Seize the Moments for Subdiffraction Incoherent Imaging

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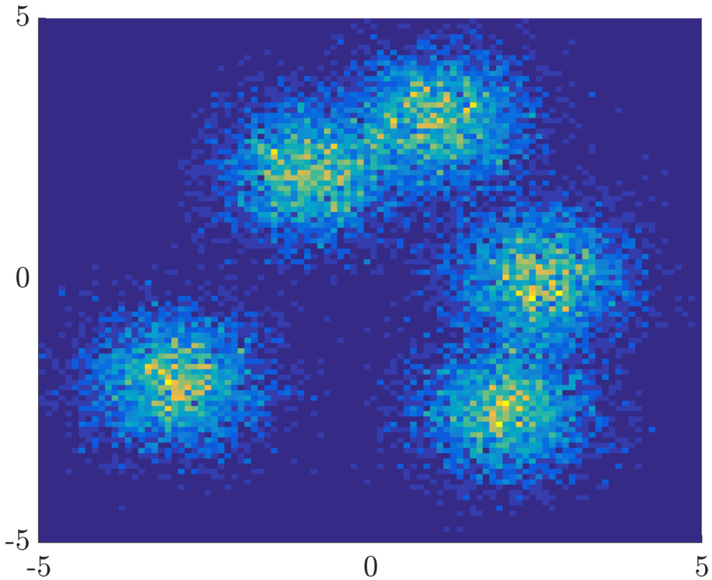
Paris, July 2018

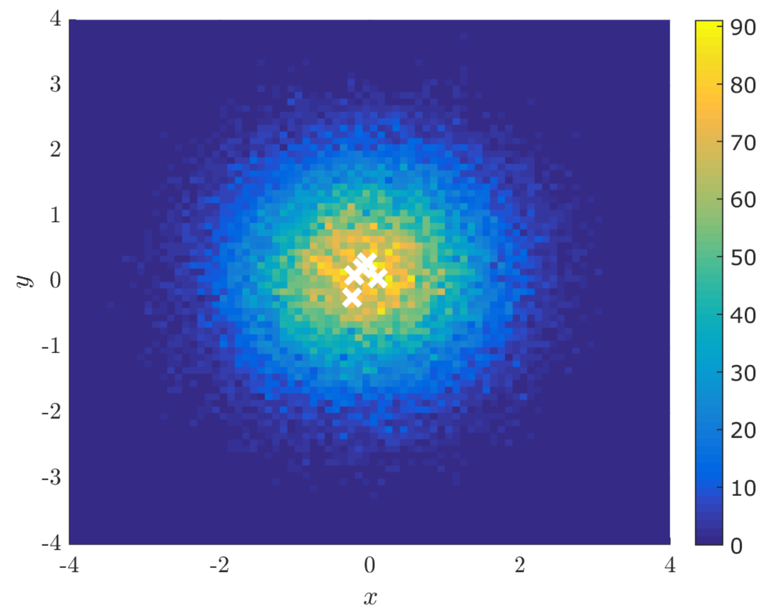
Imaging of Spatially Incoherent Sources

- Fundamental problems: **Diffraction Limit** and **Photon Shot Noise**
- Image processing?

Sparse (Good Regime, PALM, STED, compressed sensing, etc.)

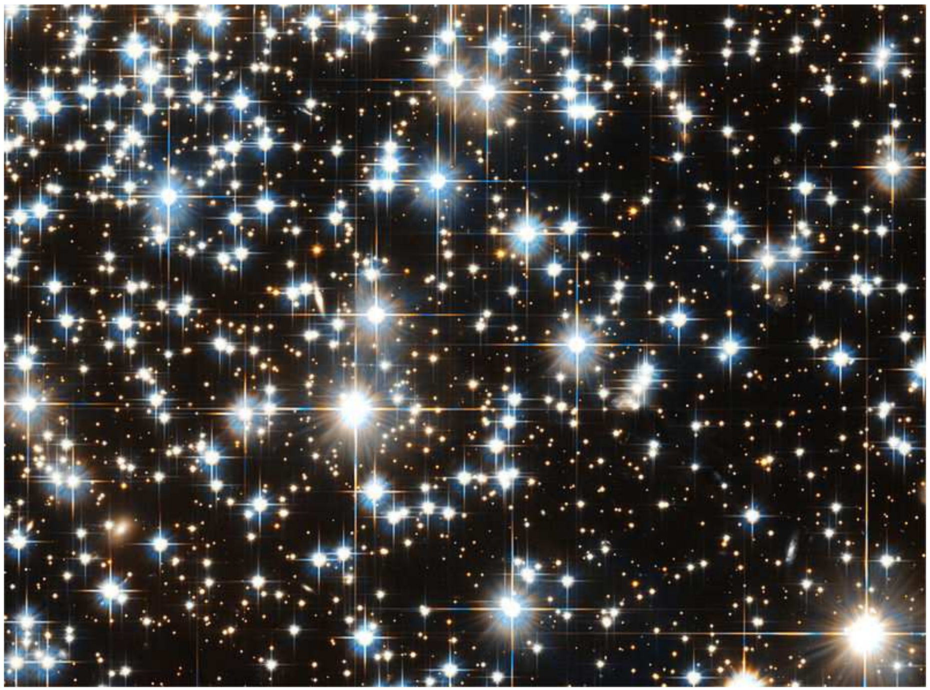
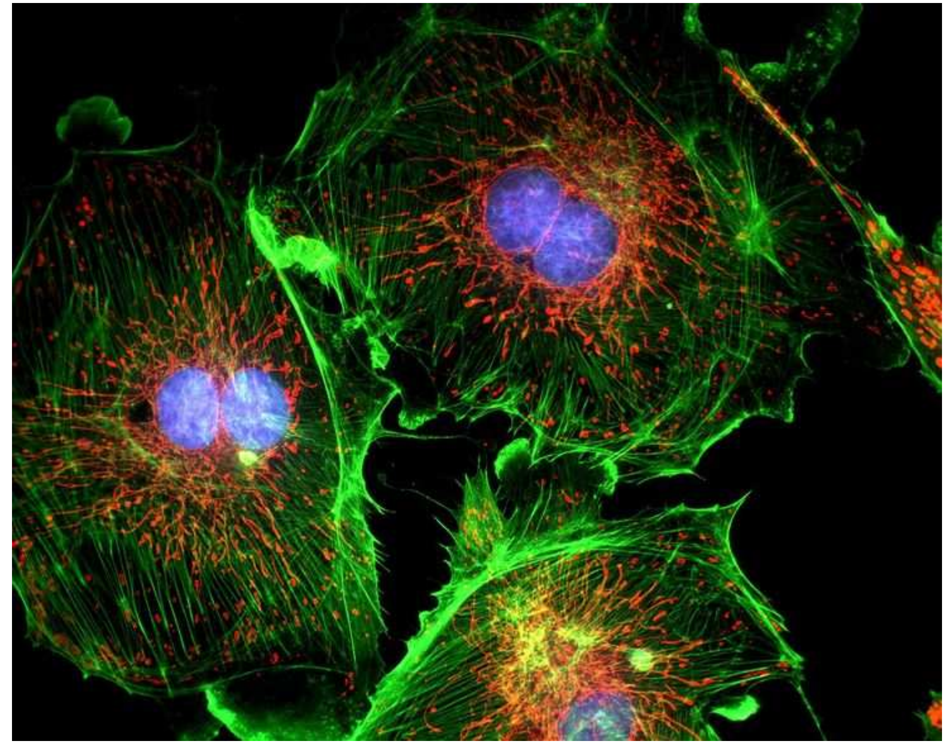


Subdiffraction (Worst Nightmare)



- Limits to image processing (**classical Cramér-Rao bound**)
- Enhancements by **better measurements in the far-field**
- Discovered via **quantum metrology**

Telescopes, Fluorescence Microscopy

(images stolen from the internet)

Publications

Selected for a Viewpoint in *Physics*
PHYSICAL REVIEW X **6,** 031033 (2016)

Quantum Theory of Superresolution for Two Incoherent Optical Point Sources

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(Received 8 November 2015; revised manuscript received 4 January 2016; published 29 August 2016)

PRL **117,** 190801 (2016) PHYSICAL REVIEW LETTERS week ending 4 NOVEMBER 2016

Far-Field Superresolution of Thermal Electromagnetic Sources at the Quantum Limit

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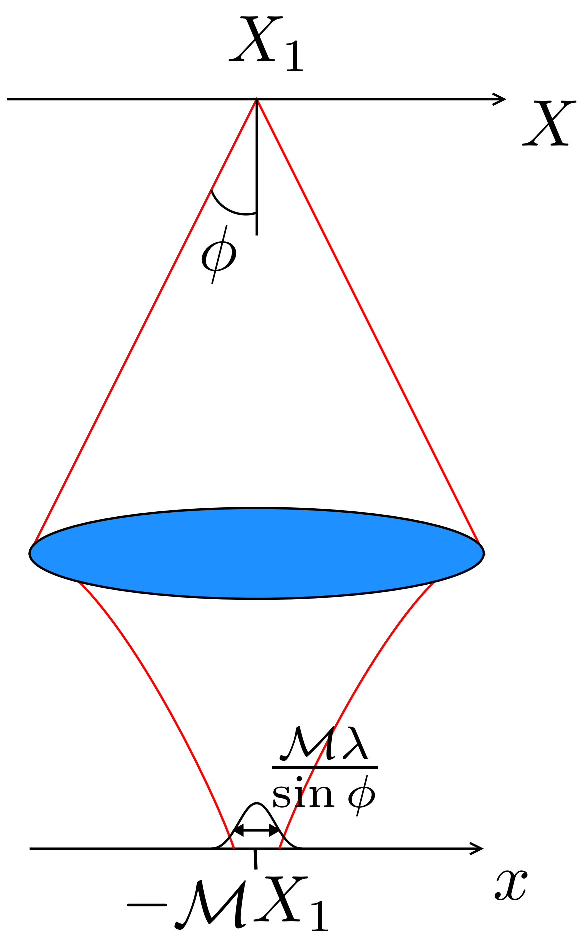
[2]*Department of Physics, National University of Singapore, 2 Science Drive 3, 117551 Singapore*
(Received 6 July 2016; published 4 November 2016)

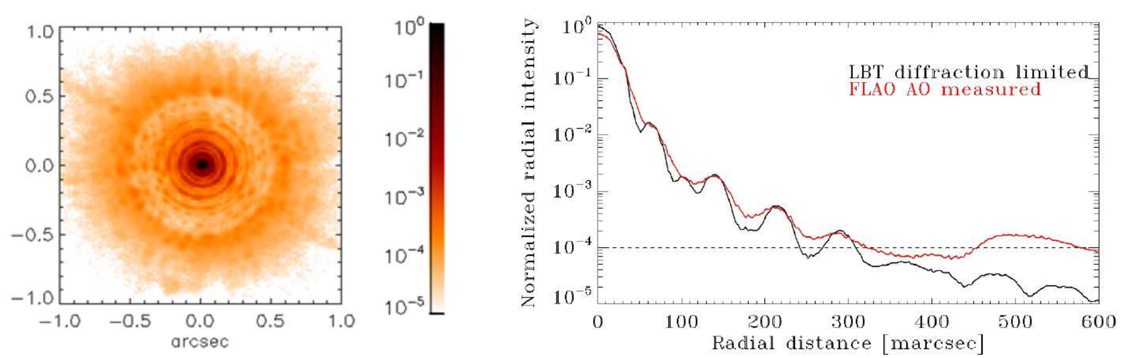
1. Tsang, Nair, Lu, PRX 6, 031033 (2016).
2. Nair, Tsang, OE 24, 3684 (2016).
3. Tsang, Nair, Lu, SPIE 10029, 1002903 (2016).
4. Nair, Tsang, PRL 117, 190801 (2016).
5. Ang, Nair, Tsang, PRA 95, 063847 (2017).
6. Tsang, NJP 19, 023054 (2017).
7. Yang, Nair, Tsang, Simon, Lvovsky, PRA 96, 063829 (2017).
8. Tsang, JMO 65, 104 (2018).
9. Tsang, PRA 97, 023830 (2018).
10. Lu, Krovi, Nair, Guha, Shapiro, arXiv:1802.02300 (2018).
11. Tsang, arXiv:1806.02781 (2018).

Diffraction Limit



Diffraction-limited

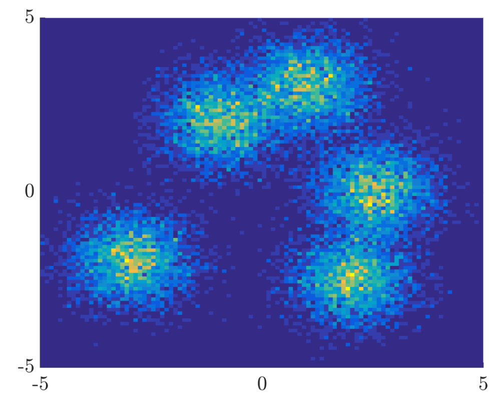
1. Fluorescence microscopy
2. Space telescopes (Webb, \$10 billion)
3. Ground-based telescopes corrected by **adaptive optics**:
   (a) Large Binocular Telescope (LBT) (Strehl ratio $> 80\%$, \$120 million)
   (b) Giant Magellan Telescope (GMT)
   (c) Thirty Meter Telescope (TMT)
   (d) European Extremely Large Telescope (E-ELT) ($>$\$1 billion each)

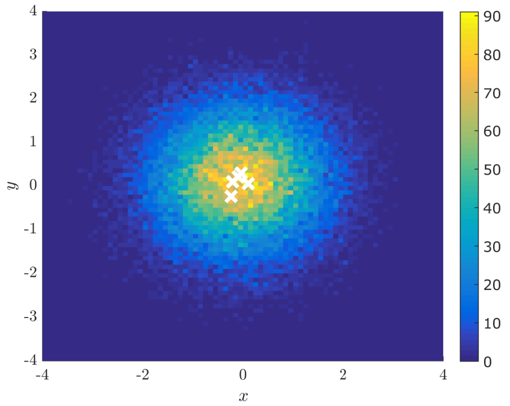


Esposito *et al.*, SPIE **8149**, 814902 (2011).

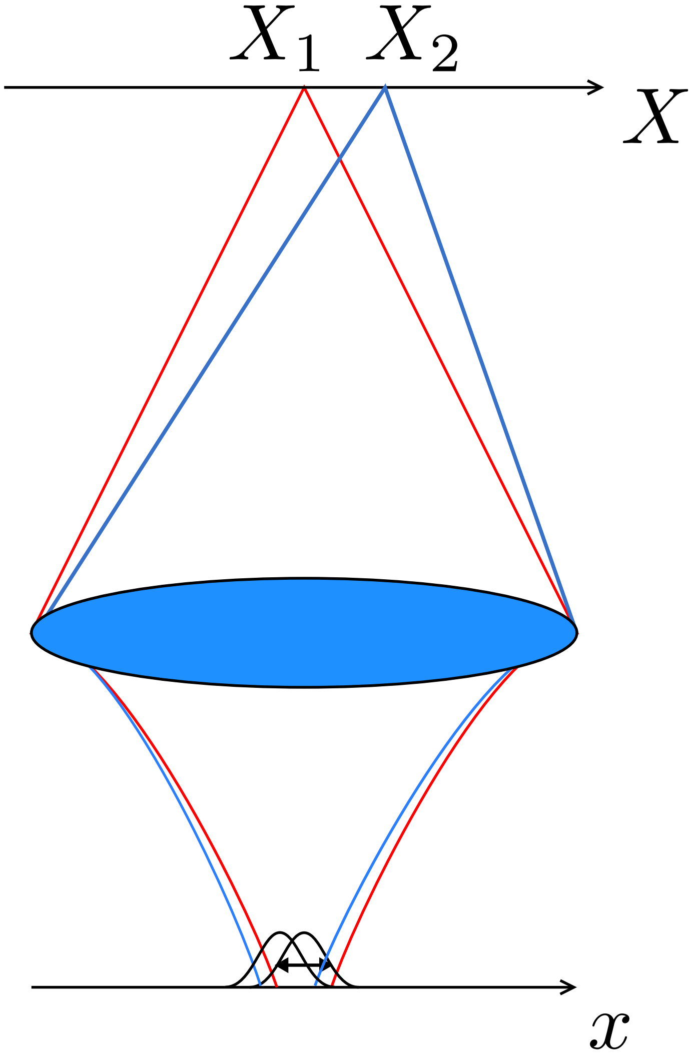
Photon Shot Noise

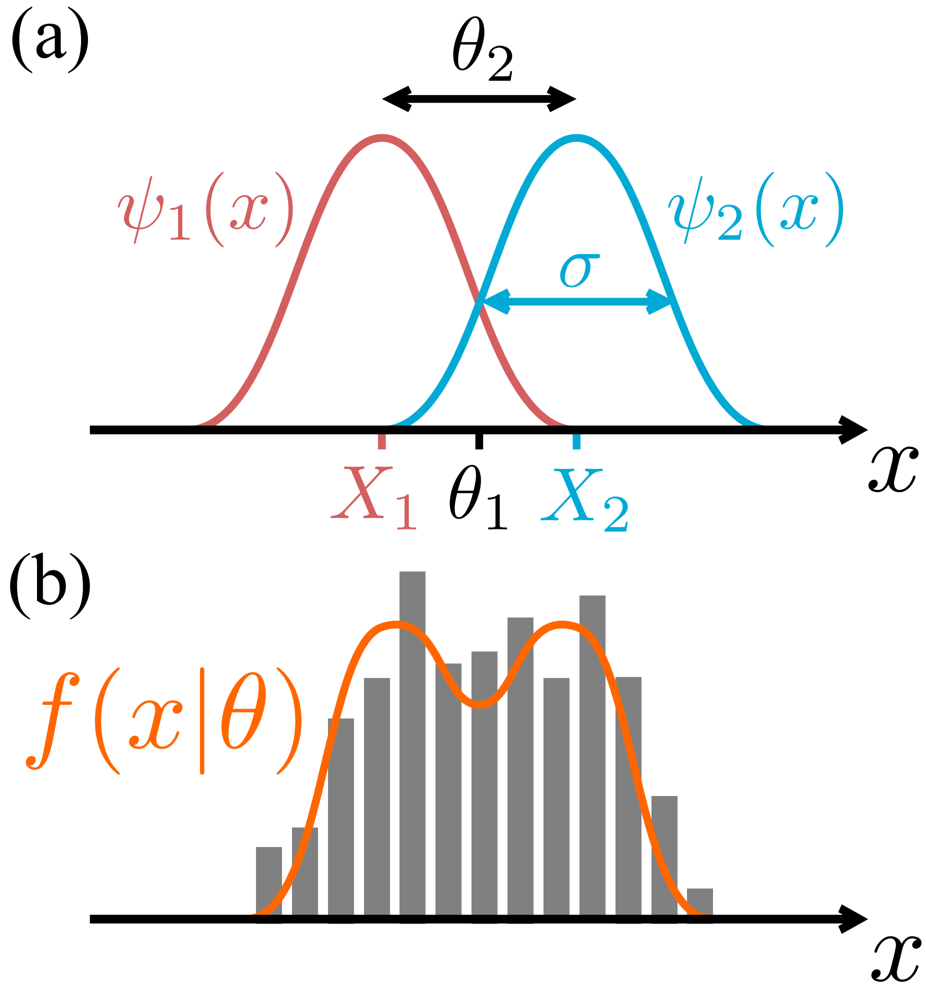
- ☐ Superresolution via image processing (e.g., SVD, basis pursuit) is possible but **extremely sensitive to noise**
- ☐ EMCCD: efficiency $> 90\%$, extremely low dark counts
- ☐ Thermal sources (stars, etc.)
  - **Poisson**, bunching negligible at optical
  - Goodman, *Statistical Optics*; Zmuidzinas, JOSA A **20**, 218 (2003)
- ☐ Fluorophores (GFP, dye molecules, quantum dots, etc.)
  - **Poisson**, negligible anti-bunching
  - Pawley *ed.*, *Handbook of Biological Confocal Microscopy*; Ram, Ober, Ward, PNAS (2006)





Two Point Sources



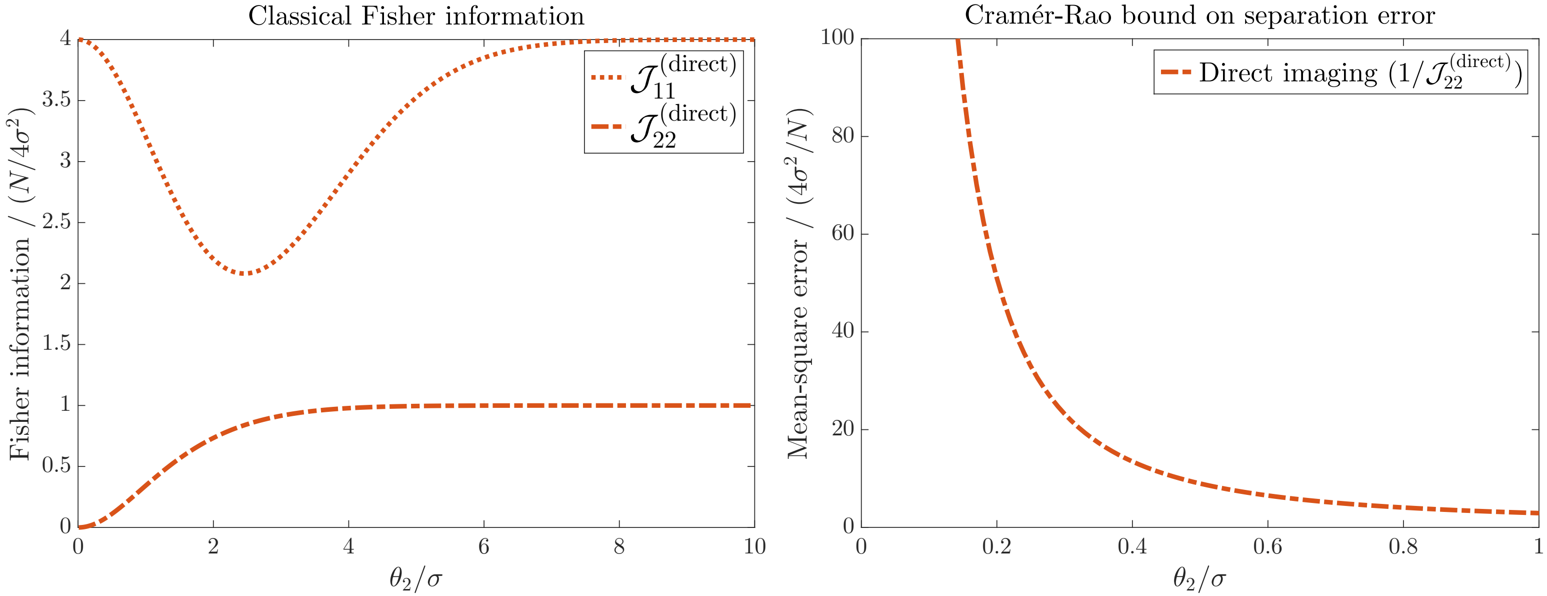


Cramér-Rao Bound for Direct Imaging

- Cramér-Rao bound:

$$\text{MSE} \geq J^{-1}, \qquad J_{\mu\nu} = N \int_{-\infty}^{\infty} dx \frac{1}{f(x|\theta)} \frac{\partial f(x|\theta)}{\partial \theta_\mu} \frac{\partial f(x|\theta)}{\partial \theta_\nu}. \tag{1}$$

- J is diagonal for centroid and **separation** of the two sources.



- **Rayleigh's curse**
- Tsai and Dunn, Lincoln Lab. Tech. Note AD-A073462 (1979); Bettens *et al.*, Ultramicroscopy **77**, 37 (1999); Van Aert *et al.*, J. Struct. Biol. **138**, 21 (2002); Ram, Ward, Ober, PNAS **103**, 4457 (2006).

Quantum Cramér-Rao Bound

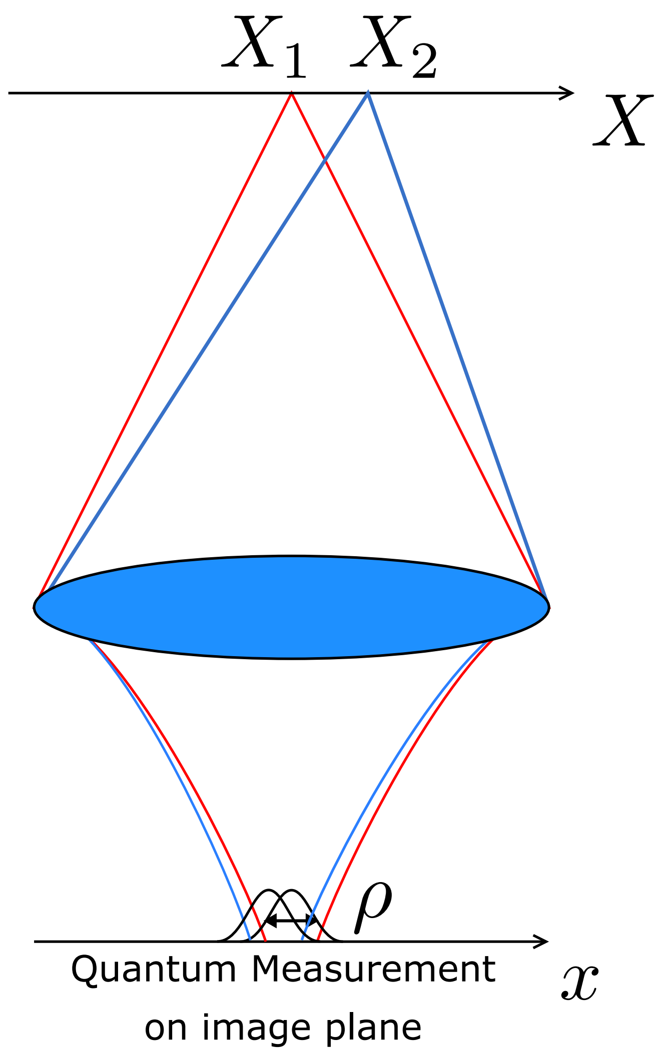
- Quantum: Direct imaging is just one of the **infinitely many possible measurements.**
- Helstrom (1967), Nagaoka (1989), Braunstein & Caves (1994): For any measurement,

$$\mathsf{MSE}_\mu \geq \left(J^{-1}\right)_{\mu\mu} \boxed{\geq \left(K^{-1}\right)_{\mu\mu}\,,} \tag{2}$$

$$K_{\mu\nu} = M\,\mathrm{Re}\left(\mathrm{tr}\, L_\mu L_\nu \rho\right), \tag{3}$$

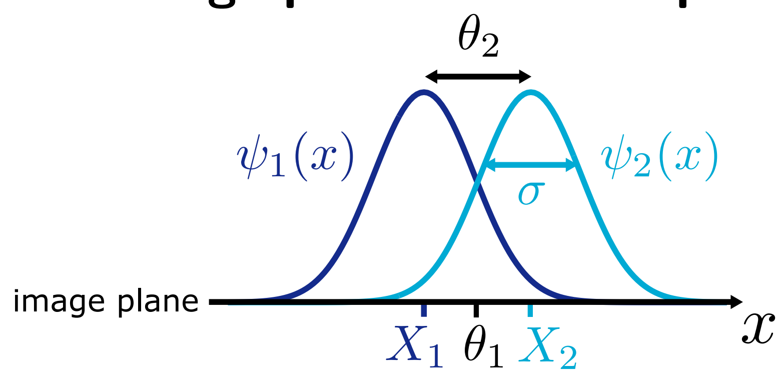
$$\frac{\partial \rho}{\partial \theta_\mu} = \frac{1}{2}\left(L_\mu \rho + \rho L_\mu\right). \tag{4}$$

- $K(\rho)$ is the quantum Fisher information, **the ultimate amount of information in the photons.**



Quantum Optics

- Thermal optical source: **average photon number per mode** $\epsilon \ll 1$



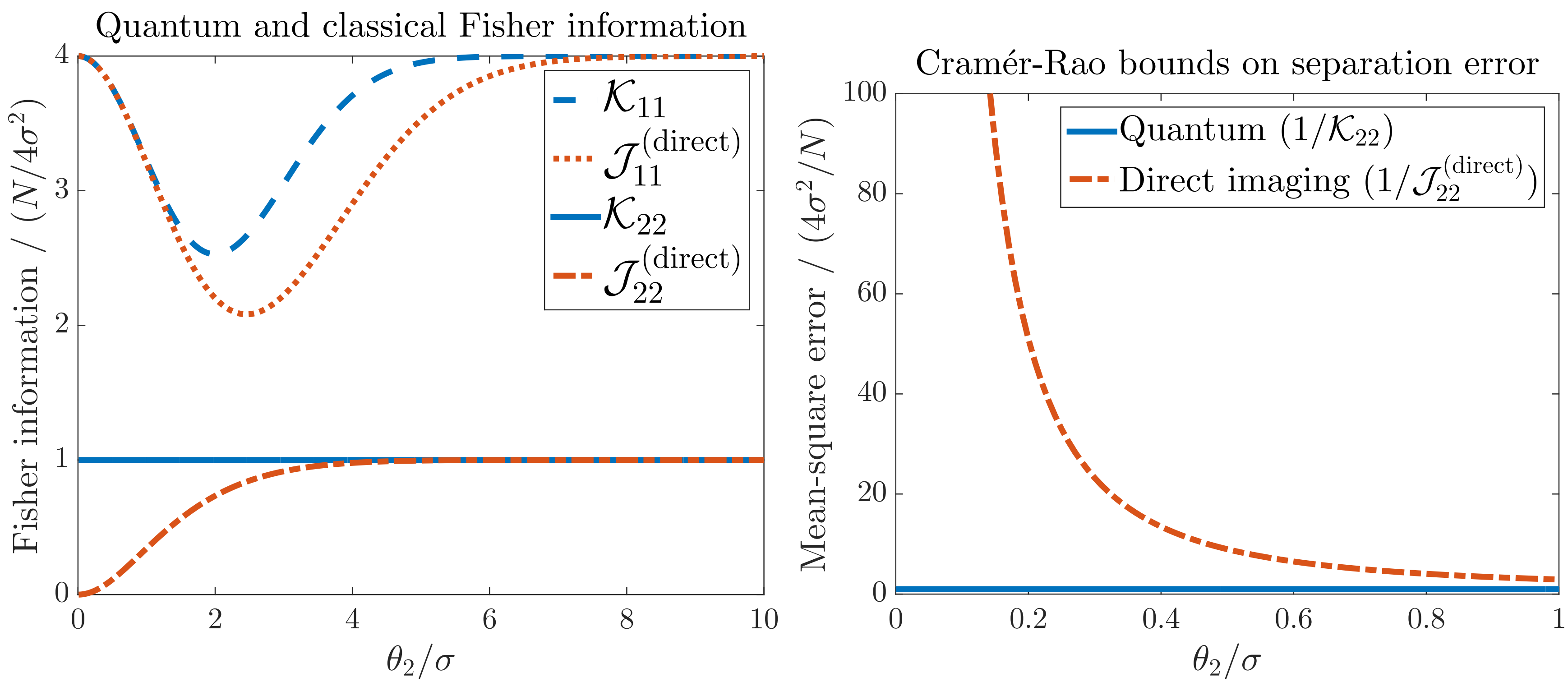
- Quantum state in M temporal modes $= \rho^{\otimes M}$,

$$\boxed{\rho = (1-\epsilon)\,|\text{vac}\rangle\langle\text{vac}| + \frac{\epsilon}{2}\left(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|\right) + O(\epsilon^2),}$$

$$\boxed{|\psi_s\rangle \equiv \int_{-\infty}^{\infty} dx \psi(x - X_s)\,|x\rangle\,.}$$

- derived from zero-mean Gaussian P function, mutual coherence, paraxial
- Multiphoton coincidence: **rare in practice**, little info as $\epsilon \ll 1$ (**homeopathy**)
- Hanbury Brown-Twiss: **obsolete for decades** (poor SNR)
- **Agrees with Poisson model**
- Similar model in Gottesman, Jennewein, Croke, PRL **109**, 070503 (2012); Tsang, PRL **107**, 270402 (2011).

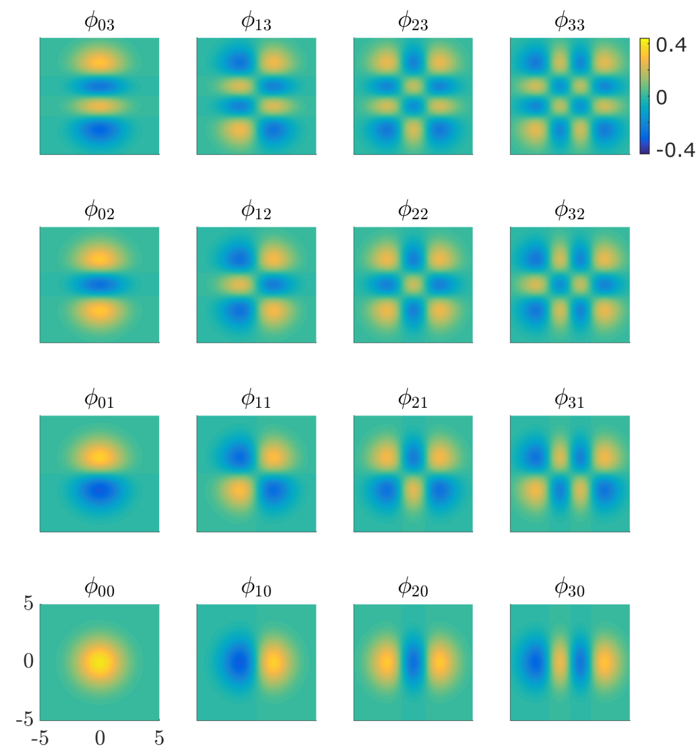
Plenty of Room at the Bottom



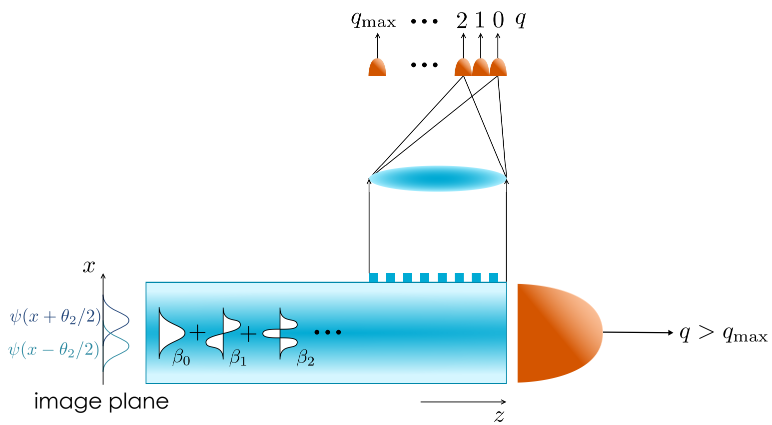
- Tsang, Nair, Lu, PRX **6**, 031033 (2016).
- **arbitrary** ϵ: Nair and Tsang, PRL **117**, 190801 (2016); Lupo and Pirandola (York), *ibid.* **117**, 190802 (2016); Editors' Suggestions.

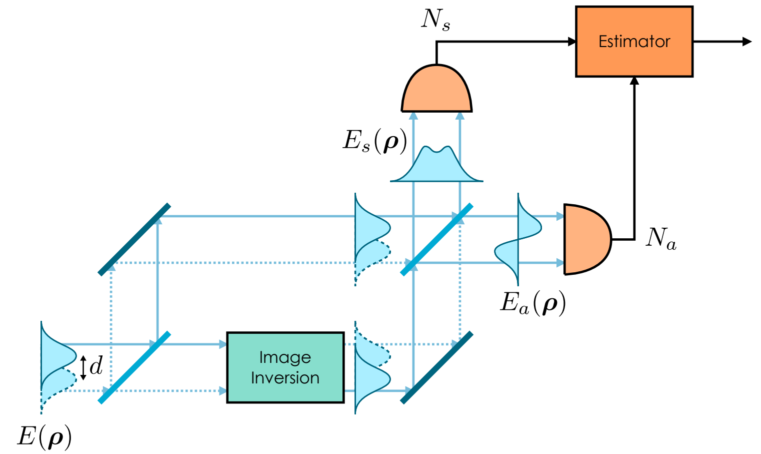
Optimal Measurement

- Measurement in **Hermite-Gaussian (TEM)** basis is optimal in Fisher information.
  - Gaussian PSF: Tsang, Nair, Lu, PRX (2016).
  - any even PSF: Rehacek *et al.*, OL **42**, 231 (2017).
- Finite N [Tsang, JMO **65**, 104 (2018)]:
  - proved MSE $\leq 4 \times$ QCRB
  - Simulations: $< 2 \times$ QCRB
  - Similar results with Bayesian/minimax

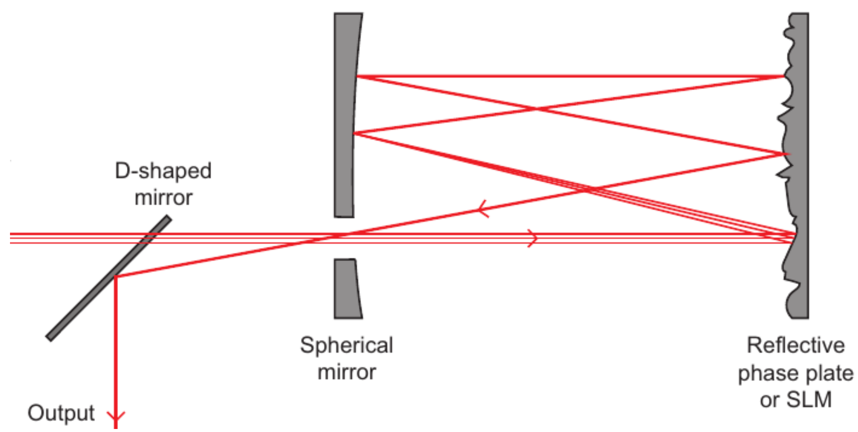


Spatial-Mode Demultiplexing (SPADE)



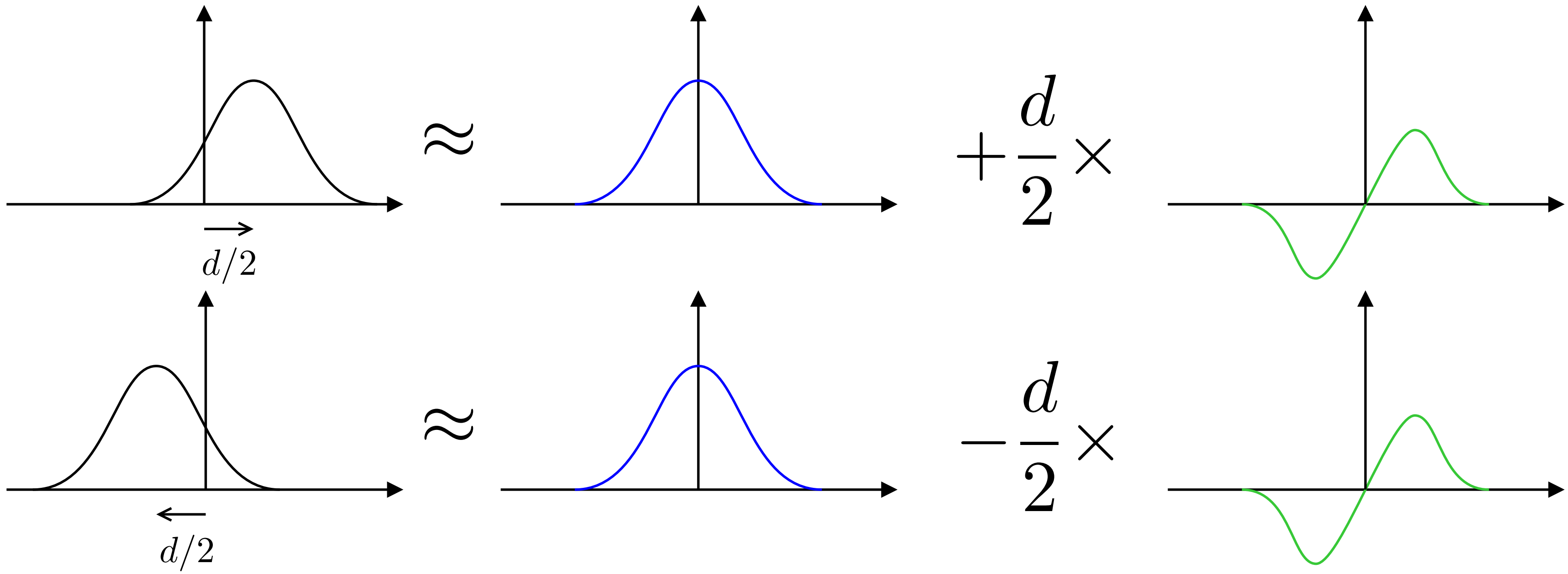


- Nair and Tsang, OE **24**, 3684 (2016): SuperLocalization via Image-inVERsion interferometry (SLIVER)
- Many other ways (in optical comm., photonic circuits, etc.)
- Labroille *et al.* (CAILabs/Bell/LKB), OE **22**, 15599 (2014):



- **Classical sources**, **Far-field linear optics/photon counting**, **Important applications**

Elementary Explanation

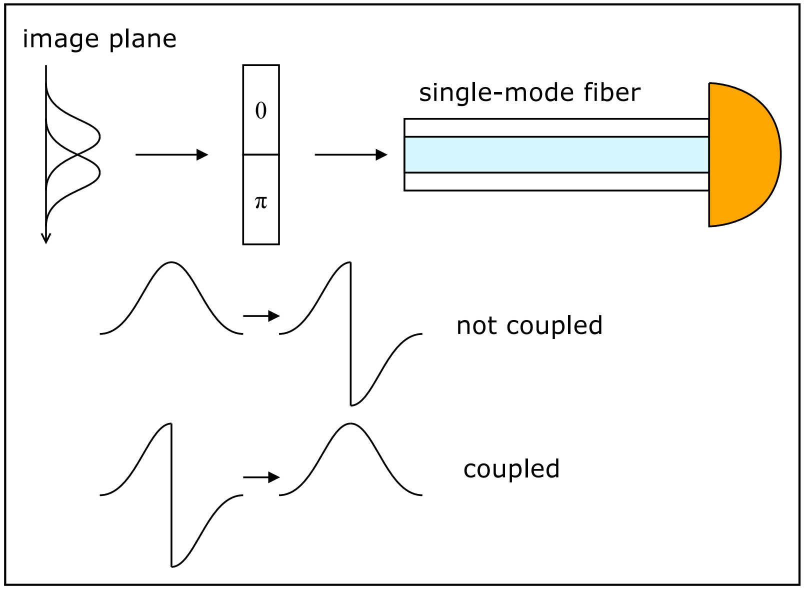


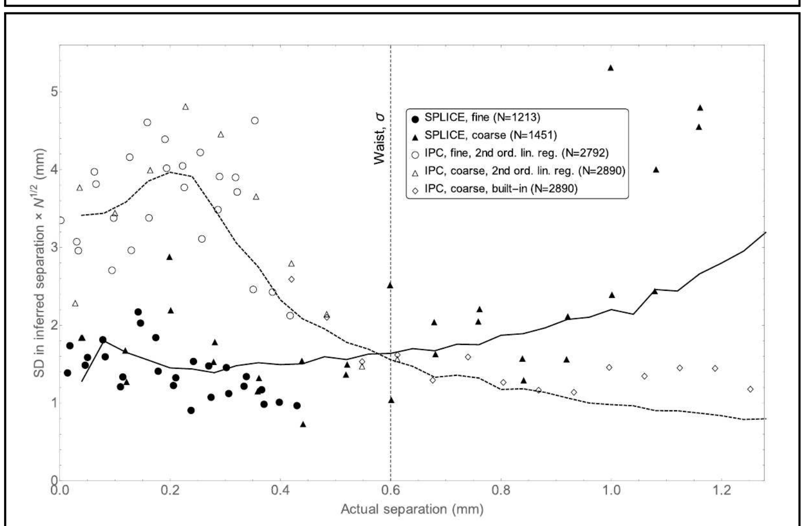
- **Incoherent sources**: energy in **1st-order mode** is

$$\propto \left(\frac{d}{2}\right)^2 + \left(-\frac{d}{2}\right)^2 = \frac{d^2}{2}. \tag{5}$$

- **0th-order mode** is just **background noise**; filtering it improves SNR.
- **Coherent noise filtering**

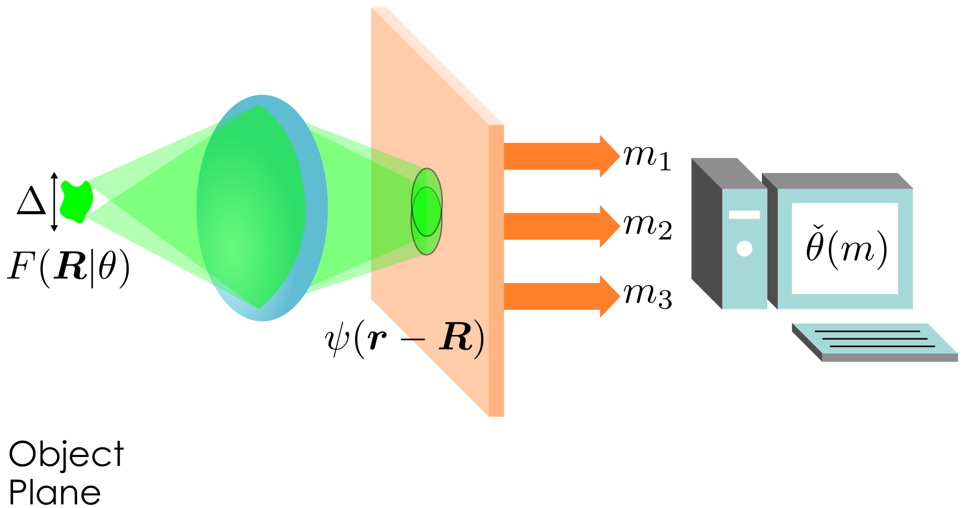
Experimental Demonstrations

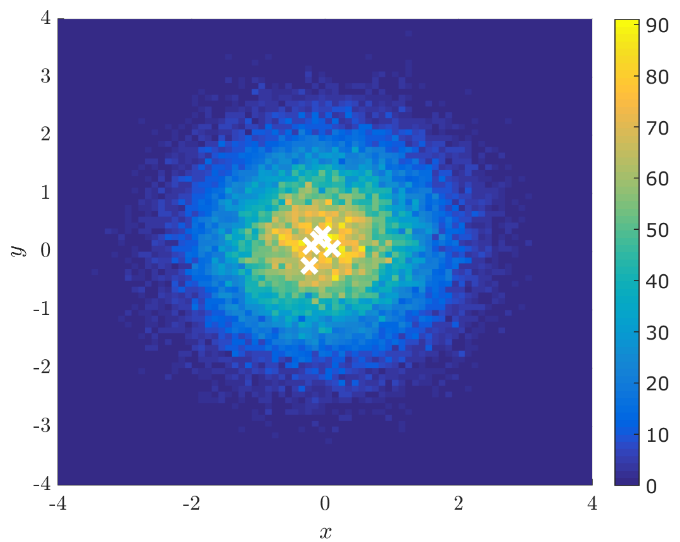




1. Tham, Ferretti, Steinberg (Toronto), PRL **118**, 070801 (2017).
   - $\sim 2 \times$ QCRB
2. Tang, Durak, Ling (CQT Singapore), OE **24**, 22004 (2016).
3. Yang, Taschilina, Moiseev, Simon, Lvovsky (Calgary), Optica **3**, 1148 (2016).
4. Paúr, Stoklasa, Hradil, Sánchez-Soto, Rehacek (Palacky/Madrid/Max Planck), Optica **3**, 1144 (2016).
5. Parniak *et al.* (Warsaw), arXiv:1803.07096 (2018).
6. Donohue *et al.* (Paderborn), arXiv:1805.2491 (2018).

Arbitrary Source Distribution



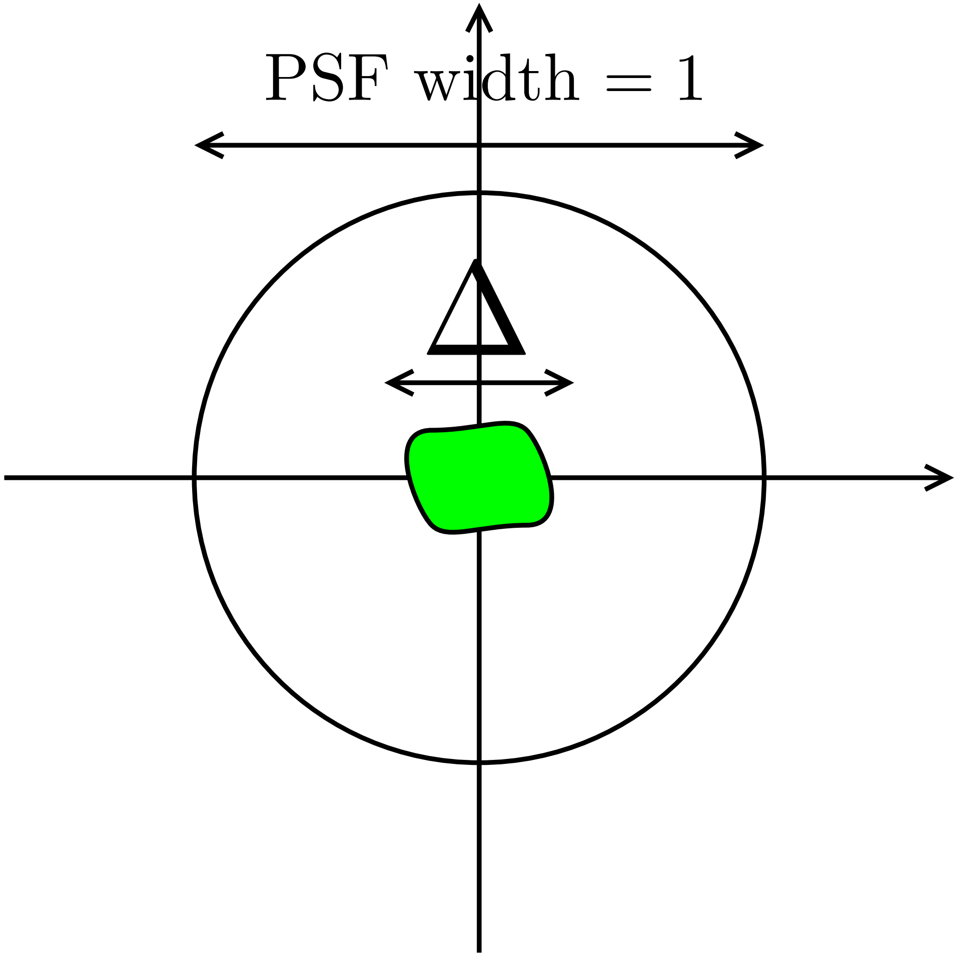


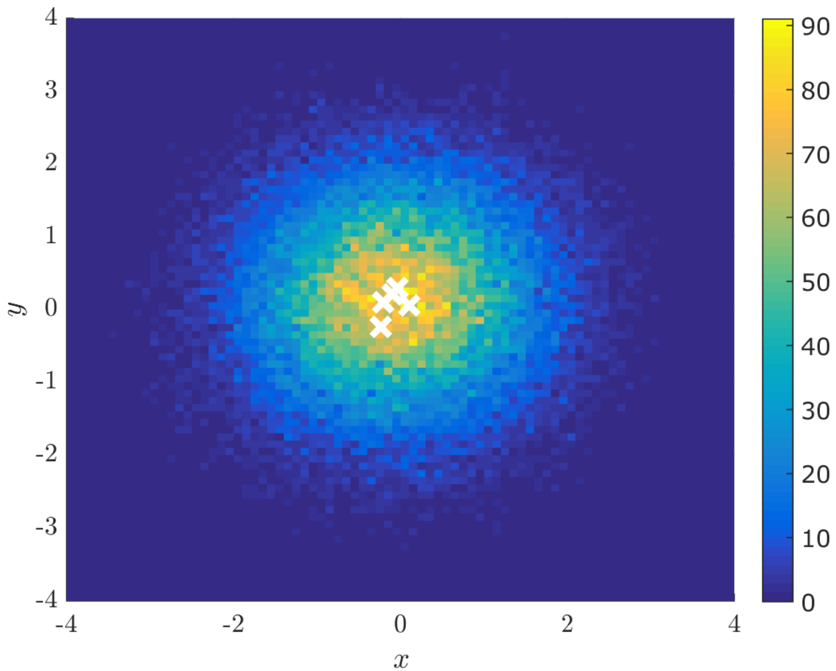
- Direct imaging of arbitrary sources with distribution $= F(X|\theta)$:

$$\boxed{f(x|\theta) = \int dX |\psi(x-X)|^2 F(X|\theta),} \qquad \boxed{J_{\mu\nu} = \int_{-\infty}^{\infty} dx \frac{1}{f} \frac{\partial f}{\partial \theta_\mu} \frac{\partial f}{\partial \theta_\nu}.} \tag{6}$$

- Parameterization?
- Simple analytic J?

Defining Subdiffraction Regime





$$\boxed{\text{object support width} \equiv \Delta \ll 1.} \tag{7}$$

Cramér-Rao Bound for Direct Imaging

- **Define parameters as object moments**:

$$\boxed{\theta_\mu = \int dX X^\mu F(X|\theta)} \tag{8}$$

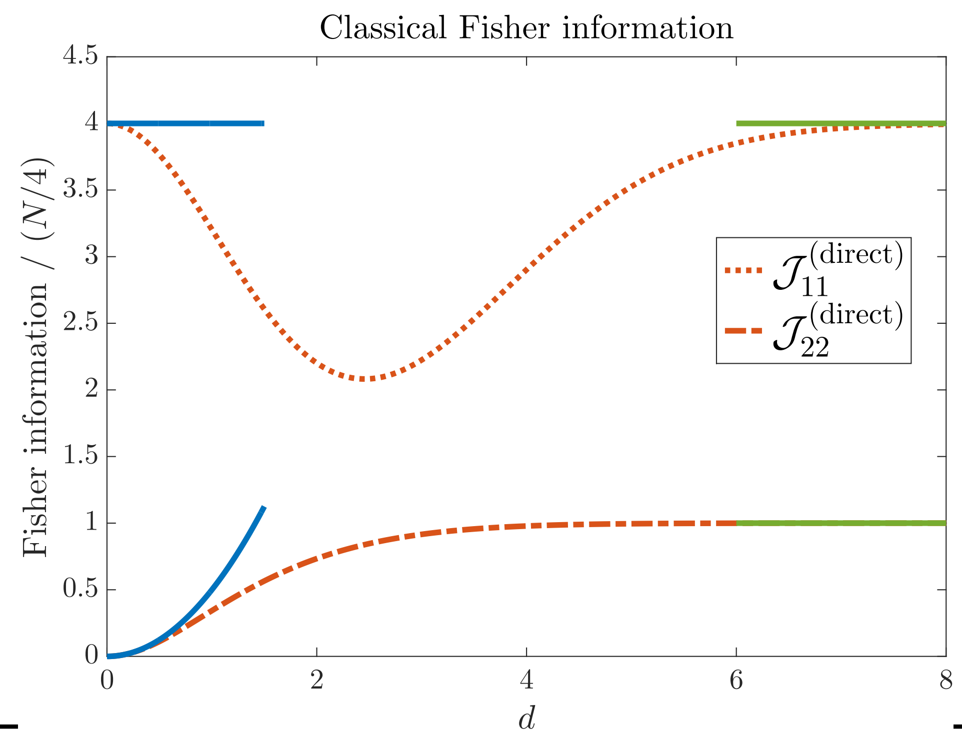
- CRB in subdiffraction regime:

$$\boxed{(J^{-1})_{\mu\nu} = \frac{O(1)}{N}.} \tag{9}$$

- Tsang, NJP **19**, 023054 (2017); PRA **97**, 023830 (2018).

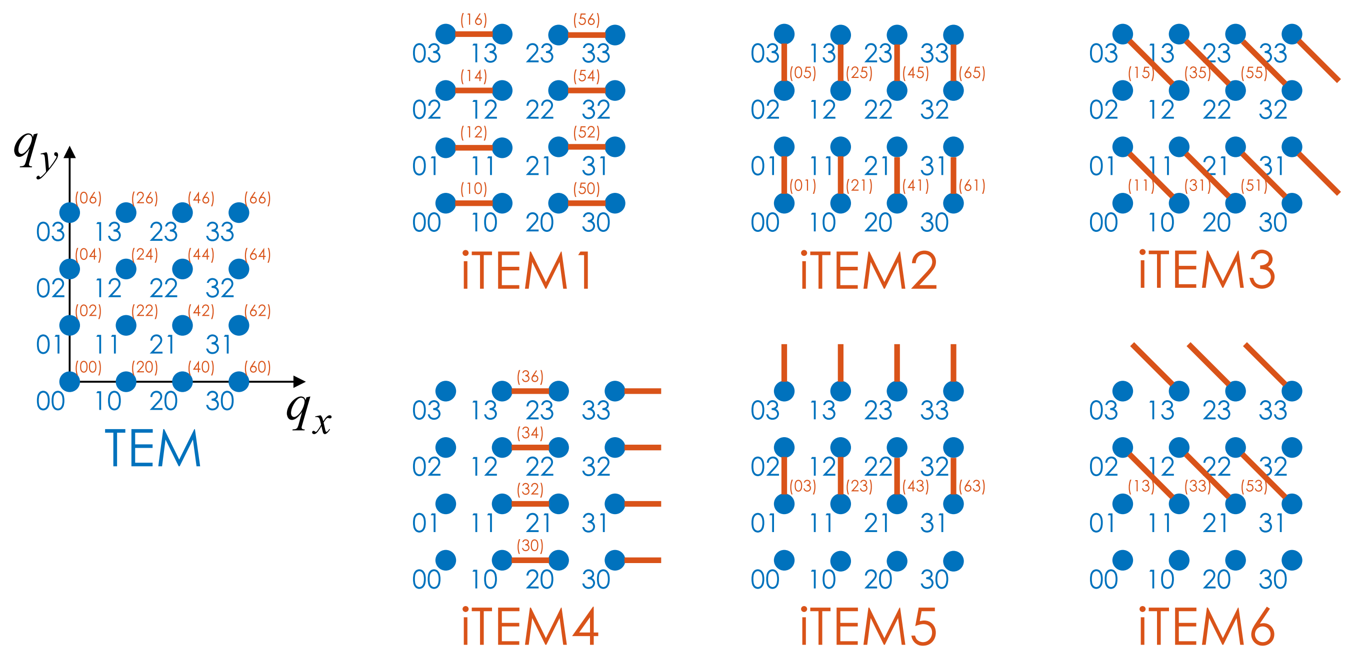
- **Special case**: for 2 point sources with separation d, $\theta_2 = d^2/4$,

$$J^{(d)} = \left(\frac{\partial\theta_2}{\partial d}\right)^2 J_{22} = NO(d^2). \tag{10}$$



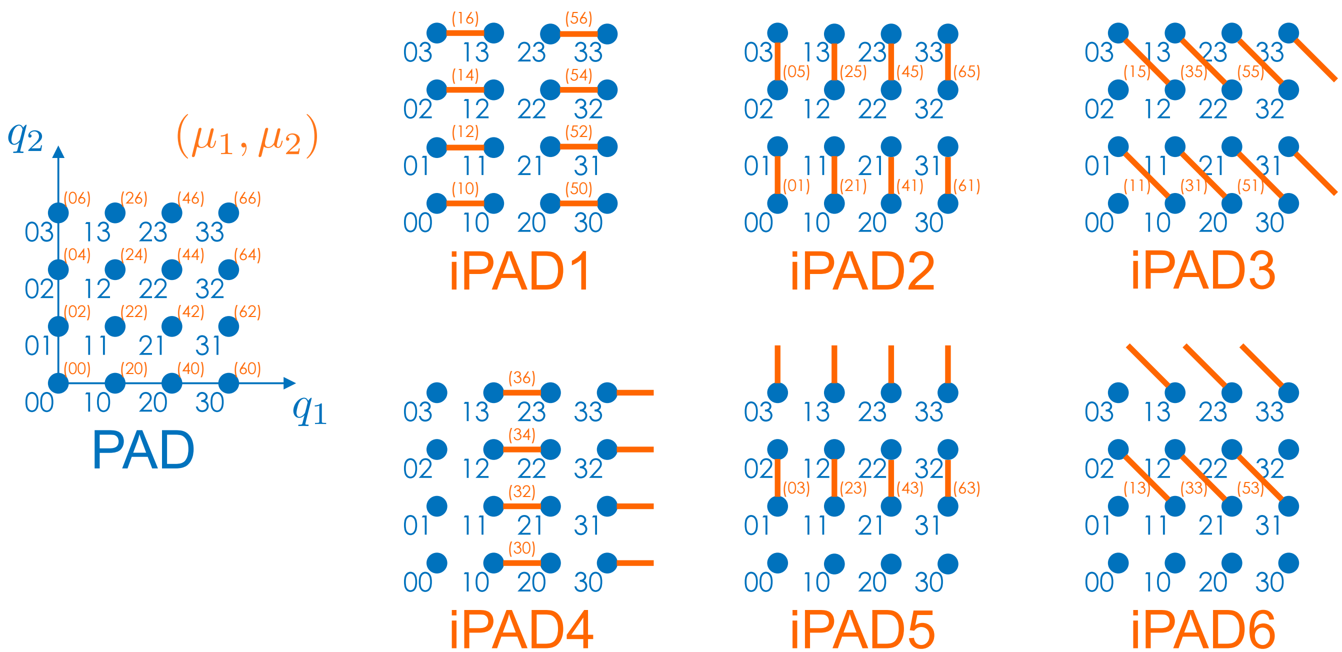
SPADE for Moment Estimation in 2D Imaging

- Gaussian PSF [Tsang, NJP (2017)]:
  - For moments with even μ_1 & even μ_2: TEM basis
    - See also Yang *et al.*, Optica (2016)
  - For other moments: interference of pairs of TEM modes



More General PSFs

- Any centrosymmetric and separable PSF [Tsang, PRA (2018)]:
  - For moments with even μ_1, even μ_2: **“PSF-adapted” (PAD) basis** (Rehacek *et al.* OL (2017), generalizes TEM)
  - For other moments: interference of pairs of PAD modes



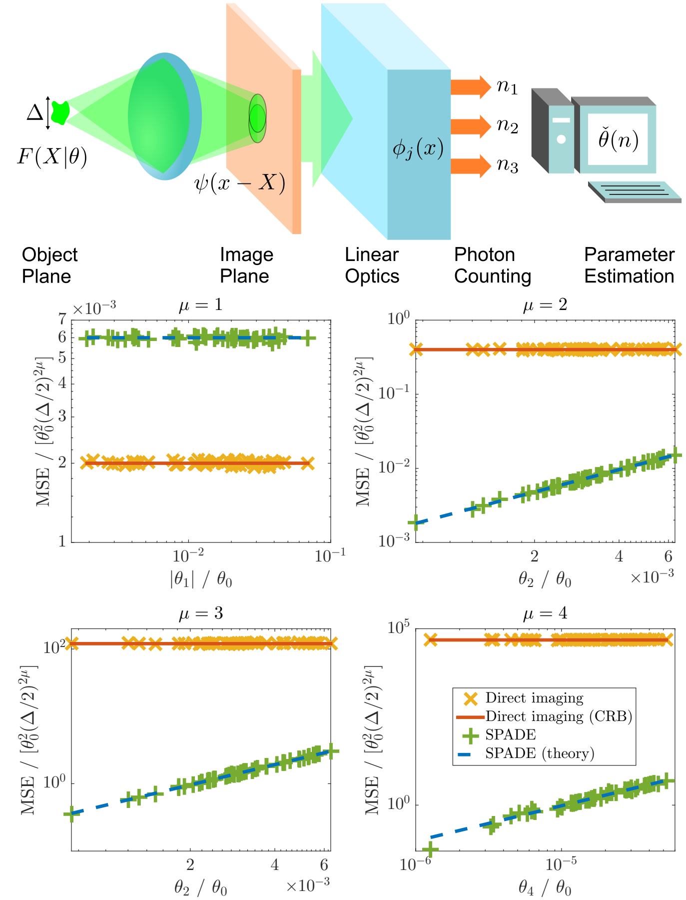
Performance of SPADE

- MSE of SPADE:

$$\boxed{\mathrm{MSE}_{\mu\mu} = \underbrace{\frac{O\left(\Delta^{2\lfloor \mu/2 \rfloor}\right)}{N}}_{\text{variance}} + \underbrace{O\left(\Delta^{2\mu+4}\right)}_{\text{bias}^2}.}$$

  much lower than direct-imaging CRB when

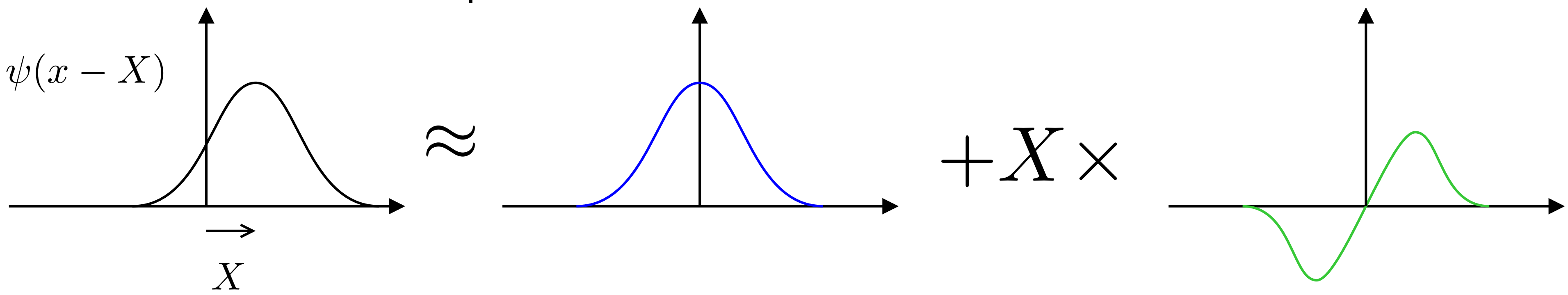
  - $\Delta \ll 1$ (subdiffraction)
  - $\mu \equiv \sum_j \mu_j \geq 2$ (2nd- and higher-order moments)
  - bias is negligible.
- **Caveat**: $\theta_\mu^2 = O(\Delta^{2\mu})$, *fractional* error is still large, need many photons
- **Object size and shape estimation**



Tsang, NJP (2017); PRA (2018).

Elementary Explanation

- Wavefunction from each point source:



- For one point source,

$$\text{Energy in first-order mode} \propto X^2. \tag{11}$$

- A distribution of **incoherent** sources:

$$\text{Total energy in first-order mode} \propto \int dX F(X|\theta) X^2. \tag{12}$$

Quantum Limit

- One-photon density operator:

$$\rho_1(\theta) = \int dX F(X|\theta) \left|\psi_X\right\rangle \left\langle\psi_X\right| , \qquad \left|\psi_X\right\rangle = \int_{-\infty}^{\infty} dx \psi(x - X) \left|x\right\rangle . \tag{13}$$

  - mixed state
  - infinite number of spatial modes
  - infinite number of parameters

- **Quantum Cramér-Rao bound** [Tsang, arXiv:1806.02781 (2018)]:

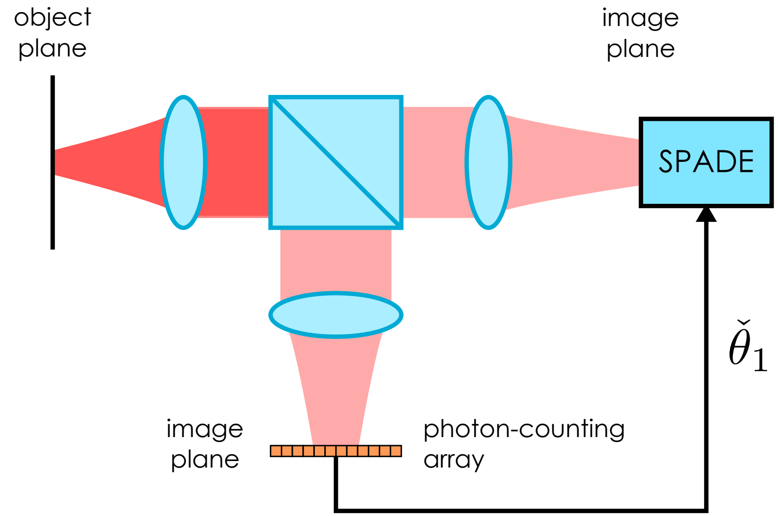
$$\boxed{(J^{-1})_{\mu\mu} \geq \frac{O(\Delta^{2\lfloor \mu/2 \rfloor})}{N}?} \tag{14}$$

  - Choose a nice Choi-Kraus representation/purification of ρ_1

- See also Zhou and Jiang (Yale), arXiv:1801.02917 (2018)

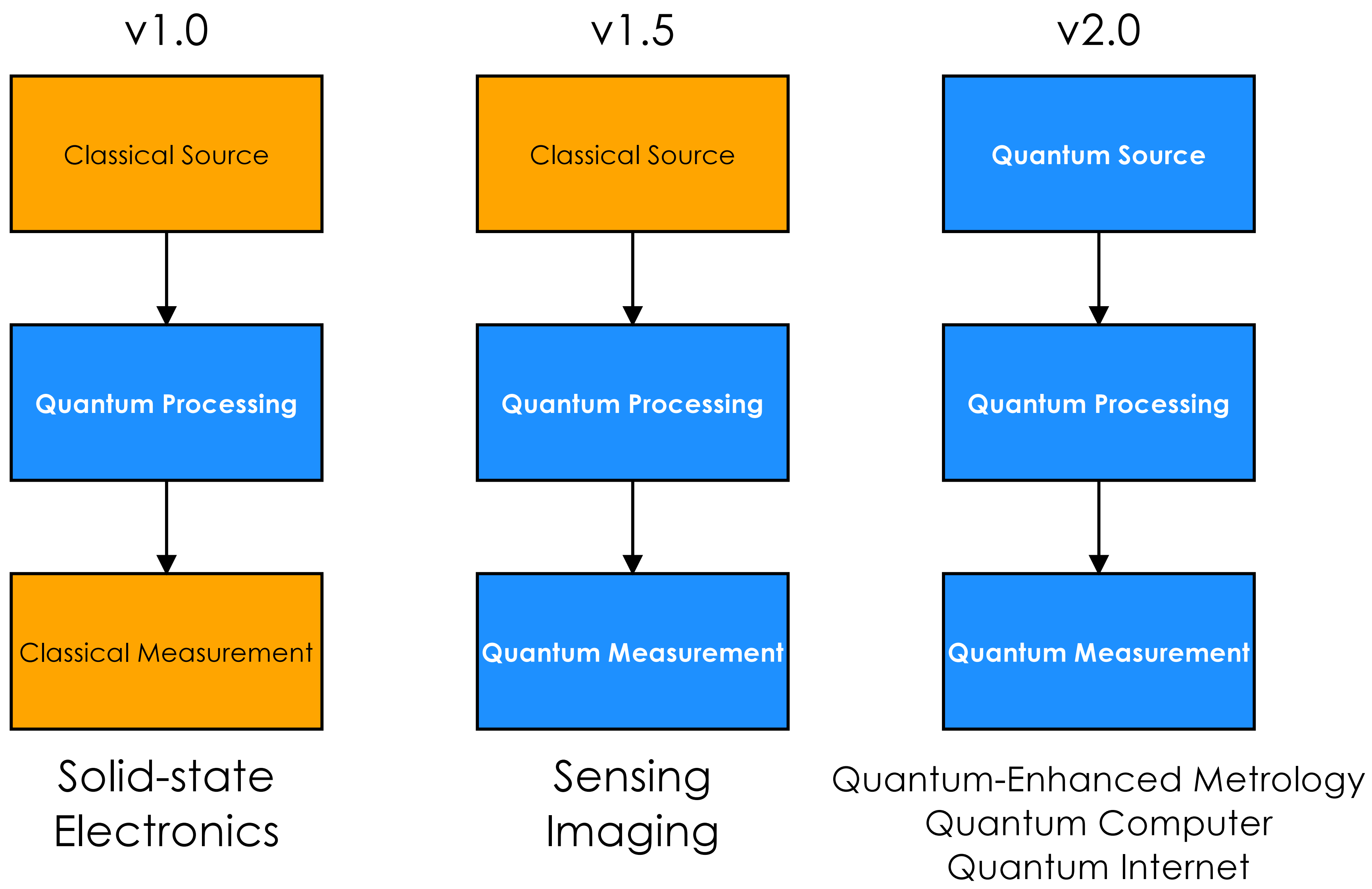
Future Directions

- Clean up math, multiparameter problems, exactly optimal measurements, etc.
- QCRB for 3D localization?
  - Backlund, Shechtman, Walsworth (Harvard/Technion), arXiv:1803.01776 (2018)
  - Yu and Prasad (New Mexico), arXiv:1805.09227 (2018)
  - Napoli, Tufarelli, Piano, Leach, Adesso (Nottingham), arXiv:1805.04116 (2018)
- Control?

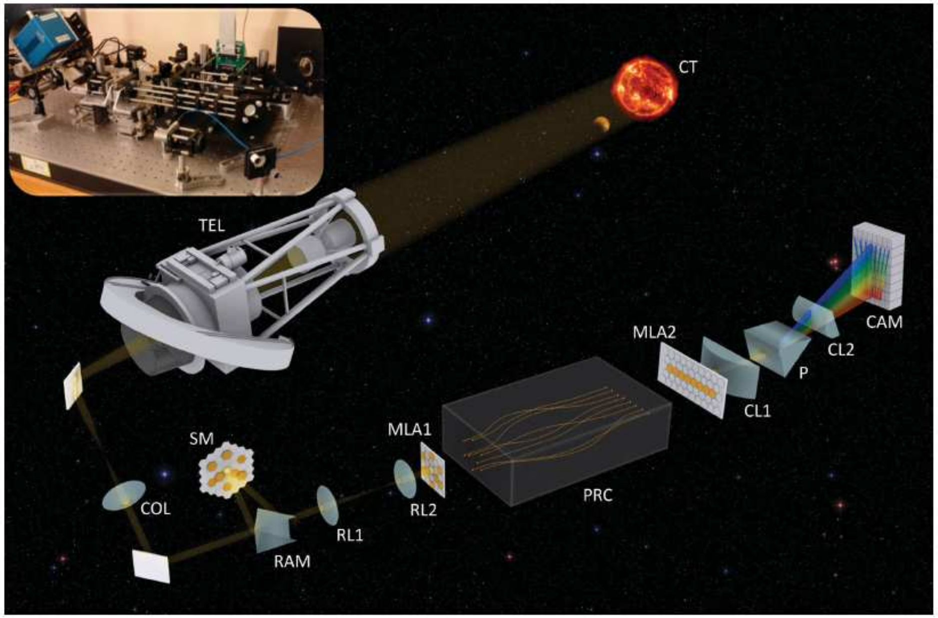


- **Experiments**
- **FAQ**: `https://sites.google.com/site/mankeitsang/news/rayleigh/faq`

Quantum Technology 1.5

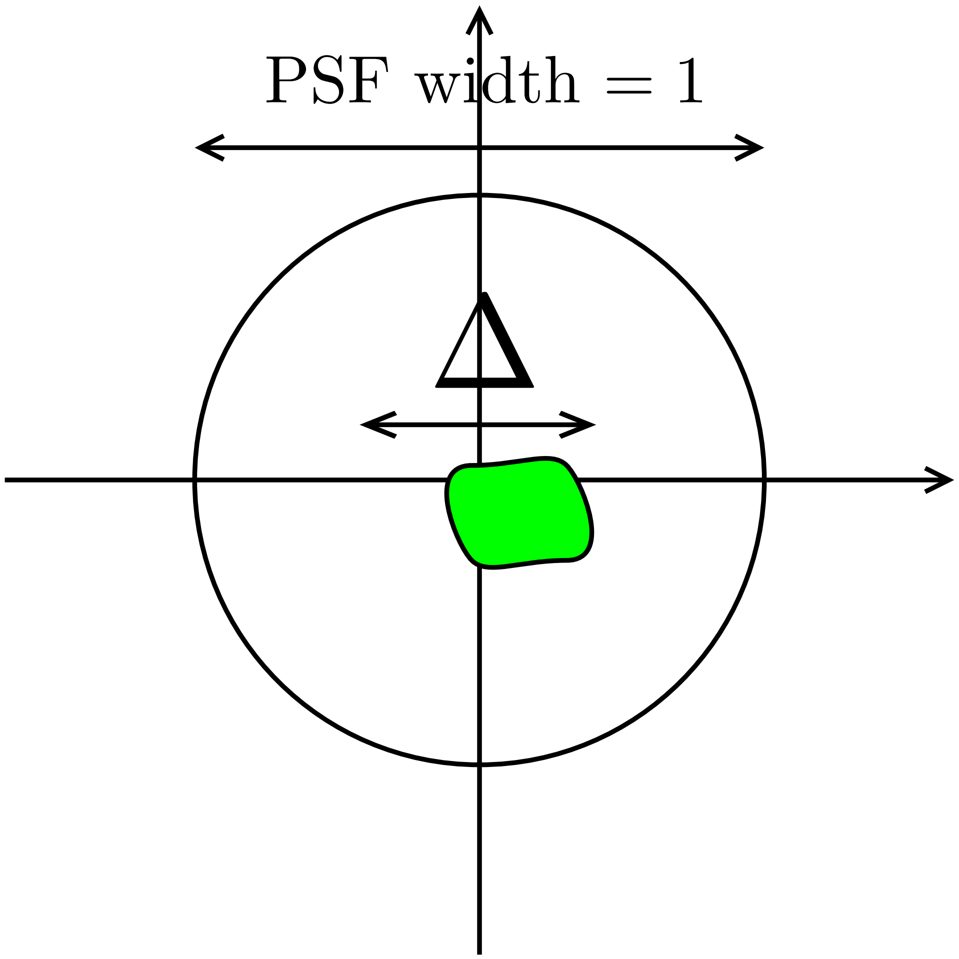


- **Astrophotonics**: photonic circuits for stellar interferometry
- **Conventional wisdom**: less sensitive to atmospheric turbulence
- **Our work**: Fundamental advantage with diffraction + photon shot noise
- Singapore: **fluorescence microscopy**



"Dragonfly," Jovanovic *et al.*, Mon. Not. R. Astron. Soc. **427**, 806 (2012)

Misalignment



$$\Delta = \textbf{centroid displacement} + \textbf{object size} \ll 1 \quad (15)$$

Rough Sketch

$$\rho_1 = \int dX F(X|\theta) e^{-ikX} |\psi\rangle \langle\psi| e^{ikX} \tag{16}$$

$$= \sum_{q,p} \underbrace{\int dX F(X|\theta) X^{q+p}}_{\text{moment matrix}} \frac{(-ik)^q}{q!} |\psi\rangle \langle\psi| \frac{(ik)^p}{p!} \tag{17}$$

$$= \sum_{q,p} \underbrace{\theta_{q+p}}_{\text{Cholesky}} \frac{(-ik)^q}{q!} |\psi\rangle \langle\psi| \frac{(ik)^p}{p!} \tag{18}$$

$$= \sum_{q,p,r} \underbrace{\Lambda_{qr}\Lambda_{pr}} \frac{(-ik)^q}{q!} |\psi\rangle \langle\psi| \frac{(ik)^p}{p!} \tag{19}$$

$$= \sum_r A_r |\psi\rangle \langle\psi| A_r^\dagger, \quad A_r \equiv \sum_q \Lambda_{qr} \frac{(-ik)^q}{q!} \quad \text{(Choi-Kraus)}, \tag{20}$$

$$\tilde{K}_{\mu\nu} = \mathrm{tr}\,\Pi \frac{\partial\Lambda}{\partial\theta_\mu} \frac{\partial\Lambda^\top}{\partial\theta_\mu}, \quad \Pi_{pq} \equiv \langle\psi| \frac{(ik)^p}{p!} \frac{(-ik)^q}{q!} |\psi\rangle\,. \tag{21}$$