Ziv-Zakai Error Bounds for Quantum Parameter Estimation: Supplementary Material

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This Supplementary Material contains detailed derivations of some of the results presented in the main text. Section I derives the classical Ziv-Zakai bounds, Sec. II calculates a quantum Ziv-Zakai bound (QZZB) when the photon-number distribution can be approximated as continuous and Gaussian, Sec. III calculates the bound for coherent states, and Sec. IV calculates the bound for rectangle states.

I. DERIVATION OF THE CLASSICAL ZIV-ZAKAI BOUNDS

Let

$$\sigma \equiv |\tilde{X}(Y) - X| \tag{S1}$$

be a nonnegative random variable. The mean-square error becomes

$$\Sigma = \int_0^\infty ds P_\sigma(s) s^2, \tag{S2}$$

where $P_\sigma(s)$ is the probability density of σ. With

$$P_\sigma(s) = -\frac{d}{ds} \Pr(\sigma \geq s) \tag{S3}$$

and integration by parts, Σ becomes

$$\Sigma = 2\int_0^\infty ds s \Pr(\sigma \geq s) = \frac{1}{2}\int_0^\infty d\tau \tau \Pr\left(|\tilde{X} - X| \geq \frac{\tau}{2}\right). \tag{S4}$$

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Since both τ and $\Pr(|\tilde{X}-X|\geq\tau/2)$ are nonnegative, one can find a lower bound on Σ by bounding $\Pr(|\tilde{X}-X|\geq\tau/2)$. Rewrite $\Pr(|\tilde{X}-X|\geq\tau/2)$ as follows:

$$\Pr\left(|\tilde{X}-X|\geq\frac{\tau}{2}\right)=\Pr\left(\tilde{X}-X>\frac{\tau}{2}\right)+\Pr\left(\tilde{X}-X\leq-\frac{\tau}{2}\right) \tag{S5}$$

$$=\int_{-\infty}^{\infty}dx_0P_X(x_0)\Pr\left(\tilde{X}-X>\frac{\tau}{2}\middle|X=x_0\right)+\int_{-\infty}^{\infty}dx_1P_X(x_1)\Pr\left(\tilde{X}-X\leq-\frac{\tau}{2}\middle|X=x_1\right), \tag{S6}$$

and let

$$x_0=x, \qquad x_1=x+\tau. \tag{S7}$$

This yields

$$\Pr\left(|\tilde{X}-X|\geq\frac{\tau}{2}\right)=\int_{-\infty}^{\infty}dx\left[P_X(x)\Pr\left(\tilde{X}>x+\frac{\tau}{2}\middle|X=x\right)+P_X(x+\tau)\Pr\left(\tilde{X}\leq x+\frac{\tau}{2}\middle|X=x+\tau\right)\right] \tag{S8}$$

$$=\int_{-\infty}^{\infty}dx[P_X(x)+P_X(x+\tau)]\times\left\{P_0\Pr\left(\tilde{X}>x+\frac{\tau}{2}\middle|X=x\right)+P_1\Pr\left(\tilde{X}\leq x+\frac{\tau}{2}\middle|X=x+\tau\right)\right\}, \tag{S9}$$

where

$$P_0\equiv\frac{P_X(x)}{P_X(x)+P_X(x+\tau)}, \qquad P_1\equiv\frac{P_X(x+\tau)}{P_X(x)+P_X(x+\tau)}=1-P_0. \tag{S10}$$

Now consider a binary hypothesis testing problem with hypotheses

$$\mathcal{H}_0:X=x, \qquad \mathcal{H}_1:X=x+\tau, \tag{S11}$$

prior probabilities

$$\Pr(\mathcal{H}_0)=P_0, \qquad \Pr(\mathcal{H}_1)=P_1, \tag{S12}$$

observation probability densities

$$P_Y(y|\mathcal{H}_0)=P_{Y|X}(y|x), \qquad P_Y(y|\mathcal{H}_1)=P_{Y|X}(y|x+\tau), \tag{S13}$$

and the following suboptimal decision rule:

$$\text{choose } \mathcal{H}_0 \text{ if } \tilde{X} \leq x + \frac{\tau}{2}, \qquad \text{choose } \mathcal{H}_1 \text{ if } \tilde{X} > x + \frac{\tau}{2}. \tag{S14}$$

The error probability of this hypothesis testing problem is then precisely given by the expression in the curly brackets in Eq. (S9). This expression is lower-bounded by the minimum error probability of the hypothesis testing problem, denoted by $\Pr_e(x, x+\tau)$, which does not depend on $\tilde{X}$, and one obtains

$$\Pr\left(|\tilde{X} - X| \geq \frac{\tau}{2}\right) \geq \int_{-\infty}^{\infty} dx [P_X(x) + P_X(x+\tau)] \Pr_e(x, x+\tau). \tag{S15}$$

The left-hand side is a monotonically decreasing function of τ, so a tighter bound can be obtained if we fill the valleys of the right-hand side as a function of τ. Denoting this valley-filling operation as $\mathcal{V}$:

$$\mathcal{V} f(\tau) \equiv \max_{\eta \geq 0} f(\tau + \eta), \tag{S16}$$

one gets

$$\Pr\left(|\tilde{X} - X| \geq \frac{\tau}{2}\right) \geq \mathcal{V} \int_{-\infty}^{\infty} dx [P_X(x) + P_X(x+\tau)] \Pr_e(x, x+\tau), \tag{S17}$$

$$\Sigma \geq \frac{1}{2} \int_0^{\infty} d\tau \tau \mathcal{V} \int_{-\infty}^{\infty} dx [P_X(x) + P_X(x+\tau)] \Pr_e(x, x+\tau). \tag{S18}$$

This is a Ziv-Zakai bound. Another version that relates the mean-square error to an equally-likely-hypothesis-testing problem can be obtained from Eq. (S8):

$$\Pr\left(|\tilde{X} - X| \geq \frac{\tau}{2}\right) \geq \int_{-\infty}^{\infty} dx 2 \min[P_X(x), P_X(x+\tau)] \times \left\{ \frac{1}{2} \Pr\left(\tilde{X} > x + \frac{\tau}{2} \middle| X = x\right) + \frac{1}{2} \Pr\left(\tilde{X} \leq x + \frac{\tau}{2} \middle| X = x + \tau\right) \right\}. \tag{S19}$$

The expression in the curly brackets is now the error probability of the same hypothesis testing problem as before, except that the prior probabilities are

$$\Pr(\mathcal{H}_0) = \Pr(\mathcal{H}_1) = \frac{1}{2}. \tag{S20}$$

Another Ziv-Zakai bound follows:

$$\Sigma \geq \frac{1}{2} \int_0^{\infty} d\tau \tau \mathcal{V} \int_{-\infty}^{\infty} dx 2 \min[P_X(x), P_X(x+\tau)] \Pr_e^{el}(x, x+\tau), \tag{S21}$$

where $\Pr_e^{el}(x, x+\tau)$ now denotes the minimum error probability with equally likely hypotheses. Note that these bounds make no assumption about the estimate.

II. QUANTUM ZIV-ZAKAI BOUND FOR APPROXIMATELY GAUSSIAN PHOTON-NUMBER DISTRIBUTIONS

Consider the QZZB for optical phase estimation with a uniform prior window:

$$\Sigma_Z \equiv \frac{1}{2}\int_0^W d\tau\tau\left(1-\frac{\tau}{W}\right)\left[1-\sqrt{1-F(\tau)}\right], \tag{S22}$$

$$F(\tau) \equiv \left|\sum_n |C_n|^2 \exp(in\tau)\right|^2, \tag{S23}$$

where $|C_n|^2$ is the photon-number distribution. If $|C_n|^2$ can be approximated as a Gaussian distribution and the sum in Eq. (S23) as a continuous Fourier transform,

$$|C_n|^2 \approx \frac{1}{\sqrt{2\pi}\Delta N_j}\exp\left[-\frac{(n-N_j)^2}{2\Delta N_j^2}\right], \tag{S24}$$

$$F(\tau) \approx \left|\int dn|C_n|^2\exp(in\tau)\right|^2 = \exp\left(-\Delta N_j^2\tau^2\right). \tag{S25}$$

Since $1-\sqrt{1-F} \geq F/2$,

$$\Sigma_Z \gtrsim \frac{1}{4}\int_0^W d\tau\tau\left(1-\frac{\tau}{W}\right)F(\tau) = \frac{1}{8\Delta N_j^2} \text{ for } W\to\infty, \tag{S26}$$

which is lower than the quantum Cramér-Rao bound (QCRB) by a constant factor of 2. This shows that the two bounds can differ significantly only when $|C_n|^2$ cannot be well approximated by a Gaussian.

For multiple copies, the fidelity can be written as

$$F(\tau) = \left|\sum_{n_1,\ldots,n_\nu} |C_{n_1}|^2|C_{n_2}|^2\ldots|C_{n_\nu}|^2\exp\left(i\sum_j n_j\tau\right)\right|^2, \tag{S27}$$

which is the squared magnitude of the Fourier transform with respect to the total photon number $\sum_j n_j$. By virtue of the central limit theorem, the total-photon-number statistics will become approximately Gaussian with variance $\nu\Delta N_j^2$ in the limit of large ν. This means that, regardless of the form of $|C_n|^2$, the QZZB and the QCRB will become comparable in the limit of large ν.

III. QUANTUM ZIV-ZAKAI BOUND FOR COHERENT STATES

With the inequalities

$$\tau\left(1-\frac{\tau}{W}\right) \geq \frac{W}{4}\sin\frac{\pi\tau}{W}, \qquad 0 \leq \tau \leq W, \tag{S28}$$

$$1-\sqrt{1-F} \geq \frac{1}{2}F, \qquad 0 \leq F \leq 1, \tag{S29}$$

the QZZB becomes

$$\Sigma_Z \equiv \frac{1}{2}\int_0^W d\tau\tau\left(1-\frac{\tau}{W}\right)\left[1-\sqrt{1-F(\tau)}\right] \tag{S30}$$

$$\geq \frac{W}{16}\int_0^W d\tau \sin\frac{\pi\tau}{W}F(\tau). \tag{S31}$$

For a coherent state and $W = 2\pi$,

$$\Sigma_Z \geq \frac{\pi}{8}\exp(-2N)\int_0^{2\pi} d\tau \sin\frac{\tau}{2}\exp\left(2N\cos\tau\right). \tag{S32}$$

Changing the integration variable to $u \equiv \cos(\tau/2)$ and using the identity $\cos\tau = 2\cos^2(\tau/2) - 1$, one obtains

$$\Sigma_Z \geq \frac{\pi}{2}\exp(-4N)\int_{-1}^{1} du \exp(4Nu^2), \tag{S33}$$

which leads to Eq. (16) in the text.

IV. QUANTUM ZIV-ZAKAI BOUND FOR RECTANGLE STATES

Using Eq. (S31) with $W = 2\pi$, one obtains

$$\Sigma_Z \geq \frac{\pi}{8(M+1)^2}I_M, \tag{S34}$$

$$I_M \equiv \int_0^{2\pi} d\tau \frac{\sin^2(M+1)\tau/2}{\sin\tau/2}. \tag{S35}$$

It can be shown using trigonometric identities that

$$\sin^2\frac{(M+1)\tau}{2} = \sin^2\frac{(M-1)\tau}{2} + 4\sin\frac{\tau}{2}\cos\frac{M\tau}{2}\sin\frac{(M-1)\tau}{2} + 4\sin^2\frac{\tau}{2}\cos^2\frac{M\tau}{2}. \tag{S36}$$

The integral I_M then satisfies the recursive relation

$$I_M = I_{M-2} + \frac{16M}{(2M-1)(2M+1)} \tag{S37}$$

$$= I_{M-2} + 4\left[\frac{1}{2(M-1)+1} + \frac{1}{2M+1}\right], \tag{S38}$$

with $I_{-1} = 0$ and $I_0 = 4$. Hence

$$I_M = 4 \sum_{k=0}^{M} \frac{1}{2k+1}. \tag{S39}$$

For large M, the discrete sum can be approximated by an integral:

$$I_M \approx 4 \int_0^M dk \frac{1}{2k+1} = 2 \ln(2M+1). \tag{S40}$$