

## Time-Symmetric Quantum Theory of Smoothing

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Smoothing is an estimation technique that takes into account both past and future observations and can be more accurate than filtering alone. In this Letter, a quantum theory of smoothing is constructed using a time-symmetric formalism, thereby generalizing prior work on classical and quantum filtering, retrodiction, and smoothing. The proposed theory solves the important problem of optimally estimating classical Markov processes coupled to a quantum system under continuous measurements, and is thus expected to find major applications in future quantum sensing systems, such as gravitational wave detectors and atomic magnetometers.

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Estimation theory is concerned with the inference of unknown signals, given their *a priori* statistics as well as noisy observations [1]. Depending on the time at which the signal is to be estimated relative to the observation time interval, estimation problems can be divided into four classes: *Prediction*, the estimation of a signal at time  $\tau$  given observations before  $\tau$ ; *Filtering*, given observations before and up to  $\tau$ ; *Smoothing*, given observations before and after  $\tau$ ; and *Retrodiction*, given observations after  $\tau$  [2]. Among the four classes, prediction and filtering have received the most attention, given their importance in applications that require real-time knowledge of a system, such as control, weather forecast, and quantitative finance. If we allow delay in the estimation, however, we can take into account the more advanced observations to produce a more accurate estimation of the signal some time in the past via smoothing techniques. For this reason, smoothing is mainly used in communication and sensing applications, when accuracy is paramount but real-time data are not required.

Conventional quantum theory can be regarded as a prediction theory. The quantum state in the Schrödinger picture represents our maximal knowledge of a system given prior observations. In particular, the quantum filtering theory developed by Belavkin and others [3,4] can be regarded as a generalization of the classical nonlinear filtering theory devised by Stratonovich and Kushner [5]. Quantum smoothing and retrodiction theories, on the other hand, have been proposed by Aharonov *et al.* as an alternative formulation of quantum mechanics [6], Barnett *et al.* for the purpose of parameter estimation [7], and Yanagisawa for initial quantum state estimation [8]. In this Letter, I generalize these earlier results on classical and quantum estimation to a quantum theory of smoothing for continuous waveform estimation. I am primarily interested in the estimation of classical random processes, such as gravitational waves and magnetic fields, coupled to a quantum object, such as a quantum mechanical oscillator or an atomic spin ensemble, under continuous measure-

ments. Previous studies on the use of filtering for these estimation problems [9] model the classical signals in terms of constant parameters or waveforms with deterministic evolution, but it is more desirable to model them as Markov processes for generality and robustness, in which case smoothing can be significantly more accurate than filtering [1]. Quantum estimation of a random optical phase process has recently been studied by Wiseman and co-workers [10,11] and Tsang *et al.* [12], but a general quantum smoothing theory is still lacking. The theory proposed here is thus expected to find important applications in future quantum sensing systems, such as gravitational wave detectors and atomic magnetometers.

Consider the estimation problem schematically shown in Fig. 1. A vectoral classical random process  $x_t \equiv [x_1(t), \dots, x_n(t)]^T$  is coupled to a quantum system. The backaction of the quantum system on the classical system that produces  $x_t$  is assumed to be negligible, so that the statistics of  $x_t$  remain unperturbed and classical. This assumption should be satisfied for the purpose of sensing and avoids the contentious issue of quantum backaction on classical systems [13]. The quantum system is measured continuously, via a weak measurement operator  $\hat{M}(dy_t)$ , where  $dy_t \equiv [dy_1(t), \dots, dy_m(t)]^T$  is the vectoral measurement outcome at time  $t$ . Define the observations in the time interval  $[t_1, t_2]$  as  $dy_{[t_1, t_2]} \equiv \{dy_t, t_1 \leq t < t_2\}$ . My ultimate goal is to calculate the fixed-interval smoothing probability

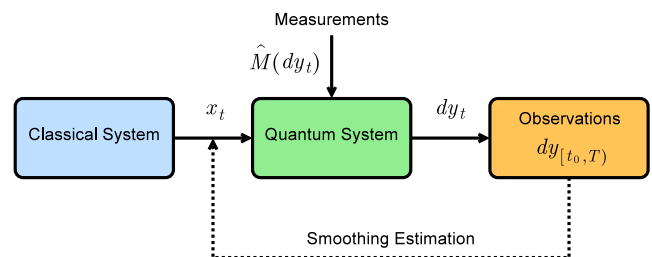


FIG. 1 (color online). Schematic of the continuous waveform estimation problem.

density  $P(x_\tau|dy_{[t_0,T]})$  at time  $\tau$ , conditioned upon past and future observations in the time interval  $t_0 \leq \tau \leq T$  so that the conditional expectations of  $x_\tau$  and the associated errors can be determined.

Central to my derivation is the use of a hybrid classical-quantum density operator  $\hat{\rho}_t(x_t)$ , which provides joint classical and quantum statistics at time  $t$  [13,14]. The classical probability density for  $x_t$  and the unconditional density operator can be determined from the hybrid operator by

$$P(x_t) = \text{tr}[\hat{\rho}_t(x_t)], \quad \hat{\rho}_t = \int dx_t \hat{\rho}_t(x_t), \quad (1)$$

respectively. To derive the smoothing density, I will need the conditional hybrid density operator  $\hat{\rho}_\tau(x_\tau|dy_{[t_0,\tau]})$  given past observations, and also a hybrid effect operator,  $\hat{E}_\tau(dy_{[\tau,T]}|x_\tau)$ , which determines the joint statistics of future observations  $dy_{[\tau,T]}$  given an arbitrary hybrid density operator  $\hat{\rho}_\tau(x_\tau)$  at time  $\tau$ ,

$$P[dy_{[\tau,T]}|\hat{\rho}_\tau(x_\tau)] = \int dx_\tau \text{tr}[\hat{E}_\tau(dy_{[\tau,T]}|x_\tau)\hat{\rho}_\tau(x_\tau)]. \quad (2)$$

The smoothing probability density is then

$$\begin{aligned} P(x_\tau|dy_{[t_0,T]}) &= P(x_\tau|dy_{[t_0,\tau]}, dy_{[\tau,T]}) \\ &= \frac{P(x_\tau, dy_{[\tau,T]}|dy_{[t_0,\tau]})}{P(dy_{[\tau,T]}|dy_{[t_0,\tau]})} \\ &= \frac{\text{tr}[\hat{E}_\tau(dy_{[\tau,T]}|x_\tau)\hat{\rho}_\tau(x_\tau|dy_{[t_0,\tau]})]}{\int dx_\tau \text{tr}[\hat{E}_\tau(dy_{[\tau,T]}|x_\tau)\hat{\rho}_\tau(x_\tau|dy_{[t_0,\tau]})]}. \end{aligned} \quad (3)$$

To calculate the conditional hybrid density operator  $\hat{\rho}_\tau(x_\tau|dy_{[t_0,\tau]})$ , which also solves the filtering problem, first consider the conditional density operator  $\hat{\rho}_\tau(x_\tau|x_{[t_0,\tau]})$  in discrete time, which describes the quantum state given a particular trajectory of  $x_{[t_0,\tau]} \equiv \{x_{t_0}, x_{t_0+\delta t}, \dots, x_{\tau-\delta t}\}$ ,

$$\hat{\rho}_{t+\delta t}(x_{t+\delta t}|\delta y_{[t_0,t]}, \delta y_t) = \frac{\int dx_t P(x_{t+\delta t}|x_t)\mathcal{K}(x_t)\mathcal{J}(\delta y_t)\hat{\rho}_t(x_t|\delta y_{[t_0,t]})}{\int dx_t \text{tr}[\mathcal{J}(\delta y_t)\hat{\rho}_t(x_t|\delta y_{[t_0,t]})]}. \quad (9)$$

Assuming Gaussian measurements, the measurement operator in the continuous limit is [3,4,15]

$$\hat{M}(dz_t) \propto \hat{1} + \sum_\mu \gamma_\mu(t) \left[ \frac{1}{2}(dz_t)_\mu \hat{c}_\mu - \frac{dt}{8} \hat{c}_\mu^\dagger \hat{c}_\mu \right], \quad (10)$$

where  $\gamma_\mu$  is assumed to be positive,  $dz_t$  is a vectoral observation process, and  $\hat{c}$  is a vector of arbitrary operators. Defining  $dy_t \equiv U dz_t$  and  $\hat{C} \equiv U \hat{c}$ ,  $U$  being a unitary matrix, the measurement operator can be cast into an equivalent but slightly more useful form as

$$\hat{M}(dy_t) \propto \hat{1} + \frac{1}{2} dy_t^T R^{-1}(t) \hat{C} - \frac{dt}{8} \hat{C}^{\dagger T} R^{-1}(t) \hat{C}, \quad (11)$$

$$\hat{\rho}_\tau(x_{[t_0,\tau]}) = \mathcal{K}(x_{\tau-\delta t}) \dots \mathcal{K}(x_{t_0+\delta t}) \mathcal{K}(x_{t_0}) \hat{\rho}_{t_0}, \quad (4)$$

where  $\hat{\rho}_{t_0}$  is the initial *a priori* density operator,  $\mathcal{K}(x_t) \equiv \exp[\delta t \mathcal{L}(x_t)]$  is a superoperator that governs the quantum system evolution for the time interval  $\delta t$  independent of the measurement process,  $\mathcal{L}$  is a superoperator in Lindblad form, and  $x_t$  acts as a parameter of the evolution. Averaging over trajectories of  $x_{[t_0,\tau]}$ , the hybrid density operator  $\hat{\rho}_\tau(x_\tau)$  can be expressed as

$$\hat{\rho}_\tau(x_\tau) = \int dx_{\tau-\delta t} \dots dx_{t_0} \hat{\rho}_\tau(x_{[t_0,\tau]}) P(x_{[t_0,\tau]}, x_\tau). \quad (5)$$

This expression can be verified by substituting it into Eqs. (1). If  $x_t$  is a Markov process,  $P(x_{[t_0,\tau]}, x_\tau) = P(x_{[t_0,\tau]}) = P(x_\tau|x_{\tau-\delta t}) \dots P(x_{t_0+\delta t}|x_{t_0}) P(x_{t_0})$ ,  $P(x_{t_0})$  being the initial *a priori* probability density. Rearranging the terms in Eqs. (4) and (5),  $\hat{\rho}_\tau(x_\tau)$  can be solved by iterating the formula

$$\hat{\rho}_{t+\delta t}(x_{t+\delta t}) = \int dx_t P(x_{t+\delta t}|x_t) \mathcal{K}(x_t) \hat{\rho}_t(x_t), \quad (6)$$

with the initial condition  $\hat{\rho}_{t_0}(x_{t_0}) = \hat{\rho}_{t_0} P(x_{t_0})$ .  $P(x_{t+\delta t}|x_t)$  for an important class of Markov processes can be determined from the Itô stochastic differential equation [1]

$$dx_t = A(x_t, t)dt + B(x_t, t)dW_t, \quad (7)$$

where  $dW_t$  is a vectoral Wiener increment with  $\mathcal{E}\{dW_t\} = 0$  and  $\mathcal{E}\{dW_t dW_t^T\} \equiv Q(t)dt$ .

To calculate the *a posteriori* hybrid state after a measurement, the quantum Bayes theorem [4] can be generalized as

$$\hat{\rho}_t(x_t|\delta y_t) = \frac{\mathcal{J}(\delta y_t)\hat{\rho}_t(x_t)}{\int dx_t \text{tr}[\mathcal{J}(\delta y_t)\hat{\rho}_t(x_t)]}, \quad (8)$$

where  $\mathcal{J}(\delta y_t)\hat{\rho} \equiv \hat{M}(\delta y_t)\hat{\rho}\hat{M}^\dagger(\delta y_t)$ . The evolution of the hybrid density operator conditioned upon past observations  $\delta y_{[t_0,t]} \equiv \{\delta y_{t_0}, \delta y_{t_0+\delta t}, \dots, \delta y_{t-\delta t}\}$  is therefore given by

where  $R$  is a real positive-definite matrix with eigenvalues  $1/\gamma_\mu$ . The stochastic master equation for  $\hat{\rho}_t(x_t = x|dy_{[t_0,t]}) \equiv \hat{F}(x, t)$  in the Itô sense is hence

$$\begin{aligned} d\hat{F} &= dt \left\{ \mathcal{L}(x)\hat{F} - \sum_\mu \frac{\partial}{\partial x_\mu} (A_\mu \hat{F}) \right. \\ &\quad + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} [(BQB^T)_{\mu\nu} \hat{F}] \\ &\quad + \frac{1}{8} (2\hat{C}^T R^{-1} \hat{F} \hat{C}^\dagger - \hat{C}^{\dagger T} R^{-1} \hat{C} \hat{F} - \hat{F} \hat{C}^{\dagger T} R^{-1} \hat{C}) \left. \right\} \\ &\quad + \frac{1}{2} [d\eta_t^T R^{-1} (\hat{C} - \langle \hat{C} \rangle_{\hat{F}}) \hat{F} + \text{H.c.}], \end{aligned} \quad (12)$$

where  $d\eta_t \equiv dy_t - dt\langle\hat{C} + \hat{C}^\dagger\rangle_{\hat{F}}/2$  is a real vectorial Wiener increment with covariance matrix  $Rdt$ ,  $\langle\hat{C}\rangle_{\hat{F}} \equiv \int dx \text{tr}[\hat{C}\hat{F}(x, t)]$ , and H.c. denotes the Hermitian conjugate. Equation (12) solves the filtering problem for the hybrid classical-quantum system and generalizes the Kushner equation [1,5] and the Belavkin equation [3]. The continuous phase estimation theory proposed in Ref. [11] may be considered as a special case of Eq. (12). A linear version of the master equation for an unnormalized  $\hat{F}(x, t)$ , analogous to the classical Zakai equation [16], is

$$\begin{aligned} d\hat{f} = dt & \left\{ \mathcal{L}(x)\hat{f} - \sum_{\mu} \frac{\partial}{\partial x_{\mu}} (A_{\mu}\hat{f}) \right. \\ & + \frac{1}{2} \sum_{\mu, \nu} \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} [(BQB^T)_{\mu\nu}\hat{f}] \\ & + \frac{1}{8} (2\hat{C}^T R^{-1} \hat{f} \hat{C}^\dagger - \hat{C}^{\dagger T} R^{-1} \hat{C} \hat{f} - \hat{f} \hat{C}^{\dagger T} R^{-1} \hat{C}) \\ & \left. + \frac{1}{2} (dy_t^T R^{-1} \hat{C} \hat{f} + \text{H.c.}), \right. \end{aligned} \quad (13)$$

and  $\hat{F}(x, t)$  is given by  $\hat{f}(x, t) / \int dx \text{tr}[\hat{f}(x, t)]$ .

To solve for  $\hat{E}_{\tau}(dy_{[\tau, T]}|x_{\tau})$ , rewrite Eq. (2) in discrete time as

$$\begin{aligned} P[\delta y_{[\tau, T]}|\hat{\rho}_{\tau}(x_{\tau})] &= \int dx_{\tau} \text{tr}[\hat{E}_{\tau}(\delta y_{[\tau, T]}|x_{\tau})\hat{\rho}_{\tau}(x_{\tau})] \quad (14) \\ &= \int dx_T \text{tr} \left[ \int dx_{T-\delta t} P(x_T|x_{T-\delta t}) \mathcal{K}(x_{T-\delta t}) \mathcal{J}(\delta y_{T-\delta t}) \dots \right. \\ & \quad \left. \times \int dx_{\tau} P(x_{\tau+\delta t}|x_{\tau}) \mathcal{K}(x_{\tau}) \mathcal{J}(\delta y_{\tau}) \hat{\rho}_{\tau}(x_{\tau}) \right]. \quad (15) \end{aligned}$$

Comparing Eq. (14) with Eq. (15) and defining the adjoint of a superoperator  $\mathcal{O}$  as  $\mathcal{O}^*$ , such that  $\text{tr}[\hat{E}(\mathcal{O}\hat{\rho})] = \text{tr}[(\mathcal{O}^*\hat{E})\hat{\rho}]$ , the hybrid effect operator can be expressed as

$$\begin{aligned} \hat{E}_{\tau}(\delta y_{[\tau, T]}|x_{\tau}) &= \mathcal{J}^*(\delta y_{\tau}) \mathcal{K}^*(x_{\tau}) \int dx_{\tau+\delta t} P(x_{\tau+\delta t}|x_{\tau}) \dots \\ & \quad \times \mathcal{J}^*(\delta y_{T-\delta t}) \mathcal{K}^*(x_{T-\delta t}) \\ & \quad \times \int dx_T P(x_T|x_{T-\delta t}) \hat{1}. \quad (16) \end{aligned}$$

The stochastic master equation for an unnormalized  $\hat{E}_t(dy_{[t, T]}|x_t = x) \propto \hat{g}(x, t)$  in continuous time becomes

$$\begin{aligned} -d\hat{g} = dt & \left[ \mathcal{L}^*(x)\hat{g} + \sum_{\mu} A_{\mu} \frac{\partial}{\partial x_{\mu}} \hat{g} + \frac{1}{2} \sum_{\mu, \nu} (BQB^T)_{\mu\nu} \right. \\ & \quad \times \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} \hat{g} + \frac{1}{8} (2\hat{C}^{\dagger T} \hat{g} R^{-1} \hat{C} - \hat{g} \hat{C}^{\dagger T} R^{-1} \hat{C} \\ & \quad \left. - \hat{C}^{\dagger T} R^{-1} \hat{C} \hat{g}) \right] + \frac{1}{2} (dy_t^T R^{-1} \hat{g} \hat{C} + \text{H.c.}), \quad (17) \end{aligned}$$

which is the adjoint equation of Eq. (13), to be solved backward in time using the backward Itô rule and the final

condition  $\hat{g}(x, T) \propto \hat{1}$ . The smoothing probability density is hence

$$h(x, \tau) \equiv P(x_{\tau} = x | dy_{[t_0, T]}) = \frac{\text{tr}[\hat{g}(x, \tau)\hat{f}(x, \tau)]}{\int dx \text{tr}[\hat{g}(x, \tau)\hat{f}(x, \tau)]}. \quad (18)$$

This form of smoothing, which combines the solutions of adjoint Eqs. (13) and (17), has a pleasing time symmetry and can be regarded as a generalization of the classical nonlinear two-filter smoothing theory proposed by Pardoux [17].

Equations (12), (13), (17), and (18) are the central results of this Letter and form the basis of a general quantum prediction, filtering, smoothing, and retrodiction theory for continuous waveform estimation. One way of solving them is to convert them to stochastic partial differential equations for quasiprobability distributions. For quantum systems with continuous degrees of freedom, the Wigner distribution is especially helpful. Let  $f(q, p, x, t)$  and  $g(q, p, x, t)$  be the Wigner distributions of  $\hat{f}(x, t)$  and  $\hat{g}(x, t)$ , respectively. They have the desired property  $\int dq dp g(q, p, x, t) f(q, p, x, t) \propto \text{tr}[\hat{g}(x, t)\hat{f}(x, t)]$ , which is unique among generalized quasiprobability distributions [18]. The smoothing density can then be rewritten as

$$h(x, \tau) = \frac{\int dq dp g(q, p, x, \tau) f(q, p, x, \tau)}{\int dx dq dp g(q, p, x, \tau) f(q, p, x, \tau)}. \quad (19)$$

As an illustration of the smoothing theory, consider the estimation of a classical force, say  $x_1(t)$ , acting on a quantum mechanical harmonic oscillator, and the position of the oscillator is monitored, via an optical phase-locked loop for example [10–12]. Let  $\mathcal{L}\hat{\rho} = -\mathcal{L}^*\hat{\rho} = -(i/\hbar) \times [\hat{H}, \hat{\rho}]$ ,  $\hat{H} = (\hat{p}^2 + \omega^2 \hat{q}^2)/2 - x_1 \hat{q}$ , and  $\hat{C} = \hat{q}$ . The linear stochastic equations for the Wigner distributions become

$$\begin{aligned} df = dt & \left\{ -p \frac{\partial f}{\partial q} + (\omega^2 q - x_1) \frac{\partial f}{\partial p} - \sum_{\mu} \frac{\partial}{\partial x_{\mu}} (A_{\mu} f) \right. \\ & \quad \left. + \frac{1}{2} \sum_{\mu, \nu} \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} [(BQB^T)_{\mu\nu} f] + \frac{\hbar^2}{8R} \frac{\partial^2 f}{\partial p^2} \right\} + \frac{dy_t q}{R} f, \quad (20) \end{aligned}$$

and

$$\begin{aligned} -dg = dt & \left[ p \frac{\partial g}{\partial q} - (\omega^2 q - x_1) \frac{\partial g}{\partial p} + \sum_{\mu} A_{\mu} \frac{\partial g}{\partial x_{\mu}} \right. \\ & \quad \left. + \frac{1}{2} \sum_{\mu, \nu} (BQB^T)_{\mu\nu} \frac{\partial^2 g}{\partial x_{\mu} \partial x_{\nu}} + \frac{\hbar^2}{8R} \frac{\partial^2 g}{\partial p^2} \right] + \frac{dy_t q}{R} g. \quad (21) \end{aligned}$$

These equations are then identical to the classical forward and backward Zakai equations [16,17]. If  $x_t$  is Gaussian and the initial  $f$  is Gaussian, the means and covariances of the Gaussian  $f$ ,  $g$ , and  $h$  can be obtained using the Mayne-

Fraser-Potter two-filter smoother [12,19], which calculates those of  $f$  and  $g$  using forward and backward Kalman-Bucy filters [1], and then combines them to give the means and covariances of  $h$ . As is well known in classical estimation theory, unless  $x_1$  is constant, the smoothing estimates and covariances cannot be obtained from a filtering theory alone. The reduced estimation errors associated with quantum smoothing can in principle be verified experimentally in future quantum sensing systems.

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