle Y|U|\phi\rangle_B \) is called **POVM** (Positive Operator-Valued Measure).

Given two density operators \( \rho_0 \) and \( \rho_1 \),

\[ \Pr(Y|\mathcal{H}_0) = \text{tr}[E(Y)\rho_0], \quad \Pr(Y|\mathcal{H}_1) = \text{tr}[E(Y)\rho_1], \]  

(12)

what is the POVM that minimizes the error probabilities?
Quantum Error Bounds


- Given constraint $P_{10} \leq \alpha$,

\[
P_{01} \geq \begin{cases} 
1 - \left[ \sqrt{\alpha F} + \sqrt{(1 - \alpha)(1 - F')} \right]^2, & \alpha < F, \\
0, & \alpha \geq F.
\end{cases}
\]  \hspace{2cm} (13)

- Average error probability:

\[
\min_{E(Y)} P_e = \frac{1}{2} \left( 1 - ||P_1 \rho_1 - P_0 \rho_0||_1 \right) \geq \frac{1}{2} \left( 1 - \sqrt{1 - 4P_0 P_1 F} \right),
\]  \hspace{2cm} (14)

\[
||A||_1 \equiv \text{tr} \sqrt{AA^\dagger}, \quad F \equiv \left( \text{tr} \sqrt{\sqrt{\rho_0} \rho_1 \sqrt{\rho_0}} \right)^2.
\]  \hspace{2cm} (15)

- For pure states,

\[
F = |\langle \psi_0 | \psi_1 \rangle|^2,
\]  \hspace{2cm} (16)

and there exist POVMs such that the fidelity bounds are saturated.
Distinguishing a photon in two possible polarizations (assume $P_0 = P_1 = 1/2$):

$$|\psi_0\rangle = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix},$$

$$|\psi_1\rangle = \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix},$$

$$F = \cos^2 2\theta,$$

$$\min_{E(Y)} P_e = \frac{1}{2} (1 - \sin 2\theta).$$

Two Linear Polarizations:

Bloch Sphere:
Consider projective measurement in the following basis:

\[
|0\rangle = \left( \frac{1}{\sqrt{2}} \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad |1\rangle = \left( \frac{1}{\sqrt{2}} \right) \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

(19)

Two Linear Polarizations:

Error probabilities:

\[
P_{10} = |\langle 1|\psi_0 \rangle|^2 = \frac{1}{2} (\cos \theta - \sin \theta)^2, \quad P_{01} = |\langle 0|\psi_1 \rangle|^2 = \frac{1}{2} (\cos \theta - \sin \theta)^2,
\]

(20)

\[
P_e = \frac{1}{2} (1 - \sin 2\theta) = \min_{E(Y)} P_e.
\]

(21)

This attains Helstrom’s bound.
Coherent-State Discrimination

- two coherent states

\[ |\alpha_{0,1} \rangle = \exp\left(-\frac{|\alpha_{0,1}|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha_{0,1}^n}{\sqrt{n!}} , \quad F = \exp(-|\alpha_0 - \alpha_1|^2). \] (22)


Photon Counter

\[ \mathcal{H}_1 : |\alpha_1 \rangle \quad \rightarrow \quad |\alpha_1 - \alpha_0 \rangle \quad \rightarrow \quad \text{nonzero count: choose } \mathcal{H}_1 \]
\[ \mathcal{H}_0 : |\alpha_0 \rangle \quad \rightarrow \quad |0 \rangle \quad \rightarrow \quad \text{zero count: choose } \mathcal{H}_0 \]

\[ D(-\alpha_0) \]

\[ P_{10} = 0, \quad P_{01} = \exp(-|\alpha_0 - \alpha_1|^2), \] (23)

\[ P_e = \frac{1}{2} \exp(-|\alpha_0 - \alpha_1|^2) = \frac{1}{2} F \approx 2 \min_{E(Y)} P_e \text{ if } F \ll 1. \] (24)

- near-optimal if \(|\alpha_0 - \alpha_1|^2 \gg 1\).
Optimal Coherent-State Receiver

- **Dolinar receiver** [Dolinar, MIT RLE Quart. Prog. Rep. 111, 115-120 (1973)]:

  $$\mathcal{H}_1 : |\alpha_1/\sqrt{M}, \ldots, \alpha_1/\sqrt{M}\rangle$$
  $$\mathcal{H}_0 : |\alpha_0/\sqrt{M}, \ldots, \alpha_0/\sqrt{M}\rangle$$

- **Cook et al., Nature 446 774, (2007):**
How to apply Helstrom bound to waveform detection?

- Force is time-varying
- Continuous quantum dynamics
- Continuous quantum measurements at the same time as the force perturbation.
Discretize time, and take continuous limit at the end.

Measurements and dynamics described by a sequence of completely positive maps:

\[
\Pr(Y|\mathcal{H}_j) = \text{tr} \left[ \mathcal{J}_j(y_{M}) \mathcal{K}_j \ldots \mathcal{J}_j(y_1) \mathcal{K}_j \rho_j(t_0) \right],
\]

(25)

In terms of Kraus operators:

\[
\mathcal{K}\rho \equiv \sum_z K(z) \rho K^\dagger(z), \quad \mathcal{J}(y)\rho \equiv \sum_z J(y, z) \rho J^\dagger(y, z).
\]

(26)

Purification (Naimark/Kraus): CP maps can always be written as unitary operations with projective measurements in a larger Hilbert space:

\[
\mathcal{K}\rho = \text{tr}_B \left[ U (\rho \otimes |\phi\rangle_B \langle \phi|) U^\dagger \right], \quad \mathcal{J}(y)\rho = B \langle y| U (\rho \otimes |\phi\rangle_B \langle \phi|) U^\dagger |y\rangle_B.
\]

(27)

Principle of deferred measurements:

The waveform detection problem can be written as a quantum-state discrimination problem:

\[ \rho_0 = U_0 |\psi\rangle \langle \psi | U_0^\dagger, \]
\[ U_0 = \mathcal{T} \exp \left[ \frac{1}{i\hbar} \int dt H_0(t) \right], \]
\[ \rho_1 = U_1 |\psi\rangle \langle \psi | U_1^\dagger, \]
\[ U_1 = \mathcal{T} \exp \left[ \frac{1}{i\hbar} \int dt H_1(x(t), t) \right], \]

(28)
(29)

\[ F = \left| \langle \psi | U_0^\dagger U_1 |\psi\rangle \right|^2. \]

(30)
Quantum Bound for Waveform Detection

- Lower bound in terms of fidelity:

\[
P_e \geq \frac{1}{2} \left( 1 - \sqrt{1 - 4P_0 P_1 F} \right),
\]

\( F = \left| \langle \psi | U_0^\dagger U_1 | \psi \rangle \right|^2 = \left| \langle \psi | T \exp \left( \frac{1}{i \hbar} \int dt H_1(t) \right) | \psi \rangle \right|^2, \)

\( H_I(t) \equiv U_0^\dagger(t, t_0) [H_1(t) - H_0(t)] U_0(t, t_0). \)

- Suppose \( H_1 - H_0 = -qx(t), U_0^\dagger qU_0 \) is linear with respect to initial positions/momenta, and |\( \psi \rangle \) has Gaussian Wigner representation. Then

\[
F = \exp \left[ -\frac{1}{\hbar^2} \int dt \int dt' x(t) \langle : \Delta q_0(t) \Delta q_0(t') : \rangle x(t') \right],
\]

\[
q_0(t) = U_0^\dagger(t, t_0) q U_0(t, t_0), \quad \Delta q_0(t) = q_0(t) - \langle q_0(t) \rangle.
\]

Quantum Receiver Design

- **error exponent**: measures the decay rate of error:

\[
\Gamma \equiv - \lim_{T \to \infty} \frac{1}{T} \ln P_e, \quad \Gamma^{(\text{Helstrom})} \equiv - \lim_{T \to \infty} \frac{1}{T} \ln \min_{E(Y)} P_e = - \lim_{T \to \infty} \frac{1}{T} \ln F. \tag{36}
\]

- Homodyne detection of \( A_{\text{out}} \) is sub-optimal: \( \Gamma^{(\text{homodyne})} = \frac{1}{2} \Gamma^{(\text{Helstrom})} \).

- Kennedy receiver is near-optimal if \( A_{\text{out}} \) is in coherent state:

\[
\Gamma^{(\text{Kennedy})} = \Gamma^{(\text{Helstrom})}. \tag{37}
\]

![Quantum Receiver Diagram](image)

- **Require Quantum Noise Cancellation** [M. Tsang and C. M. Caves, PRL 105, 123601 (2010); PRX 2, 031016 (2012)], a coherent feedforward control technique, if measurement backaction noise is significant.

- **Estimation** is trivial in theory, but with technical imperfections/homodyne detection, likelihood-ratio test is necessary.
Continuous noisy measurement of mechanical oscillator energy
- Is the energy classical (continuous) or quantum (discrete)?
- Focus on estimation (calculation of likelihood ratio).


Continuous Quantum Measurements

- **Sequential** measurements of a quantum system:

\[
\Pr(Y|\mathcal{H}_j) = \text{tr} \left[ J_j(y_M)K_j \ldots J_j(y_1)K_j \rho_j(t_0) \right],
\]  
(38)

- $\mathcal{K}$ and $\mathcal{J}$ are **completely-positive maps**. In terms of **Kraus operators**:

\[
\mathcal{K}\rho \equiv \sum_z K(z)\rho K^\dagger(z), \quad \mathcal{J}(y)\rho \equiv \sum_z J(y, z)\rho J^\dagger(y, z).
\]  
(39)

- Stick with **smaller** Hilbert space; easier for numerical analysis.

- **infinitesimal CP map** (Lindblad):

\[
\mathcal{K}\rho = \rho + \delta t\mathcal{L}\rho + o(\delta t).
\]  
(40)

- For **weak measurements with Gaussian noise**,

\[
\mathcal{J}(\delta y)\rho = \tilde{P}(\delta y) \left[ \rho + \frac{\delta y}{2R} \left( c\rho + \rho c^\dagger \right) + \frac{\delta t}{8Q} \left( 2c\rho c^\dagger - c^\dagger c\rho - \rho c^\dagger c \right) \right] + o(\delta t),
\]  
(41)

\[
\tilde{P}(\delta y) = \mathcal{N}(0, R\delta t).
\]  
(42)

Suppose noise variance $R$ is the same in both hypothesis. Then it can be shown that

$$\Lambda = \frac{\text{tr} f_1(T)}{\text{tr} f_0(T)},$$

(43)

where $f_1$ and $f_0$ obey the quantum Duncan-Mortensen-Zakai (DMZ) equation:

$$df_j = dt\mathcal{L}_j f_j + \frac{dy}{2R} \left( c_j f_j + f_j c_j^\dagger \right) + \frac{dt}{8Q_j} \left( 2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j \right).$$

(44)

some stochastic calculus:

$$d \text{tr} f_j = \text{tr} df_j = \frac{dy}{2R} \text{tr} \left( c_j f_j + f_j c_j^\dagger \right) = \frac{dy}{R} \frac{\text{tr} \left( c_j f_j + f_j c_j^\dagger \right)}{2 \text{tr} f_j} \text{tr} f_j,$$

(45)

$$\ln \text{tr} f_j(t) = \int_{t_0}^{T} \frac{dy}{R} \mu_j - \int_{t_0}^{T} \frac{dt}{2R} \mu_j^2, \quad \mu_j \equiv \frac{1}{\text{tr} f_j} \text{tr} \left( \frac{c_j + c_j^\dagger}{2} f_j \right).$$

(46)
Final result:

\[
\Lambda(T) = \exp \left[ \int_{t_0}^{T} \frac{dy}{R} (\mu_1 - \mu_0) - \int_{t_0}^{T} \frac{dt}{2R} (\mu_1^2 - \mu_0^2) \right].
\] (47)

\(\mu_j\) is the expected value of \((c_j + c_j^\dagger)/2\) given the observation record, assuming \(\mathcal{H}_j\) is true, can be calculated by quantum filters.

Similar formula exists for continuous measurements with Poisson noise.


Applications

$\mathcal{H}_0$:

$\mathcal{H}_1$:


Given \( y \) and likelihood function \( P(y|x) \), estimate \( x \).

Let estimate be \( \tilde{x}(y) \).

Mean-square error:

\[
\mathbb{E}(\delta x^2) \equiv \int dy [\tilde{x}(y) - x]^2 P(y|x).
\]

For unbiased estimates, \( x = \int dy \tilde{x}(y) P(y|x) \), Cramér-Rao bound:

\[
\mathbb{E}(\delta x^2) \geq J^{-1},
\]

Fisher information:

\[
J \equiv \int dy P(y|x) \left( \frac{\partial \ln P(y|x)}{\partial x} \right)^2.
\]

If \( P(y|x) \) is Gaussian:

\[
P(y|x) = \frac{1}{\sqrt{(2\pi)^K \det R}} \exp \left[ -\frac{1}{2} (y - Cx)^\top R^{-1} (y - Cx) \right],
\]

CRB is attainable using maximum-likelihood estimation.
Other Tracking Examples
Quantum Parameter Estimation

Quantum:

\[ P(y|x) = \text{tr} [E(y)\rho_x] . \] (52)

Quantum Cramér-Rao bound (QCRB) (valid for any POVM but may not be achievable):

\[ \mathbb{E}(\delta x^2) \geq 1/J^{(Q)} , \]
\[ J^{(Q)} \equiv \text{tr} \left( \Delta h^\dagger \Delta h \rho_x \right) , \]
\[ \frac{\partial \rho_x}{\partial x} = \frac{1}{2} \left( h \rho_x + \rho_x h^\dagger \right) , \]
\[ \Delta h \equiv h - \text{tr} \left( h \rho_x \right) . \] (53) (54)

Example: Optical Phase Estimation

- Optical phase modulation:

\[ \rho_x = \exp(i\hat{n}x)\rho_0 \exp(-i\hat{n}x), \quad \frac{\partial \rho_x}{\partial x} = i[\hat{n}, \rho_x], \quad \hat{h} = 2i\hat{n}, \quad \mathbb{E}(\delta x^2) \geq \frac{1}{4 \langle \Delta n^2 \rangle}. \quad (55) \]

Generalizations of Classical Cramér-Rao Bound

- Error covariance matrix for multiple parameters with prior distribution $P(x)$:

$$
\Sigma \equiv \mathbb{E} \left[ (\tilde{x} - x)(\tilde{x} - x)^\top \right] \equiv \int dydx P(y|x) P(x) (\tilde{x} - x)(\tilde{x} - x)^\top. \quad (56)
$$

- Define loss function in terms of positive-semidefinite matrix $\Lambda$ as

$$
C(\tilde{x}, x) = (\tilde{x} - x)^\top \Lambda (\tilde{x} - x). \quad (57)
$$

- Average cost/Bayes risk

$$
C \equiv \mathbb{E} [C(\tilde{x}, x)] = \text{tr} (\Lambda \Sigma) \geq 0. \quad (58)
$$

- Bayesian Cramér-Rao bound (Van Trees) for any $\Lambda$:

$$
C \geq \text{tr} (\Lambda J^{-1}), \quad J = J(Y) + J(X), \quad J(Y) = \mathbb{E} \left[ \frac{\partial \ln P(y|x)}{\partial x_j} \frac{\partial \ln P(y|x)}{\partial x_k} \right], \quad J(X) = \mathbb{E} \left[ \frac{\partial \ln P(x)}{\partial x_j} \frac{\partial \ln P(x)}{\partial x_k} \right]. \quad (59)
$$

- More compact way: $\Sigma \geq J^{-1}$; i.e., $\Sigma - J^{-1}$ is positive-semidefinite.
Revisiting Church of Larger Hilbert Space

- How to deal with continuous sensing?
  \[ x(t) \]

- Discretize time, purification in larger Hilbert space, principle of deferred measurements:

\[ \text{(a)} \quad \text{CP} \quad \text{CP} \quad \ldots \quad \text{CP} \quad \text{CP} \]

\[ \text{(b)} \quad U \quad U \quad \ldots \quad U \quad \text{Ancillae} \]
Fisher information matrix $J(t, t')$

$$
\int dt dt' \Lambda(t, t') \left\{ \mathbb{E} \left[ \delta x(t) \delta x(t') \right] - J^{-1}(t, t') \right\} \geq 0,
$$

(61)

$J^{-1}(t, t')$ is defined by $\int dt' J(t, t') J^{-1}(t', \tau) = \delta(t - \tau)$.

Two components:

$$
J(t, t') = J(Q)(t, t') + J(X)(t, t').
$$

(62)

$J(Q)$ is a two-time quantum covariance function:

$$
J(Q)(t, t') = \frac{4}{\hbar^2} \mathbb{E} \left\{ \text{tr} \left[ : \Delta h(t) \Delta h(t') : \rho_0 \right] \right\}, \quad \hat{h}(t) \equiv \hat{U}^\dagger(t, t_0) \frac{\partial \hat{H}(t)}{\partial x(t)} \hat{U}(t, t_0).
$$

$J(X)$ incorporates a priori waveform information

$$
J(X)(t, t') = \mathbb{E} \left\{ \frac{\delta \ln P[x]}{\delta x(t)} \frac{\delta \ln P[x]}{\delta x(t')} \right\}.
$$

(63)

Example 1: Adaptive Optical Phase Estimation


\[ \phi(t) = \int_{-\infty}^{\infty} d\tau g(t - \tau) x(\tau), \quad (64) \]

\[ \mathbb{E}[\delta x^2(t)] \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{4|g(\omega)|^2 S_{\Delta I}(\omega) + 1/S_x(\omega)}, \quad (65) \]

\[ \text{e.g. } S_{\Delta I}^{\text{coh}}(\omega) = \frac{\bar{P}}{\hbar \omega_0}, \quad S_x^{\text{OU}}(\omega) = \frac{\kappa}{\omega^2 + \epsilon^2}. \quad (66) \]

Filtering: Berry and Wiseman, PRA 65, 043803 (2002); 73, 063824 (2006).

Filtering does not saturate QCRB!
Classical Estimation

- **Filtering, Prediction**: real-time or advanced estimation

- **Smoothing**: delayed estimation, more accurate when $x(t)$ is a stochastic process
Smoothing for Quantum Sensing

\[ df = dt \mathcal{L}(x)f + \frac{dt}{8} \left( 2C^{\top}R^{-1}fC^{\dagger} - C^{\dagger}\mathcal{T}R^{-1}Cf - fC^{\dagger}\mathcal{T}R^{-1}C \right) + \frac{1}{2} dy^{\top}R^{-1}(Cf + fC^{\dagger}) \]

\[ -dg = dt \mathcal{L}^*(x)g + \frac{dt}{8} \left( 2C^{\dagger}\mathcal{T}R^{-1}gC - C^{\dagger}\mathcal{T}R^{-1}Cg - gC^{\dagger}\mathcal{T}R^{-1}C \right) + \frac{1}{2} dy^{\top}R^{-1}(C^{\dagger}g + gC) \]

\[ h(x, t) = P(x_t = x|Y_{\text{past}}, Y_{\text{future}}) = \frac{\text{tr} \left[ g(x, t)f(x, t) \right]}{\int dx \left( \text{numerator} \right)} \]

M. Tsang, PRL 102, 250403 (2009).
Experimental Demonstration


- Wheatley et al., PRL 104, 093601 (2010): very close to QCRB for coherent state:

- QCRB can be lowered by squeezing.
Squeezed-Light Phase Estimation

Yonezawa et al., Science 337, 1514 (2012)
Squeezed Light in Gravitational-Wave Detector

Example 2: Optomechanical Force Estimation

\[ \hat{H}_I = -\hat{q} f, \]

\[ \mathbb{E}[\delta f^2(t)] \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(4/\hbar^2)S_\Delta q(\omega) + 1/S_f(\omega)}. \]  

- Smoothing can't saturate QCRB due to the presence of **measurement backaction noise**
- **Standard Quantum Limit** (Braginsky, Caves et al.): backaction noise cannot be removed.
Quantum Noise Cancellation (QNC)

- Coherent Feedforward Quantum Control
- See also Kimble et al., PRD 65, 022002 (2001).
Noise Cancellation

Inside noise-canceling headphones

Sound waves created by headphone speaker
Noise created by external source

= Silence

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Quantum Non-Demolition (QND) Observables

Consider Heisenberg picture:

\[
\frac{dq(t)}{dt} = \frac{p(t)}{m}, \quad \frac{dp(t)}{dt} = -m\omega^2 q(t). \tag{68}
\]

Since \([q,p] \neq 0\),

\[
[q(t), q(t')] \neq 0 \text{ for } t \neq t'. \tag{69}
\]

\(q(t)\) and \(q(t')\) are incompatible observables. uncertainty principle (roughly) says measurement of one will disturb the other.

Pair the harmonic oscillator with another with negative mass:

\[
\frac{dq'(t)}{dt} = -\frac{p'(t)}{m}, \quad \frac{dp'(t)}{dt} = m\omega^2 q'(t), \tag{70}
\]

\[
\frac{d[q(t) + q'(t)]}{dt} = \frac{[p(t) - p'(t)]}{m}, \quad \frac{d[p(t) - p'(t)]}{dt} = -m\omega^2 [q(t) + q'(t)]. \tag{71}
\]

Since \([q + q', p - p'] = 0\),

\[
[q(t) + q'(t), q(t') + q'(t')] = 0, \quad [p(t) - p'(t), p(t') - p'(t')] = 0, \tag{72}
\]

\[
[q(t) + q'(t), p(t') - p'(t')] = 0. \tag{73}
\]

These observables that commute with each other in Heisenberg picture are called QND observables.
Quantum-Mechanics-Free Subsystems

- **Positive-mass oscillator** \( \{q, p\} \) and **negative-mass oscillator** \( \{q', p'\} \)

\[
Q = q + q', \quad P = \frac{p + p'}{2}
\]

- **Conjugate Pairs**

\[
\Phi = \frac{q - q'}{2}, \quad \Pi = p - p'
\]

- **Dynamical Pairs**

**QND observables**: commute with each other at different times in the Heisenberg picture,

\[
[X_j(t), X_k(t')] = 0.
\] (74)

No backaction noise if measurements are made at the times at which they commute.

- Measurements of QND observables are also called **backaction-evading measurements**.


- QND observables are equivalent to **classical stochastic processes** according to spectral theorem; i.e., they can be measured to any arbitrary accuracy.

- By choosing an appropriate Hamiltonian, it is possible to make a subsystem of observables **QND**

- **obey any linear/nonlinear classical dynamics**.


- QNC + Smoothing saturate QCRB for coherent state.
- Optical squeezing of input light can lower the QCRB.
Example 3: Magnetometry

- **Linear Gaussian smoother** for magnetometry: Petersen and Mølmer, PRA **74**, 043802 (2006).
- **QNC**:
  - Wasilewski et al., PRL **104**, 133601 (2010).
If $P(y|x)$ is not a Gaussian, CRB often grossly underestimate the achievable estimation error.

Consider $M$ repeated measurements, such that $P(y_1, \ldots, y_M|x) = \prod_{m=1}^{M} P(y_m|x)$.

◆ $\Sigma$ v.s. $M$ (log-log plot) for a classical phase estimation problem:

Van Trees and Bell, Bayesian Bounds for Parameter Estimation and Nonlinear Filtering/Tracking (Wiley, Hoboken, 2007)

Ziv-Zakai bound:

$$\mathbb{E}(\delta x^2) \geq \frac{1}{2} \int_{0}^{\infty} d\tau \int_{-\infty}^{\infty} dx 2 \min \left[ P_X(x), P_X(x+\tau) \right] P_e(x, x+\tau),$$

(75)

$P_X(x)$ is prior, $P_e(x, x+\tau)$ is the error probability of a binary hypothesis testing problem with $P(y|H_0) = P(y|x)$, $P(y|H_1) = P(y|x+\tau)$, and $P_0 = P_1 = 1/2$. 
Quantum Ziv-Zakai Bounds

- If \( P(y|x) = \text{tr} \left[ E(y) \rho_x \right] \), for any POVM \( E(y) \),

\[
P_e(x, x + \tau) \geq \frac{1}{2} \left[ 1 - \frac{1}{2} \| \rho_x - \rho_{x+\tau} \|_1 \right] \geq \frac{1}{2} \left[ 1 - \sqrt{1 - F(\rho_x, \rho_{x+\tau})} \right].
\]

(76)

We immediately obtain quantum Ziv-Zakai bounds on \( \mathbb{E}(\delta x^2) \).


- Can be used to prove Heisenberg limit:

\[
\mathbb{E}(\delta x^2) \geq \frac{C}{\langle n \rangle^2}.
\]

(77)

see also Giovannetti et al., PRL 108, 260405 (2012); Hall et al., PRA 85, 041802(R) (2012).

- Compared with QCRB \( \mathbb{E}(\delta x^2) \geq 1/4 \langle \Delta n^2 \rangle \), Heisenberg limit is much higher than QCRB when \( \langle \Delta n^2 \rangle \gg \langle n \rangle^2 \).

Hybrid Quantum Systems

Kippenberg and Vahala, Science 321, 1172 (2008)

Choi et al., Nature 468, 412 (2010)


Sayrin et al., Nature 477, 73 (2011)

Haroche and Raimond, Exploring the Quantum


Neeley et al., Nature 467, 570 (2010)
Electro-Optic Modulation

- \[ \epsilon = \epsilon_0 \left( 1 + \chi^{(1)} + \chi^{(2)} E + \chi^{(3)} : E E + \ldots \right) \]
- \[ \chi^{(2)} \text{ (Pockels): } \Delta \phi (V) \propto V \text{: e.g., Lithium Niobate (LiNbO}_3\text{)} \]
- Optical:
  - transparent between 350nm-5\text{\(\mu\)m}
  - intrinsic \( Q \sim 10^6 \) resonator at 1.55\text{\(\mu\)m} [Ilchenko et al., JOSAB 20, 333 (2003)]
  - 10dB squeezing [Vahlbruch et al. PRL 100, 033602 (2008)]

- Microwave:
  - intrinsic \( \epsilon_l \approx 28, \epsilon_t \approx 45, Q \approx 2.3 \times 10^3 \) at 9GHz, 300K [Bourreau et al., EL 22, 399 (1986)], loss should decrease with temp.
  - Cu half-wave resonator: \( Q \approx 100 \) at 9GHz, 300K [Ilchenko et al.]
  - 26.5GHz EOM with Nb electrode on LiNbO\text{\(3\)} at 4.2K [Yoshida et al., IEEE TMMT 47, 1201 (1999)]
Three-Wave Mixing

$$\omega_{\text{micro}} + \omega_{\text{opt}} = \Omega_{\text{opt}}$$ (78)

- Up-conversion
- Parametric amplification
- Down-conversion
- Spontaneous parametric down conversion

Spatial mode matching
Resonant Enhancement

- How to enhance desired processes and suppress parasitic ones?

Similar with microwave resonator


Analogy with Cavity Optomechanics

Optical/microwave photons ↔ microwave/RF phonons:

Eichenfield et al., Nature 462, 78 (2009)


Analogy with Cavity Optomechanics

Optical Photons $\leftrightarrow$ Microwave/RF Photons:

\[
\hat{H}_I \propto \phi(\hat{V}) \hat{a}^\dagger \hat{a}
\]
\[
\hat{V} \propto \hat{b} + \hat{b}^\dagger
\]

Optical Photons $\leftrightarrow$ Microwave/RF Phonons:

\[
\hat{H}_I \propto \phi(\hat{q}) \hat{a}^\dagger \hat{a}
\]
\[
\hat{q} \propto \hat{b} + \hat{b}^\dagger
\]

(79)

(80)
Interaction picture:

\[
H_I = \hbar g \left( b e^{-i \omega_b t} + b^{\dagger} e^{i \omega_b t} \right) \left[ \alpha e^{i(\omega_a - \omega_b) t} + a^{\dagger} e^{i \omega_a t} \right] \left[ \alpha e^{-i(\omega_a - \omega_b) t} + a e^{-i \omega_a t} \right]
\] (81)

Rotating-wave approximation:

\[
H_I \approx \hbar g \alpha \left( b a^{\dagger} + b^{\dagger} a \right)
\] (82)

Exchange photons:

Wilson-Rae et al., PRL 99, 093901 (2007); Marquardt et al., 093902 (2007).
Recent Experiments of Optomechanical Cooling

- Chan et al., Nature 478, 89 (2011)

- Rocheleau et al., Nature 463, 72 (2010)
\[ H_I \approx g \sqrt{N_{\text{pump}}} \left( a^\dagger b + ab^\dagger \right) \]  \hspace{1cm} (83)

\[ g = \eta \frac{\omega_a n^3 r}{2d} \sqrt{\frac{\hbar \omega_b}{2C}}, \]  \hspace{1cm} (84)

\[ G \equiv \frac{g^2 N_{\text{pump}}}{\gamma_a \gamma_b} \]  \hspace{1cm} (85)

Cooling : \( G \gg 1 \)  \hspace{1cm} (86)

Conversion : \( G = 1 \)  \hspace{1cm} (87)
\[
\frac{da}{dt} = ig\alpha b - \frac{\Gamma_a}{2} a + \sqrt{\gamma_a} A_{in} + \sqrt{\gamma'_a} A', \quad (88)
\]
\[
\frac{db}{dt} = ig\alpha^* a - \frac{\Gamma_b}{2} b + \sqrt{\gamma_b} B_{in} + \sqrt{\gamma'_b} B', \quad (89)
\]
\[
A_{out} = \sqrt{\gamma_a} a - A_{in}, \quad B_{out} = \sqrt{\gamma_b} b - B_{in}, \quad (90)
\]
\[
\Gamma_{a,b} = \gamma_{a,b} + \gamma'_{a,b}, \quad G_0 \equiv \frac{g^2 N_{pump}}{\Gamma_a \Gamma_b}, \quad \eta \equiv \frac{\gamma_a \gamma_b}{\Gamma_a \Gamma_b}. \quad (91)
\]
Plots

\[
\frac{\Gamma_a}{2} \quad \frac{\Gamma_b}{2}
\]

\[
\text{Res} \quad \text{Im}s
\]

\[
R(0)/\eta
\]

\[
4G_0/(1+G_0)^2
\]

M. Tsang, PRA 84, 043845 (2011)
Photon-pair creation/annihilation:

\[ H_I \approx g \sqrt{N_{\text{pump}}} \left( a^\dagger b^\dagger + ab \right) , \]

\[ G \equiv \frac{g^2 N_{\text{pump}}}{\gamma_a \gamma_b} \]

Oscillation: \( G \geq 1 \),

Entangled Photon Pairs: \( G \ll 1 \)
Parametric Amplification/Oscillation

\[
\frac{da}{dt} = ig_\alpha b^+ - \frac{\Gamma_a}{2} a + \sqrt{\gamma_a} A_{\text{in}} + \sqrt{\gamma'_a} A',
\]
\[
\frac{db}{dt} = ig_\alpha a^+ - \frac{\Gamma_b}{2} b + \sqrt{\gamma_b} B_{\text{in}} + \sqrt{\gamma'_b} B',
\]
\[
A_{\text{out}} = \sqrt{\gamma_a} a - A_{\text{in}},
\]
\[
B_{\text{out}} = \sqrt{\gamma_b} b - B_{\text{in}}.
\]
\[ R(0)/\eta \ (\text{dB}) \]

\[ 10\log_{10}\frac{4G_0}{(1-G_0)^2} \]

\[ G_0 \]

\[ -\frac{\Gamma_a}{2}, -\frac{\Gamma_b}{2} \]

\[ \text{Im} s \]

\[ \text{Re} s \]
Rokhsari et al., Opt. Express 13, 5293 (2005):

\[
H_I \approx g \sqrt{N_{\text{pump}}} \left( e^{-i\bar{\theta}} a + e^{i\bar{\theta}} a^\dagger \right) \left( e^{-i\delta} b + e^{i\delta} b^\dagger \right),
\]

(99)

\[
\bar{\theta} \equiv \frac{\theta_+ + \theta_-}{2}, \quad \delta \equiv \frac{\theta_+ - \theta_-}{2}.
\]

(100)

- \( \theta_\pm \) are phases of the pump beams and control which quadratures are coupled.
- \( \chi^{(3)} \) (Kerr): \( \phi(V) \propto \chi^{(3)} V^2 \), backaction-evading microwave energy measurement:

\[
H_I \propto \chi^{(3)} V^2 a^\dagger a.
\]

(101)
Coupling Strength

\[ G = \frac{g^2 N_{\text{pump}}}{\gamma_a \gamma_b}, \quad g = \eta \frac{\omega_a n^3 r}{2d} \sqrt{\frac{\hbar \omega_b}{2C}}. \]  

(102)

- Ilchenko et al., JOSAB 20, 333 (2003) \((\gamma_a \approx 2\pi \times 90 \text{ MHz}, \gamma_b \approx 2\pi \times 50 \text{ MHz}, d \approx 150 \mu\text{m})\):

|Fig. 1. Experimental setup: (1) LiNbO\(_3\) optical cavity, (2) microwave resonator, (3) microwave feeding strip line, and (4) diamond coupling prism. Inset: geometric characteristics of the nonlinear optical cavity.|

\[ g \approx 20 \text{ Hz}, \quad G \approx 2 \times 10^{-5} \text{ at 2 mW pump} \]  

(103)

- \(g\) can be improved by \(\sim 10^1 - 10^2\), \(\gamma_b\) reduced by \(\sim 10^3\) using superconducting microwave resonator
- \(r\) in BaTiO\(_3\) and KTN is higher than LiNbO\(_3\) by \(10^1 - 10^2\)
## Competition: Electro-optomechanics, Atoms

M. Tsang, PRA 81, 063837 (2010); 84, 043845 (2011).

Regal and Lehnert, J. Phys.: Conf. Series 264, 012025 (2011);

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<td>Experiment</td>
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<td>N/A</td>
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Hoffman et al., arXiv:1108.4153 (2011); Hafezi et al., PRA 85, 020302(R) (2012)
Waveform Detection

Waveform Estimation

Parameter Estimation Beyond CRB

Quantum Electro-optics
- M. Tsang, PRA 81, 063837 (2010); 84, 043845 (2011).

Open-System Quantum Metrology

Imaging
Collaborations

- **Cavity electro-optics**: with Aaron Danner at National U. Singapore
- **Parameter estimation for optomechanical force sensing**: with Warwick Bowen at U. Queensland
- **Time-varying optical phase estimation**: with Hidehiro Yonezawa at U. Tokyo, Elanor Huntington et al. at U. New South Wales
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