Editors' Suggestion

Quantum limit to subdiffraction incoherent optical imaging

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(Received 8 August 2018; published 3 January 2019)

The application of quantum estimation theory to the problem of imaging two incoherent point sources has recently led to new insights and better measurements for incoherent imaging and spectroscopy. To establish a more general limit beyond the case of two sources, here I evaluate a quantum bound on the Fisher information that can be extracted by any far-field optical measurement about the moments of a subdiffraction object. The bound matches the performance of a spatial-mode-demultiplexing (SPADE) measurement scheme in terms of its scaling with the object size, indicating that SPADE is close to quantum-optimal. Coincidentally, the result is also applicable to the estimation of diffusion parameters with a quantum probe subject to random displacements.

DOI: 10.1103/PhysRevA.99.012305

I. INTRODUCTION

The fundamental resolution of optical imaging can be framed as a problem of quantum estimation [1]: With any measurement permitted by quantum mechanics, how well can one estimate unknown parameters from the light? While Helstrom laid the foundation of quantum estimation theory and first applied it to incoherent imaging [1], it was not until recently that this approach yielded genuine surprises on the age-old topic. Through the computation of the quantum Fisher information (QFI), it was found that the separation between two sub-Rayleigh incoherent point sources can be estimated much more accurately than previously realized [2]. This discovery has since led to new insights and better measurements for incoherent imaging and spectroscopy [2–26]. Experimental demonstrations have also been reported [27–35].

Generalizing such results for arbitrary source distributions is much more difficult, as the quantum state may depend on infinitely many spatial modes and infinitely many parameters. Some progress has been made in Refs. [3,4], which evaluate the performance of a spatial-mode-demultiplexing (SPADE) measurement for estimating the moments of any subdiffraction object. Reference [3] also proves quantum bounds for location and scale parameters and conjectures that SPADE may be quantum-optimal for general imaging. A similar conjecture was raised earlier by Krovi, Guha, and Shapiro in Ref. [11]. Zhou and Jiang have recently taken a major step towards proving the conjectures [21]: Using novel arguments that do not resort to the QFI, they propose limits on the scaling of the Fisher information with respect to the object size for any moment parameter. Their bounds may have issues concerning their precise values and validity, however, as elaborated in Appendix E.

To derive a limit using more standard quantum estimation theory, here I evaluate an upper bound on the QFI for the moment-estimation problem. The result matches the performance of SPADE evaluated in Refs. [3,4] in terms of the object-size scaling, indicating that SPADE is close to quantum-optimal. While the end result looks similar to those of Zhou and Jiang, the use of the QFI here leads to a bound that overcomes the issues in Ref. [21] and sets a more absolute and computable quantum limit for incoherent imaging.

II. QUANTUM OPTICS AND QUANTUM ESTIMATION THEORY

Consider the far-field imaging of quasimonochromatic incoherent optical sources, as depicted by Fig. 1. The quantum state of light in M temporal modes can be modeled as the tensor product $\rho^{\otimes M}$, where

$$\rho = (1-\epsilon)\rho_0 + \epsilon\rho_1, \tag{2.1}$$

$\epsilon \ll 1$ is the expected photon number per temporal mode, ρ_0 is the vacuum state, ρ_1 is the one-photon state, and $O(\epsilon^2)$ terms are ignored [2,36,37]. Assuming scalar paraxial waves [38] and the imaging of sources in one transverse dimension for simplicity, the one-photon state on the image plane is given by [2,3]

$$\rho_1 = \int dX F(X|\theta) e^{-i\hat{k}X} |\psi\rangle\langle\psi| e^{i\hat{k}X}, \tag{2.2}$$

$$|\psi\rangle = \int dk \Psi(k) |k\rangle, \tag{2.3}$$

where $F(X|\theta)$ is the normalized object intensity distribution with $\int dX F(X|\theta) = 1$, X is the object-plane coordinate, $\theta = (\theta_1, \theta_2, \dots)$ is a vector of unknown parameters, $\hat{k}$ is the one-photon spatial-frequency (momentum) operator, $|k\rangle$ is the one-photon eigenket that satisfies $\hat{k}|k\rangle = k|k\rangle$ and $\langle k|k'\rangle = \delta(k-k')$, and $\Psi(k)$ is the optical transfer function (OTF) of the imaging system [38]. The diffraction limit introduces a finite bandwidth to $\Psi(k)$, and the Fourier transform of $\Psi(k)$ gives the point-spread function. X and $\hat{k}$ are normalized with respect to the magnification factor and the OTF bandwidth

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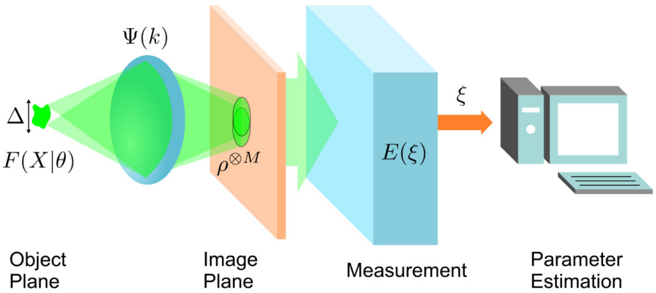

FIG. 1. A far-field incoherent optical imaging system. $F(X|\theta)$ is the object intensity distribution, Δ is its characteristic width, $\Psi(k)$ is the optical transfer function (OTF) of the imaging system, $\rho^{\otimes M}$ is the quantum state of light in M temporal modes on the image plane, E is the POVM that models the measurement, and ξ is a measurement outcome.

such that they are unitless. While this work will focus on imaging, note that Eq. (2.2) also describes a quantum object in initial state $|\psi\rangle$ subject to random displacements with unknown statistics [39–41].

Any measurement can be modeled by a positive operator-valued measure (POVM) E [1,42], such that the probability of a measurement outcome ξ conditioned on θ is

$$P(\xi|\theta) = \operatorname{tr} E(\xi)\rho^{\otimes M}, \tag{2.4}$$

where tr denotes the operator trace. If the measurement consists of passive linear optics and photon counting, the standard Poisson model in optical astronomy and fluorescence microscopy [4,7,43–50] is retrieved in the "ultraviolet" limit of $\epsilon \to 0$ and $M \to \infty$, with $N \equiv M\epsilon$, the expected photon number in all modes, held constant [2].

Denoting the partial derivative with respect to θ_μ by the comma notation $P_{,\mu} \equiv \partial P/\partial\theta_\mu$, the Fisher information matrix is given by

$$J_{\mu\nu}(P) \equiv \sum_\xi \frac{P_{,\mu}(\xi|\theta)P_{,\nu}(\xi|\theta)}{P(\xi|\theta)}, \tag{2.5}$$

which plays a fundamental role in parameter estimation theory and can be used to set Cramér-Rao lower error bounds [5,51]. In the context of imaging, the Fisher information has been proposed by many as the fundamental measure of resolution [46–50,52–56]. In recent years, it has become especially popular in fluorescence microscopy [46–50].

In quantum estimation theory, it is known [1,2,42] that, for any POVM,

$$J(P) \leqslant NK(\rho_1), \tag{2.6}$$

$$K_{\mu\nu}(\rho_1) = \operatorname{tr}\rho_{1,\mu}L_\nu, \quad \rho_{1,\mu} = \tfrac{1}{2}(L_\mu\rho_1 + \rho_1 L_\mu), \tag{2.7}$$

where the matrix inequality means that $NK - J$ is positive-semidefinite. Appendix A proves that Eq. (2.6) in fact holds for any thermal state with arbitrary ϵ. The QFI matrix $NK(\rho_1)$ thus serves as an even more fundamental measure of resolution that depends only on the quantum state and holds for any measurement.

III. QUANTUM BOUND BASED ON AN ALTERNATIVE CHOI-KRAUS REPRESENTATION

Define the object moment parameters as

$$\theta_\mu \equiv \int dX F(X|\theta)X^\mu, \quad \mu \in \mathbb{N}, \tag{3.1}$$

with $\theta_0 = 1$. Under benign conditions, each moment sequence determines F uniquely [57], so there is little loss of generality by parametrizing the imaging problem in terms of the moments. Expanding $\exp(-i\hat{k}X)$ in the Taylor series $\sum_{q=0}^\infty (-i\hat{k})^q X^q/q!$, I can rewrite Eq. (2.2) as

$$\rho_1 = \sum_{q=0}^\infty \sum_{p=0}^\infty \theta_{q+p}\frac{(-i\hat{k})^q}{q!}|\psi\rangle\langle\psi|\frac{(i\hat{k})^p}{p!}. \tag{3.2}$$

Assume that the support of $F(X|\theta)$ has an infinite number of points, such that $\int dX F(X|\theta)\mathcal{P}^2(X) > 0$ for any nonzero polynomial $\mathcal{P}$, and the Hankel matrix θ_{q+p} is positive-definite [57]. The Cholesky factorization can then be used to write

$$\theta_{q+p} = \sum_{r=0}^\infty \Lambda_{qr}\Lambda_{pr}, \tag{3.3}$$

where Λ is a real lower-triangular matrix with strictly positive diagonal elements [58]. Equation (3.2) becomes

$$\rho_1 = \sum_{r=0}^\infty A_r|\psi\rangle\langle\psi|A_r^\dagger, \quad A_r \equiv \sum_{q=0}^\infty \Lambda_{qr}\frac{(-i\hat{k})^q}{q!}, \tag{3.4}$$

where $\{A_r\}$ are Choi-Kraus operators [42] and $\dagger$ denotes the Hermitian conjugate. It can be shown via purification [59] that an upper bound on the QFI is

$$K(\rho_1) \leqslant \tilde{K}, \quad \tilde{K}_{\mu\nu} = 4\,\mathrm{Re}(B_\mu B_\nu + C_{\mu\nu}), \tag{3.5}$$

$$B_\mu \equiv \sum_{r=0}^\infty \langle\psi|A_r^\dagger A_{r,\mu}|\psi\rangle, \tag{3.6}$$

$$C_{\mu\nu} \equiv \sum_{r=0}^\infty \langle\psi|A_{r,\nu}^\dagger A_{r,\mu}|\psi\rangle. \tag{3.7}$$

Defining the positive-semidefinite matrix

$$\Pi_{pq} \equiv \frac{1}{p!q!}\langle\psi|(i\hat{k})^p(-i\hat{k})^q|\psi\rangle = \frac{i^{p-q}}{p!q!}\int dk|\Psi(k)|^2k^{p+q}, \tag{3.8}$$

which consists of the OTF moments, I obtain

$$B_\mu = \operatorname{tr}\Pi\Lambda\Lambda_{,\mu}^\top, \quad C_{\mu\nu} = \operatorname{tr}\Pi\Lambda_{,\mu}\Lambda_{,\nu}^\top, \tag{3.9}$$

where $\top$ denotes the transpose. Assume that the OTF magnitude is even, viz., $|\Psi(k)|^2 = |\Psi(-k)|^2$, such that Π is real and symmetric ($\Pi = \Pi^\top$), and B_μ and $C_{\mu\nu}$ are also real. To evaluate B_μ, first note that

$$\begin{aligned} B_\mu &= \operatorname{tr}\Pi\Lambda\Lambda_{,\mu}^\top = \operatorname{tr}(\Pi\Lambda\Lambda_{,\mu}^\top)^\top = \operatorname{tr}\Lambda_{,\mu}\Lambda^\top\Pi^\top \\ &= \operatorname{tr}\Pi^\top\Lambda_{,\mu}\Lambda^\top = \operatorname{tr}\Pi\Lambda_{,\mu}\Lambda^\top. \end{aligned} \tag{3.10}$$

Then the normalization of ρ_1 can be used to show

$$\operatorname{tr}\rho_1 = \operatorname{tr}\Pi\Lambda\Lambda^\top = 1, \tag{3.11}$$

$$\operatorname{tr}\rho_{1,\mu} = \operatorname{tr}\Pi\Lambda_{,\mu}\Lambda^\top + \operatorname{tr}\Pi\Lambda\Lambda_{,\mu}^\top = 2B_\mu = 0. \tag{3.12}$$

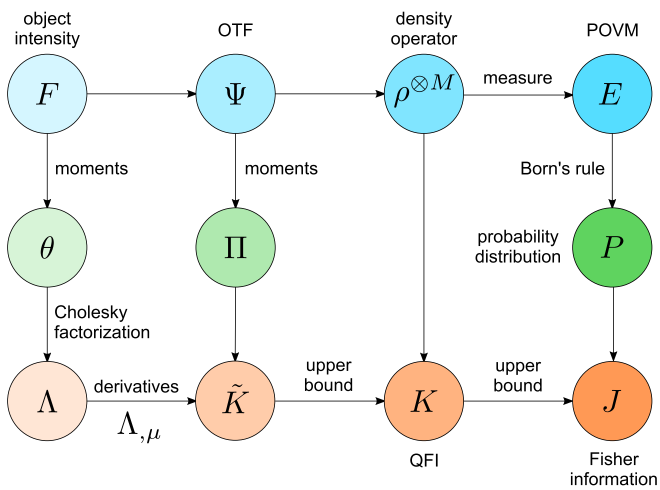

FIG. 2. Relationships among the key quantities in this work.

Hence

$$\tilde{K}_{\mu\nu} = 4C_{\mu\nu} = 4\,\mathrm{tr}\,\Pi\Lambda_{,\mu}\Lambda_{,\nu}^{\top}. \tag{3.13}$$

Figure 2 summarizes the relationships among the key quantities in this work.

As the right-hand side of Eq. (3.13) consists of infinite sums, their convergence is needed for $\tilde{K}$ to be a nontrivial upper bound on the QFI. Appendix B proves that $|\tilde{K}_{\mu\nu}| < \infty$ if $|\Psi(k)|^2$ is bandlimited or Gaussian ($\propto \exp[-k^2/(2\beta^2)]$) and $F(X|\theta)$ is any probability density with compact support in the Szegő class [60] or Gaussian ($\propto \exp[-X^2/(2\Delta^2)]$). If both are Gaussian, a further condition is $\beta\Delta < 1/2$. These are sufficient conditions but already quite general; $\tilde{K}$ may converge under more relaxed conditions.

IV. QUANTUM BOUND IN THE SUBDIFFRACTION REGIME

Although the QFI and its upper bound $\tilde{K}$ are functions of infinitely many parameters in general, the goal of this work is to show that $\tilde{K}_{\mu\mu}$ obeys a universal behavior when the parameters correspond to a subdiffraction regime. Let $\Delta > 0$ be a characteristic width of $F(X|\theta)$ around $X = 0$, such that $\theta_\mu = O(\Delta^\mu)$, where the big O notation denotes terms on the order of the argument and is defined by

$$\lim_{\Delta\to 0}\left|\frac{O[f(\Delta)]}{f(\Delta)}\right| < \infty. \tag{4.1}$$

Recall that X has been normalized with respect to the magnification factor and OTF bandwidth; the subdiffraction regime can therefore be defined by $\Delta \ll 1$ [3,4].

The dependence of the Cholesky factor Λ on θ can be studied via the recursive relation [61]

$$\Lambda_{qr} = \begin{cases} \sqrt{\theta_{2q} - \sum_{s=0}^{q-1}(\Lambda_{qs})^2}, & r = q,\\ \left(\theta_{q+r} - \sum_{s=0}^{r-1}\Lambda_{qs}\Lambda_{rs}\right)/\Lambda_{rr}, & r < q,\\ 0, & r > q,\end{cases} \tag{4.2}$$

starting from $\Lambda_{00} = \sqrt{\theta_0} = 1$. Equation (4.2) can be differentiated to give

$$\Lambda_{qr,\mu} = \begin{cases} \left(\delta_\mu^{2q} - 2\sum_{s=0}^{q-1}\Lambda_{qs}\Lambda_{qs,\mu}\right)/(2\Lambda_{qq}), & r = q,\\ \left[\delta_\mu^{q+r} - \sum_{s=0}^{r-1}(\Lambda_{qs,\mu}\Lambda_{rs} + \Lambda_{qs}\Lambda_{rs,\mu}) - \Lambda_{qr}\Lambda_{rr,\mu}\right]/\Lambda_{rr}, & r < q,\\ 0, & r > q,\end{cases} \tag{4.3}$$

where δ_b^a is the Kronecker delta. Since the diagonal elements of the Cholesky factor Λ are all strictly positive, all Λ and $\Lambda_{,\mu}$ elements are finite, and a dimensional analysis of Eqs. (4.2) and (4.3) gives

$$\Lambda_{qr} = O(\Delta^q), \tag{4.4}$$

$$\Lambda_{qr,\mu} = O(\Delta^{q-\mu}). \tag{4.5}$$

Inspecting the dependence of the Λ elements on a given θ_μ with even μ, starting from the upper-left corner and going row by row, one can see that the dependence does not appear until the diagonal element Λ_{qq} with $q = \mu/2$. In other words,

$$\Lambda_{qr,\mu} = \begin{cases} 0, & q \leqslant \mu/2, r < \mu/2,\\ 1/(2\Lambda_{qq}) = O(\Delta^{-\mu/2}), & q = r = \mu/2,\\ o(\Delta^{-\mu/2}), & q > \mu/2,\end{cases} \tag{4.6}$$

where the small o notation denotes terms that are asymptotically negligible relative to the argument and is defined by

$$\lim_{\Delta\to 0}\left|\frac{o(f(\Delta))}{f(\Delta)}\right| = 0. \tag{4.7}$$

Thus only one element in $\Lambda_{,\mu}$ is $O(\Delta^{-\mu/2})$, and the rest of the elements are all in higher orders. I can then express Eq. (3.13) as

$$\tilde{K}_{\mu\mu} = 4\sum_{s,t}\Pi_{st}\sum_r \Lambda_{tr,\mu}\Lambda_{sr,\mu} \tag{4.8}$$

$$= 4\Pi_{qq}(\Lambda_{qq,\mu})^2 + 4\sum_{s+t>\mu}\Pi_{st}\sum_r\Lambda_{tr,\mu}\Lambda_{sr,\mu}, \quad q = \frac{\mu}{2}. \tag{4.9}$$

Recall that $\hat{k}$ has been normalized with respect to the OTF width, and usual OTFs, such as bandlimited and Gaussian functions, have finite moments. Thus $\Pi_{st} = O(1)$, only $4\Pi_{qq}(\Lambda_{qq,\mu})^2$ is $O(\Delta^{-\mu})$, and the rest of the terms on the right-hand side of Eq. (4.9) are all $o(\Delta^{-\mu})$. Assuming $\tilde{K}_{\mu\mu} < \infty$, the infinite sum in Eq. (4.9) converges to $o(\Delta^{-\mu})$, and $\tilde{K}_{\mu\mu}$

can be approximated as

$$\tilde{K}_{\mu\mu} = O(\Delta^{-\mu}) \approx 4\Pi_{qq}(\Lambda_{qq,\mu})^2 = \frac{\langle\psi|\,\hat{k}^{2q}\,|\psi\rangle}{q!^2(\Lambda_{qq})^2}, \quad q = \frac{\mu}{2}. \tag{4.10}$$

If μ is odd, the dependence of Λ on a given θ_μ starts to appear only on the row $q = (\mu+1)/2$ in the elements $\Lambda_{q\,q-1}$ and Λ_{qq}. Specifically,

$$\Lambda_{qr,\mu} = \begin{cases} 0, & q \leqslant (\mu+1)/2, r < (\mu-1)/2, \\ 1/(\Lambda_{rr}) = O(\Delta^{-(\mu-1)/2}), & q = (\mu+1)/2, r = (\mu-1)/2, \\ -\Lambda_{q\,q-1}/(\Lambda_{qq}\Lambda_{q-1\,q-1}) = O(\Delta^{-(\mu-1)/2}), & q = r = (\mu+1)/2, \\ o(\Delta^{-(\mu-1)/2}), & q > (\mu+1)/2. \end{cases} \tag{4.11}$$

Now there are two $O(\Delta^{-(\mu-1)/2})$ leading-order terms in $\Lambda_{qr,\mu}$. I can express Eq. (3.13) as

$$\tilde{K}_{\mu\mu} = 4\Pi_{qq}[(\Lambda_{q\,q-1,\mu})^2 + (\Lambda_{qq,\mu})^2] + 4\sum_{s+t>\mu+1}\Pi_{st}\sum_r \Lambda_{tr,\mu}\Lambda_{sr,\mu}, \quad q = \frac{\mu+1}{2}, \tag{4.12}$$

where $4\Pi_{qq}[(\Lambda_{q\,q-1,\mu})^2 + (\Lambda_{qq,\mu})^2] = O(\Delta^{-(\mu-1)})$ and the rest of the terms are all $o(\Delta^{-(\mu-1)})$. Assuming again $\tilde{K}_{\mu\mu} < \infty$, I obtain

$$\begin{aligned}\tilde{K}_{\mu\mu} &= O(\Delta^{-(\mu-1)}) \approx 4\Pi_{qq}[(\Lambda_{q\,q-1,\mu})^2 + (\Lambda_{qq,\mu})^2] \\ &= \frac{4\,\langle\psi|\,\hat{k}^{2q}\,|\psi\rangle}{q!^2(\Lambda_{q-1\,q-1})^2}\left[1 + \left(\frac{\Lambda_{q\,q-1}}{\Lambda_{qq}}\right)^2\right], \quad q = \frac{\mu+1}{2}.\end{aligned} \tag{4.13}$$

Equation (4.10) for even μ and Eq. (4.13) for odd μ can be summarized as

$$J_{\mu\mu}(P) \leqslant NK_{\mu\mu}(\rho_1) \leqslant N\tilde{K}_{\mu\mu} = NO(\Delta^{-2\lfloor\mu/2\rfloor}), \tag{4.14}$$

which sets a lower bound on the mean-square error MSE_μ of any unbiased estimator of a moment θ_μ via the Cramér-Rao bound $\mathrm{MSE}_\mu \geqslant (J^{-1})_{\mu\mu} \geqslant 1/J_{\mu\mu}$ [51].

V. DISCUSSION

Equations (4.10), (4.13), and (4.14) are the central results of this work. The scaling of Eq. (4.14) with respect to Δ matches the performance of SPADE for moment estimation evaluated in Refs. [3,4]. The Fisher information for direct imaging is $J_{\mu\mu} = NO(1)$ in the subdiffraction regime, so substantial improvements can be obtained for $\mu \geqslant 2$ [3,4]. For $\mu = 1, 2$, the inverse of Eq. (4.14) also matches an $O(\Delta^{2\mu-2})/N$ quantum error bound computed in Appendix C via the convexity of QFI.

For a more sobering perspective, consider the signal-to-noise ratio (SNR), defined here as $\theta_\mu^2 = O(\Delta^{2\mu})$ divided by the mean-square error. Equation (4.14) then suggests that a quantum limit on the SNR is

$$\mathrm{QSNR}_\mu \equiv N\tilde{K}_{\mu\mu}\theta_\mu^2 = NO(\Delta^{2\lceil\mu/2\rceil}). \tag{5.1}$$

While it remains a significant improvement over the $NO(\Delta^{2\mu})$ SNR for direct imaging, Eq. (5.1) still decreases for smaller Δ, especially for higher moments, and decays in a roughly exponential fashion with increasing μ for a given Δ in the subdiffraction regime, as shown more carefully in Appendix D. This difficulty with higher moments is known in the context of SPADE [3,4], but the quantum limit here proves that it is fundamental for any measurement.

Although Eq. (4.14) assumes one-dimensional imaging, previous studies of two-dimensional imaging in quantum estimation theory [3,4,10,21] show no new surprises, and it is reasonable to conjecture that the quantum limit on the Fisher information becomes $NO(\Delta^{-2\lfloor|\mu|/2\rfloor})$—the same as the SPADE performance—where $|\mu| = \sum_j \mu_j$ is the total moment order [3,4].

Unlike Zhou and Jiang's Theorem 1 in Ref. [21], the quantum bound here does not depend on the POVM and is more amenable to approximation or numerical computation. The scaling of Eq. (4.14) with Δ for odd moments is also tighter than that suggested by their Theorem 1. Furthermore, their Theorem 3 makes a questionable assumption about the optimal POVM. Appendix E presents a review of Ref. [21] and highlights these issues. The use of the QFI here, on the other hand, guarantees that Eq. (4.14) holds for any POVM.

Beyond imaging, Eq. (2.2) also describes a quantum object subject to random displacements with unknown and possibly non-Gaussian statistics. Δ is then a measure of the displacement magnitude. The result here can therefore be applied to the estimation of diffusion parameters with a quantum probe in the weak-signal ($\Delta \ll 1$) regime, without the need to assume Gaussian statistics as in prior works [39–41]. Potential applications include magnetometry [39], optical interferometry [40], and optomechanical force sensing [41].

VI. CONCLUSION

I have proposed a general quantum limit to subdiffraction incoherent imaging in terms of moment estimation, going far beyond the simple example of two point sources in previous studies. This limit does not depend on the measurement and is also tight in terms of its scaling with the object size, thus setting a fundamental criterion of resolution for far-field incoherent imaging, with prime applications being observational astronomy and fluorescence microscopy.

Looking forward, many open problems still remain, including a more precise evaluation of the QFI, a more detailed comparison with SPADE, generalizations for more dimensions and other types of sources, derivations of tighter multiparameter quantum bounds, and an experimental demonstration of quantum-limited measurements for more general objects. As the light sources are classical and the measurements require only far-field linear optics and photon counting [3,4,7], a clear path towards practical applications of the quantum-inspired technology can be envisioned, with the quantum limit serving as the ultimate yardstick.

ACKNOWLEDGMENT

This work was supported by the Ministry of Education - Singapore Academic Research Fund Tier 1 Project Grant No. R-263-000-C06-112.

APPENDIX A: QUANTUM BOUNDS FOR THERMAL STATES

1. A bound on the QFI

Let $\{a_j\}$ be the bosonic annihilation operators with respect to a set of optical modes and

$$\rho = \int D\alpha \Phi(\alpha) |\alpha\rangle \langle\alpha| , \tag{A1}$$

where $\alpha \equiv (\alpha_1, \alpha_2, \dots)^\top$ is a column vector of complex amplitudes, $D\alpha \equiv \prod_j d^2\alpha_j$, Φ is the Glauber-Sudarshan distribution, and $|\alpha\rangle$ is a coherent state that satisfies $a_j |\alpha\rangle = \alpha_j |\alpha\rangle$ [1]. For a thermal state,

$$\Phi = \frac{1}{\det(\pi\Gamma)} \exp(-\alpha^\dagger \Gamma^{-1}\alpha), \tag{A2}$$

where $\Gamma > 0$ is the mutual coherence matrix. Helstrom has shown in Sec. V of Ref. [62] [see also Sec. VIII 6(b) of Ref. [1]] that the QFI is

$$K_{\mu\nu}(\rho) = \operatorname{tr} \Gamma_{,\mu} \Upsilon_\nu, \tag{A3}$$

where Υ_μ is a Hermitian matrix that satisfies

$$\Gamma_{,\mu} = \tfrac{1}{2}[(I+\Gamma)\Upsilon_\mu\Gamma + \Gamma\Upsilon_\mu(I+\Gamma)], \tag{A4}$$

and I is the identity matrix. The QFI is an upper bound on the Fisher information for any POVM [42], viz.,

$$J(P) \leqslant K(\rho^{\otimes M}) = MK(\rho). \tag{A5}$$

To obtain a simpler bound than Eqs. (A3)–(A5), diagonalize Γ in terms of its eigenvalues $\{\gamma_j\}$ and orthonormal eigenvectors $\{e_j\}$ as

$$\Gamma = \sum_j \gamma_j e_j e_j^\dagger, \tag{A6}$$

where $\{e_j\}$ includes vectors that support $\{\Gamma_{,\mu}\}$ and $\gamma_j \geqslant 0$. In terms of this basis, Eqs. (A3) and (A4) can be expressed as [62]

$$K_{\mu\nu}(\rho) = \sum_{j,l} \frac{2(e_j^\dagger \Gamma_{,\mu} e_l)(e_l^\dagger \Gamma_{,\nu} e_j)}{\gamma_j + \gamma_l + 2\gamma_j\gamma_l}. \tag{A7}$$

Let u be an arbitrary real vector and $\Gamma' \equiv \sum_\mu u_\mu \Gamma_{,\mu}$. Since $\Gamma_{,\mu}$ and therefore Γ' are Hermitian,

$$\begin{aligned}\sum_{\mu,\nu} u_\mu K_{\mu\nu}(\rho) u_\nu &= \sum_{j,l} \frac{2|e_j^\dagger \Gamma' e_l|^2}{\gamma_j+\gamma_l+2\gamma_j\gamma_l} \leqslant \sum_{j,l} \frac{2|e_j^\dagger\Gamma' e_l|^2}{\gamma_j+\gamma_l} \\ &= \epsilon \sum_{\mu,\nu} u_\mu K_{\mu\nu}(\Gamma) u_\nu, \end{aligned} \tag{A8}$$

where I have extended the definition of the QFI for any positive-definite matrix as

$$K_{\mu\nu}(\Gamma) = \frac{\operatorname{tr} \Gamma_{,\mu} L_\nu^{(\Gamma)}}{\operatorname{tr}\Gamma}, \tag{A9}$$

$$\Gamma_{,\mu} = \frac{1}{2}\big(L_\mu^{(\Gamma)}\Gamma + \Gamma L_\mu^{(\Gamma)}\big), \tag{A10}$$

and $L_\mu^{(\Gamma)}$ is a symmetric logarithmic derivative (SLD) of Γ. Equation (A8) results in

$$K(\rho) \leqslant \epsilon K(\Gamma), \quad MK(\rho) \leqslant NK(\Gamma), \tag{A11}$$

which can be combined with Eq. (A5) to give

$$J(P) \leqslant K(\rho^{\otimes M}) \leqslant NK(\Gamma). \tag{A12}$$

In other words, rather than computing $K(\rho)$ via Eqs. (A3) and (A4), one can compute a looser quantum bound given by Eqs. (A9) and (A10) in terms of the SLDs of Γ.

2. Ultraviolet and infrared limits

Let

$$\Gamma = \epsilon g, \quad \epsilon = \operatorname{tr}\Gamma, \quad \operatorname{tr} g = 1. \tag{A13}$$

In the limit $\epsilon \to 0$, $I + \Gamma \to I$, and the Υ_μ defined by Eq. (A4) becomes identical to the $L_\mu^{(\Gamma)}$ defined by Eq. (A10). Taking the ultraviolet limit $\epsilon \to 0$ while holding $N \equiv M\epsilon$ constant, I obtain

$$\lim_{\epsilon\to 0} MK_{\mu\nu}(\rho) = \lim_{\epsilon\to 0} M \operatorname{tr}\Gamma_{,\mu}\Upsilon_\nu = NK_{\mu\nu}(\Gamma), \tag{A14}$$

which means that, in the ultraviolet limit, the QFI approaches $NK(\Gamma)$, and the second inequality in Eq. (A12) becomes an equality.

One may also ask what happens in the opposite $\epsilon \to \infty$ "infrared" limit, which is more applicable to radio and microwave frequencies or scattered laser sources. Then $I + \Gamma \to \Gamma$, $\Upsilon_\mu \to \Gamma^{-1}\Gamma_{,\mu}\Gamma^{-1}$, and the $\epsilon \to \infty$ limit gives

$$\lim_{\epsilon\to\infty} K_{\mu\nu}(\rho^{\otimes M}) = M \operatorname{tr}\Gamma_{,\mu}\Gamma^{-1}\Gamma_{,\nu}\Gamma^{-1} = MJ(\Phi), \tag{A15}$$

which is equal to the classical Fisher information with respect to Φ [63]. Heterodyne detection is sufficient to achieve this quantum limit, as the Husimi distribution, which governs the heterodyne statistics, approaches Φ in the $\epsilon \to \infty$ limit. For any ϵ, the classical-simulation technique [64] can also be used to show that

$$K_{\mu\nu}(\rho^{\otimes M}) \leqslant MJ(\Phi), \tag{A16}$$

since Φ is positive.

3. Proof of Eq. (2.6) for any thermal state

Now suppose that ϵ does not depend on θ. The SLDs of Γ become the same as the SLDs of g, resulting in $K(\Gamma) = K(g)$. Following Refs. [2,3], Eq. (2.2) assumes that g is the density matrix of ρ_1 with respect to the basis $\{a_j^\dagger \lvert\text{vac}\rangle\}$. Since K is basis-independent, I can write

$$K(\Gamma) = K(g) = K(\rho_1), \tag{A17}$$

which can be combined with Eq. (A12) to give Eq. (2.6). Hence Eq. (2.6) in fact holds for any thermal state with arbitrary ϵ. The right-hand side of Eq. (2.6) is equal to the QFI for a thermal state in the ultraviolet limit, as shown by Eq. (A14); Sec. II arrives at the same result by making the $\epsilon \ll 1$ approximation at the beginning.

Consider the QFI per photon defined as

$$\kappa(\epsilon) \equiv \frac{K(\rho)}{\epsilon}, \tag{A18}$$

$$\begin{aligned}\kappa_{\mu\nu}(\epsilon) &= \sum_{j,l} \frac{2(e_j^\dagger g_{,\mu} e_l)(e_l^\dagger g_{,\nu} e_j)}{\lambda_j + \lambda_l + 2\epsilon\lambda_j\lambda_l} \\ &= \sum_{j,l} \frac{2\langle e_j\rvert \rho_{1,\mu} \lvert e_l\rangle \langle e_l\rvert \rho_{1,\nu} \lvert e_j\rangle}{\lambda_j + \lambda_l + 2\epsilon\lambda_j\lambda_l},\end{aligned} \tag{A19}$$

where $\{\lambda_j \equiv \gamma_j/\epsilon\}$ are the eigenvalues of g and also ρ_1 and $\{\lvert e_j\rangle \equiv \sum_l e_{jl} a_l^\dagger \lvert\text{vac}\rangle\}$ are the eigenkets of ρ_1. It is obvious that $\kappa(\epsilon)$ is a nonincreasing function of ϵ, viz.,

$$\kappa(\epsilon') \leqslant \kappa(\epsilon) \quad \text{if } \epsilon' > \epsilon, \tag{A20}$$

with the supremum achieved at $\lim_{\epsilon\to 0} \kappa(\epsilon) = K(\rho_1)$. This behavior is consistent with the explicit calculations of $\kappa(\epsilon)$ in Refs. [8,9] via other methods.

APPENDIX B: SUFFICIENT CONDITIONS FOR $|\tilde{K}_{\mu\nu}| < \infty$

Since the Π matrix given by Eq. (3.8) is positive-semidefinite, the $\tilde{K}$ matrix given by Eq. (3.13) is Gramian [58] and also positive-semidefinite, with

$$\tilde{K}_{\mu\mu} \geqslant 0, \quad |\tilde{K}_{\mu\nu}| \leqslant \sqrt{\tilde{K}_{\mu\mu}\tilde{K}_{\nu\nu}}. \tag{B1}$$

It suffices to prove $\tilde{K}_{\mu\mu} < \infty$ for any μ. Let

$$\tilde{\Pi} \equiv W^\top \Pi W, \quad \tilde{\Lambda}_{,\mu} \equiv W^{-1}\Lambda_{,\mu}, \tag{B2}$$

where W is a real invertible matrix. Then

$$\begin{aligned}\tilde{K}_{\mu\mu} &= 4\,\mathrm{tr}\,\Pi\Lambda_{,\mu}\Lambda_{,\mu}^\top = 4\,\mathrm{tr}\,\tilde{\Pi}\tilde{\Lambda}_{,\mu}\tilde{\Lambda}_{,\mu}^\top \\ &\leqslant 4\|\tilde{\Pi}\| \cdot \|\tilde{\Lambda}_{,\mu}\tilde{\Lambda}_{,\mu}^\top\|_1 = 4\|\tilde{\Pi}\| \cdot \|\tilde{\Lambda}_{,\mu}\|_2^2,\end{aligned} \tag{B3}$$

where $\|\cdot\|$ is the operator norm, $\|\cdot\|_1$ is the trace norm, and $\|\cdot\|_2$ is the Hilbert-Schmidt norm [65]. Thus $\tilde{K}_{\mu\mu} < \infty$ if

(1) $\tilde{\Pi}$ is bounded ($\|\tilde{\Pi}\| < \infty$), and

(2) $\tilde{\Lambda}_{,\mu}$ is Hilbert-Schmidt ($\|\tilde{\Lambda}_{,\mu}\|_2 < \infty$).

In the following, I assume

$$W_{qp} = w^q \sqrt{q!}\delta_p^q, \tag{B4}$$

where $0 < w < \infty$ is an adjustable constant to make the convergence conditions more general.

1. Sufficient conditions for $\|\tilde{\Pi}\| \leqslant \|\tilde{\Pi}\|_1 < \infty$

First I prove that $\tilde{\Pi}$ is in fact trace-class ($\|\tilde{\Pi}\|_1 < \infty$) and must therefore be bounded ($\|\tilde{\Pi}\| \leqslant \|\tilde{\Pi}\|_1 < \infty$) [65] if the OTF is bandlimited or Gaussian. In the latter case w should be chosen appropriately.

Since $\tilde{\Pi} \geqslant 0$, it is trace-class if

$$\|\tilde{\Pi}\|_1 = \mathrm{tr}\,\tilde{\Pi} = \sum_{q=0}^{\infty} \frac{w^{2q}}{q!} \int dk |\Psi(k)|^2 k^{2q} < \infty. \tag{B5}$$

Two cases are of interest:

(i) For a bandlimited OTF with support in $[-\beta, \beta]$ and $0 < \beta < \infty$,

$$\int dk |\Psi(k)|^2 k^{2q} \leqslant \beta^{2q}, \tag{B6}$$

$$\mathrm{tr}\,\tilde{\Pi} \leqslant \sum_{q=0}^{\infty} \frac{(w\beta)^{2q}}{q!} = \exp[(w\beta)^2], \tag{B7}$$

which converges for any w and β.

(ii) For a Gaussian OTF with standard deviation β [66],

$$\int dk |\Psi(k)|^2 k^{2q} = \frac{(2q)!}{q!2^q}\beta^{2q}, \tag{B8}$$

$$\mathrm{tr}\,\tilde{\Pi} = \sum_{q=0}^{\infty} \frac{(2q)!}{q!^2 2^q}(w\beta)^{2q}, \tag{B9}$$

which converges if $w\beta < 1/\sqrt{2}$ according to the ratio test [67]. Thus I should choose a w that satisfies

$$w < \frac{1}{\sqrt{2}\beta}. \tag{B10}$$

2. Sufficient conditions for $\|\tilde{\Lambda}_{,\mu}\|_2 < \infty$

Next I prove that $\tilde{\Lambda}_{,\mu}$ is Hilbert-Schmidt if $F(X|\theta)$ is any probability density with compact support in the Szegő class [60,68,69] or Gaussian. In the latter case, w should also be chosen appropriately.

Noting that $\Lambda_{,\mu}$ is lower-triangular, the Hilbert-Schmidt norm is given by

$$\begin{aligned}\|\tilde{\Lambda}_{,\mu}\|_2^2 &= \mathrm{tr}\,\tilde{\Lambda}_{,\mu}\tilde{\Lambda}_{,\mu}^\top = \sum_{q=0}^{\infty} \frac{1}{q!w^{2q}} \sum_{r=0}^{q} (\Lambda_{qr,\mu})^2 \\ &= \sum_{q=0}^{\infty} \frac{\eta_q}{q!w^{2q}},\end{aligned} \tag{B11}$$

$$\eta_q \equiv \sum_{r=0}^{q} (\Lambda_{qr,\mu})^2. \tag{B12}$$

For convenience, I normalize the object-plane coordinate X with respect to the object characteristic width $0 < \Delta < \infty$ as $X = x\Delta$, such that

$$\theta_\mu = \int dX F(X|\theta) X^\mu = \phi_\mu \Delta^\mu, \tag{B13}$$

$$\phi_\mu \equiv \int dx f(x|\theta) x^\mu, \tag{B14}$$

$$f(x|\theta) \equiv \Delta F(x\Delta|\theta), \tag{B15}$$

and ϕ_μ and $f(x|\theta)$ are independent of Δ. Define the Hankel matrix with respect to θ as

$$\Theta_{qp} = \theta_{q+p}, \tag{B16}$$

and the normalized Hankel matrix as

$$\Xi_{qp} = \phi_{q+p}. \tag{B17}$$

Define also the lower-triangular Cholesky factors Λ and V by

$$\Theta = \Lambda\Lambda^\top, \tag{B18}$$

$$\Xi = VV^\top. \tag{B19}$$

Then the matrices are related by

$$\Theta = D\Xi D, \quad \Lambda = DV, \quad D_{qp} \equiv \Delta^q \delta_p^q. \tag{B20}$$

In particular,

$$\Lambda_{qr} = \Delta^q V_{qr} = O(\Delta^q), \tag{B21}$$

which verifies Eq. (4.4). A formula for $\Lambda_{qr,\mu}$ is [70]

$$\Lambda_{qr,\mu} = \sum_{s=0}^{q} \Lambda_{qs} T_{sr} \left(\Lambda_{(q)}^{-1}\Theta_{(q),\mu}\Lambda_{(q)}^{-\top}\right)_{sr}, \tag{B22}$$

$$T_{sr} \equiv \begin{cases} 0, & s < r, \\ 1/2, & s = r, \\ 1, & s > r, \end{cases} \tag{B23}$$

where the subscript (q) denotes the $(q+1)$-by-$(q+1)$ upper-left submatrix, viz.,

$$\Lambda_{(q)rs} = \Lambda_{rs}, \quad 0 \leqslant r \leqslant q, \quad 0 \leqslant s \leqslant q, \tag{B24}$$

$$\Lambda_{(q)}^{-1} = (\Lambda_{(q)})^{-1}, \quad \Lambda_{(q)}^{-\top} = [(\Lambda_{(q)})^{-1}]^\top. \tag{B25}$$

Since

$$\Theta_{qr,\mu} = \delta_\mu^{q+r}, \tag{B26}$$

$\Theta_{(q),\mu} = 0$ if $q < \lceil \mu/2 \rceil$, and Eq. (B22) gives

$$\Lambda_{qr,\mu} = 0 \quad \text{if } q < \left\lceil \frac{\mu}{2} \right\rceil, \tag{B27}$$

which is consistent with Eqs. (4.6) and (4.11). Suppressing the subscript (q) for clarity, I can also write

$$D^{-1}\Theta_{,\mu}D^{-1} = \Delta^{-\mu}\Theta_{,\mu}, \tag{B28}$$

$$\Lambda^{-1}\Theta_{,\mu}\Lambda^{-\top} = V^{-1}D^{-1}\Theta_{,\mu}D^{-1}V^{-\top} = \Delta^{-\mu}Q, \tag{B29}$$

$$Q \equiv V^{-1}\Theta_{,\mu}V^{-\top}. \tag{B30}$$

Equation (B22) becomes

$$\Lambda_{qr,\mu} = \Delta^{q-\mu}\sum_{s=0}^{q} V_{qs}T_{sr}Q_{sr} = O(\Delta^{q-\mu}), \tag{B31}$$

which verifies Eq. (4.5). Applying the Cauchy-Schwartz inequality to Eq. (B31), I obtain

$$\begin{aligned} (\Lambda_{qr,\mu})^2 &\leqslant \Delta^{2q-2\mu}\left[\sum_{s=0}^{q}(V_{qs})^2\right]\left[\sum_{s=0}^{q}(T_{sr}Q_{sr})^2\right] \\ &= \Delta^{2q-2\mu}\phi_{2q}\sum_{s=0}^{q}(T_{sr}Q_{sr})^2. \end{aligned} \tag{B32}$$

This leads to an upper bound on Eq. (B12) given by

$$\eta_q \leqslant \Delta^{2q-2\mu}\phi_{2q}\sum_{r=0}^{q}\sum_{s=0}^{q}(T_{sr}Q_{sr})^2. \tag{B33}$$

To simplify the double sum, note that Q as defined by Eq. (B30) is symmetric with $Q_{rs} = Q_{sr}$, so it can be shown that [71]

$$\begin{aligned} \|Q\|_2^2 &= \sum_{r=0}^{q}\sum_{s=0}^{q}(Q_{sr})^2 = \sum_{s=0}^{q}(Q_{ss})^2 + 2\sum_{r=0}^{q}\sum_{s=r+1}^{q}(Q_{sr})^2 \\ &\geqslant 2\sum_{r=0}^{q}\sum_{s=0}^{q}(T_{sr}Q_{sr})^2, \end{aligned} \tag{B34}$$

leading to

$$\eta_q \leqslant \frac{\Delta^{2q-2\mu}\phi_{2q}}{2}\|Q\|_2^2. \tag{B35}$$

With Eq. (B30), $\|Q\|_2$ can be bounded as

$$\|Q\|_2 \leqslant \left\|V_{(q)}^{-1}\right\|^2 \cdot \|\Theta_{(q),\mu}\|_2 \leqslant \left\|\Xi_{(q)}^{-1}\right\|\sqrt{\mu+1}, \tag{B36}$$

where I have restored the subscript (q) for emphasis and used the facts [58,65]

$$\|AB\|_2 \leqslant \|A\| \cdot \|B\|_2, \tag{B37}$$

$$\left\|V_{(q)}^{-1}\right\| = \left\|V_{(q)}^{-\top}\right\| = \left\|V_{(q)}^{-\top}V_{(q)}^{-1}\right\|^{1/2} = \left\|\Xi_{(q)}^{-1}\right\|^{1/2}, \tag{B38}$$

$$\|\Theta_{(q),\mu}\|_2^2 = \sum_{r=0}^{q}\sum_{s=0}^{q}\left(\delta_\mu^{r+s}\right)^2 = \sum_{r=0}^{q}\sum_{s=0}^{q}\delta_\mu^{r+s} \leqslant \mu+1. \tag{B39}$$

Combining Eq. (B11), (B27), (B35), and (B36), I obtain

$$\|\tilde\Lambda_{,\mu}\|_2^2 \leqslant \frac{\mu+1}{2}\Delta^{-2\mu}\sum_{q=\lceil\mu/2\rceil}^{\infty}\zeta_q, \tag{B40}$$

$$\zeta_q \equiv \frac{\phi_{2q}}{q!}\left(\frac{\Delta}{w}\right)^{2q}\left\|\Xi_{(q)}^{-1}\right\|^2. \tag{B41}$$

Since Ξ and therefore its submatrix $\Xi_{(q)}$ are positive-definite [58], $\|\Xi_{(q)}^{-1}\|$ is the largest eigenvalue of $\Xi_{(q)}^{-1}$, which is equal to the inverse of the smallest eigenvalue of $\Xi_{(q)}$. Let λ_q be the smallest eigenvalue of $\Xi_{(q)}$. The right-hand side of Eq. (B40) converges and $\tilde\Lambda_{,\mu}$ is Hilbert-Schmidt if it passes the ratio test

$$\lim_{q\to\infty}\left|\frac{\zeta_{q+1}}{\zeta_q}\right| = \lim_{q\to\infty}\frac{1}{q+1}\frac{\Delta^2}{w^2}\frac{\phi_{2q+2}}{\phi_{2q}}\frac{\lambda_q^2}{\lambda_{q+1}^2} < 1. \tag{B42}$$

Two cases are of interest:

(a) $f(x|\theta)$ is any probability density in the Szegő class with compact support within $[x_1, x_2]$, $|x_j| < \infty$ [60,69], viz.,

$$\mathcal{S} \equiv \int_{x_1}^{x_2} dx \frac{\ln f(x|\theta)}{\sqrt{(x-x_1)(x_2-x)}} > -\infty. \tag{B43}$$

For example, any strictly positive f is in the class, as there exists a δ such that $f \geqslant \delta > 0$ and $\ln f \geqslant \ln \delta > -\infty$, leading to

$$\mathcal{S} \geqslant \ln \delta \int_{x_1}^{x_2} \frac{dx}{\sqrt{(x-x_1)(x_2-x)}} = \pi \ln \delta > -\infty. \tag{B44}$$

If Eq. (B43) is satisfied, it is known [60,69] that, for $q \to \infty$, there exist constants $\Omega > 0$ and $0 < \tau < 1$ such that

$$\lambda_q \to \Omega \sqrt{q} \tau^q, \quad \frac{\lambda_q^2}{\lambda_{q+1}^2} \to \frac{1}{\tau^2}. \tag{B45}$$

Furthermore, since $x^2 \leqslant \max(|x_1|, |x_2|)^2$ for $x \in [x_1, x_2]$,

$$\begin{aligned}\phi_{2q+2} &= \int_{x_1}^{x_2} dx f(x|\theta) x^{2q+2} \\ &\leqslant \max(|x_1|, |x_2|)^2 \int_{x_1}^{x_2} dx f(x|\theta) x^{2q} \\ &= \max(|x_1|, |x_2|)^2 \phi_{2q}.\end{aligned} \tag{B46}$$

The left-hand side of Eq. (B42) can therefore be bounded as

$$\lim_{q\to\infty} \left|\frac{\zeta_{q+1}}{\zeta_q}\right| \leqslant \lim_{q\to\infty} \frac{\Delta^2 \max(|x_1|, |x_2|)^2}{(q+1) w^2 \tau^2}, \tag{B47}$$

which approaches zero and passes the ratio test given by Eq. (B42) for any w, Δ, and $|x_j|$. Beyond the Szegő class, Eq. (B42) is also satisfied if $\lambda_q^2/\lambda_{q+1}^2 = o(q)$, or if $\lambda_q^2/\lambda_{q+1}^2 = O(q)$ and a small enough Δ/w is chosen.

(b) $f(x|\theta) \propto \exp(-x^2/2)$. Then the standard deviation of $F(X|\theta)$ is Δ and $\phi_{2q+2}/\phi_{2q} = 2q+1$. It is known that [60,68]

$$\lambda_q \to \Omega q^{1/4} \tau^{\sqrt{q}}, \quad \frac{\lambda_q^2}{\lambda_{q+1}^2} \to 1. \tag{B48}$$

Equation (B42) becomes

$$\lim_{q\to\infty} \left|\frac{\zeta_{q+1}}{\zeta_q}\right| = \frac{2\Delta^2}{w^2} < 1, \tag{B49}$$

which is satisfied if

$$w > \sqrt{2}\Delta. \tag{B50}$$

3. Summary

To summarize, Appendix B 1 shows that $\tilde{\Pi}$ is trace-class if $|\Psi(k)|^2$ is one of the following: (i) bandlimited with any choice of w, or (ii) Gaussian with $w < 1/(\sqrt{2}\beta)$, while Appendix B 2 shows that $\tilde{\Lambda}_{,\mu}$ is Hilbert-Schmidt if $F(X|\theta)$ is one of the following: (a) in the Szegő class with any choice of w, or (b) Gaussian with $w > \sqrt{2}\Delta$.

Thus the choice of w becomes an issue only if both are Gaussian. To satisfy both (ii) and (b), the standard deviations should satisfy

$$\beta\Delta < \tfrac{1}{2}, \tag{B51}$$

such that a choice within $\sqrt{2}\Delta < w < 1/(\sqrt{2}\beta)$ is possible.

Taking $\Delta \ll 1$, $\beta = O(1)$, and $w = O(1)$, $\|\tilde{\Pi}\| = O(1)$ and the right-hand side of Eq. (B40) converges to $O(\Delta^{-2\lfloor\mu/2\rfloor})$ under the conditions above. Equation (B3) becomes

$$\tilde{K}_{\mu\mu} \leqslant O(\Delta^{-2\lfloor\mu/2\rfloor}), \tag{B52}$$

which is consistent with Eq. (4.14).

With a trace-class $\tilde{\Pi}$, $\tilde{\Lambda}_{,\mu}$ is said to be square-summable with respect to $\tilde{\Pi}$ if and only if $\tilde{K}_{\mu\mu}$ converges [65]. An operator is guaranteed to be square-summable if it is bounded, and may still be so even if it is unbounded [65]. As Hilbert-Schmidt operators are a subclass of bounded operators, requiring $\tilde{\Lambda}_{,\mu}$ to be Hilbert-Schmidt may be an overkill; more relaxed conditions for the convergence of $\tilde{K}_{\mu\mu}$ may exist. Choosing a different scaling matrix W can also lead to other conditions.

APPENDIX C: QUANTUM BOUNDS VIA CONVEXITY AND CLASSICAL SIMULATION

Discretize $F(X|\theta)$ as a distribution of point sources, such that

$$F(X|\theta) = \sum_s F_s \delta(X - X_s), \tag{C1}$$

$$\rho_1 = \sum_s F_s e^{-i\hat{k}X_s} |\psi\rangle \langle\psi| e^{i\hat{k}X_s}. \tag{C2}$$

First assume that $\{F_s\}$ are known. Denoting the QFI with respect to parameters $\{X_s\}$ as $K^{(X)}$, I can use the convexity of QFI [41,72,73] to write

$$K^{(X)}(\rho_1) \leqslant G, \tag{C3}$$

$$G \equiv \sum_s F_s K^{(X)}(e^{-i\hat{k}X_s} |\psi\rangle \langle\psi| e^{i\hat{k}X_s}), \tag{C4}$$

$$G_{st} = 4F_s \beta^2 \delta_s^t, \tag{C5}$$

$$\beta = \sqrt{\langle\psi|\hat{k}^2|\psi\rangle - (\langle\psi|\hat{k}|\psi\rangle)^2}. \tag{C6}$$

With

$$\theta_\mu = \sum_s F_s X_s^\mu, \quad H_{\mu s} \equiv \frac{\partial\theta_\mu}{\partial X_s} = F_s \mu X_s^{\mu-1}, \tag{C7}$$

I can transform the Cramér-Rao bounds back to the ones with respect to θ as

$$K(\rho_1)^{-1} \geqslant H G^{-1} H^\top, \tag{C8}$$

$$(HG^{-1}H^\top)_{\mu\nu} = \frac{\mu\nu\theta_{\mu+\nu-2}}{4\beta^2} = O(\Delta^{\mu+\nu-2}). \tag{C9}$$

Hence

$$\begin{aligned}[J(P)^{-1}]_{\mu\mu} &\geqslant \frac{[K(\rho_1)^{-1}]_{\mu\mu}}{N} \geqslant \frac{(HG^{-1}H^\top)_{\mu\mu}}{N} \\ &= \frac{\mu^2\theta_{2\mu-2}}{4N\beta^2} = \frac{O(\Delta^{2\mu-2})}{N}.\end{aligned} \tag{C10}$$

The scaling of this bound with respect to Δ is looser than that of the inverse of Eq. (4.14) for $\Delta \ll 1$ and $\mu > 2$ but does not rely on the $\Delta \ll 1$ approximation.

Yet another bound can be computed by treating $\{F_s\}$ as parameters and using the classical-simulation technique [64]:

$$K^{(F)}(\rho_1) \leqslant J^{(F)}(F), \tag{C11}$$

$$J_{st}^{(F)}(F) = \sum_u \frac{1}{F_u}\frac{\partial F_u}{\partial F_s}\frac{\partial F_u}{\partial F_t} = \frac{\delta_s^t}{F_s}, \tag{C12}$$

$$K(\rho_1)^{-1} \geqslant R J^{(F)}(F)^{-1} R^\top, \tag{C13}$$

$$R_{\mu s} \equiv \frac{\partial \theta_\mu}{\partial F_s} = X_s^\mu, \tag{C14}$$

$$[R J^{(F)}(F)^{-1} R^\top]_{\mu\nu} = \theta_{\mu+\nu} = O(\Delta^{\mu+\nu}). \tag{C15}$$

This proof is a straightforward generalization of Appendix D in Ref. [4]. The final result is

$$[J(P)^{-1}]_{\mu\mu} \geqslant \frac{[K(\rho_1)^{-1}]_{\mu\mu}}{N} \geqslant \frac{[R J^{(F)}(F)^{-1} R^\top]_{\mu\mu}}{N}$$
$$= \frac{\theta_{2\mu}}{N} = \frac{O(\Delta^{2\mu})}{N}, \tag{C16}$$

the scaling of which is unfortunately looser than those of Eqs. (4.14) and (C10) for $\Delta \ll 1$.

APPENDIX D: DECAY OF THE QUANTUM SNR FOR HIGHER MOMENTS

With Eq. (4.10), the quantum SNR given by Eq. (5.1) for an even $\mu = 2q$ can be expressed in terms of the normalized quantities defined by Eqs. (B13)–(B21) as

$$\mathrm{QSNR}_{2q} = N[\chi_q \Delta^{2q} + o(\Delta^{2q})], \tag{D1}$$

$$\chi_q = \frac{\langle\psi|\,\hat{k}^{2q}\,|\psi\rangle\,\phi_{2q}^2}{q!^2 (V_{qq})^2}, \tag{D2}$$

where ϕ_{2q} is a normalized object moment and V is the Cholesky factor of the normalized Hankel matrix Ξ. For a given Δ in the subdiffraction regime, the SNR as a function of q depends on not only Δ^{2q} but also the prefactor χ_q. Here I show that the sequence $\{\chi_q : q \in \mathbb{N}\}$ is bounded under benign conditions, so the SNR must decay with q at least as quickly as the exponential Δ^{2q}.

If the OTF is bandlimited or Gaussian with bandwidth $\beta < \infty$, Eqs. (B6) and (B8) give

$$\langle\psi|\,\hat{k}^{2q}\,|\psi\rangle \leqslant \frac{(2q)!}{q!2^q}\beta^{2q}. \tag{D3}$$

If the $f(x|\theta)$ given by Eq. (B15) has a compact support within $[-1, 1]$,

$$\phi_{2q} \leqslant 1. \tag{D4}$$

As the support has been assumed to contain an infinite number of points, $\Xi_{(q)} > 0$, and $V_{qq} > 0$ is an eigenvalue of the lower-triangular Cholesky factor $V_{(q)}$ [58]. Let v be the eigenvector of $V_{(q)}$ with eigenvalue V_{qq} and $v^\top v = 1$. Then

$$V_{qq}^2 = v^\top V_{(q)}^\top V_{(q)} v \geqslant \min_{v^\top v=1} v^\top V_{(q)}^\top V_{(q)} v = \lambda_q, \tag{D5}$$

where λ_q is the smallest eigenvalue of $V_{(q)}^\top V_{(q)}$ and also $V_{(q)} V_{(q)}^\top = \Xi_{(q)}$ [58], so $\lambda_q > 0$. Substituting Eqs. (D3)–(D5) into Eq. (D2) gives

$$\chi_q \leqslant \frac{(2q)!\beta^{2q}}{q!^3 2^q \lambda_q} \equiv \chi_q'. \tag{D6}$$

$\chi_q' < \infty$ for any finite q, and if $f(x|\theta)$ is in the Szegő class, λ_q obeys Eqs. (B45) as $q \to \infty$, leading to $\lim_{q\to\infty} \chi_q' = 0$. Hence $\{\chi_q' : q \in \mathbb{N}\}$ is a bounded sequence, so is $\{\chi_q : q \in \mathbb{N}\}$, and there exists a finite $\tilde{\chi}$ such that

$$\chi_q \leqslant \tilde{\chi} < \infty, \quad \mathrm{QSNR}_{2q} \approx N\chi_q\Delta^{2q} \leqslant N\tilde{\chi}\Delta^{2q}. \tag{D7}$$

A similar decay behavior of the quantum SNR for the odd moments can be shown via the same procedure.

APPENDIX E: REVIEW OF REF. [21]

Here I summarize the essential arguments in Ref. [21], using the notation and parametrization here and focusing on the one-photon state for simplicity. Rewrite Eq. (3.2) as

$$\rho_1 = \sum_{\nu=0}^{\infty} \theta_\nu \sigma_\nu, \tag{E1}$$

$$\sigma_\nu \equiv \sum_{q=0}^{\nu} \frac{(-i\hat{k})^q\,|\psi\rangle\,\langle\psi|\,(i\hat{k})^{\nu-q}}{q!(\nu-q)!}, \tag{E2}$$

such that the probability distribution for a measurement $E_1(\xi)$ obeys

$$\pi(\xi|\theta) = \mathrm{tr}\, E_1(\xi)\rho_1 = \sum_{\nu=0}^{\infty} \theta_\nu S_\nu(\xi), \tag{E3}$$

$$S_\nu(\xi) \equiv \mathrm{tr}\, E_1(\xi)\sigma_\nu = \sum_{q=0}^{\nu} \frac{\langle\psi|\,(i\hat{k})^{\nu-q} E_1(\xi)(-i\hat{k})^q\,|\psi\rangle}{q!(\nu-q)!}. \tag{E4}$$

The Fisher information for θ_μ becomes

$$J_{\mu\mu} = N\sum_\xi \frac{\pi_{,\mu}(\xi|\theta)^2}{\pi(\xi|\theta)} = N\sum_\xi \frac{S_\mu^2(\xi)}{\pi(\xi|\theta)}$$
$$= N\Delta^{-\mu}\sum_\xi \frac{\Delta^\mu |S_\mu(\xi)|}{\pi(\xi|\theta)}|S_\mu(\xi)|. \tag{E5}$$

If

$$\frac{\Delta^\mu |S_\mu(\xi)|}{\pi(\xi|\theta)} \leqslant c_\mu = O(1), \quad \sum_\xi |S_\mu(\xi)| \equiv d_\mu < \infty, \tag{E6}$$

then

$$J_{\mu\mu} \leqslant c_\mu d_\mu N\Delta^{-\mu} = NO(\Delta^{-\mu}), \tag{E7}$$

which is essentially Theorem 1 in Ref. [21]. To prove $c_\mu = O(1)$, note that $S_\mu(\xi) \neq 0$ must hold for $\pi_{,\mu}(\xi|\theta) \neq 0$, so the expansion in Eq. (E3) must contain at least the term $\theta_\mu S_\mu(\xi)$. In other words,

$$\pi(\xi|\theta) = O(\Delta^\alpha), \quad \alpha \leqslant \mu. \tag{E8}$$

Coupled with the proof of $|S_\mu(\xi)| < \infty$ in Ref. [21] and the fact $\pi(\xi|\theta) > 0$,

$$\frac{\Delta^\mu |S_\mu(\xi)|}{\pi(\xi|\theta)} = \frac{\Delta^\mu |S_\mu(\xi)|}{O(\Delta^\alpha)} = O(\Delta^{\mu-\alpha}). \tag{E9}$$

Reference [21] also proves $d_\mu < \infty$ under reasonable conditions.

Compared with Eqs. (4.10), (4.13), and (4.14), not only is the scaling of Eq. (E7) with Δ for odd moments less tight, the value of its prefactor $c_\mu d_\mu$ also depends on the measurement and does not seem easy to compute. Without a more concrete prefactor, it would not be possible to study the SNR as a function of μ for a given Δ like Appendix D and show that higher moments are more difficult to estimate, as the prefactor may increase quickly with μ.

Reference [21] further argues that the optimal POVM that maximizes the Fisher information for a given θ_μ should satisfy

$$E_1(\xi)(-i\hat{k})^q\,|\psi\rangle = 0 \quad \text{for } q < \left\lfloor \frac{\mu}{2} \right\rfloor, \tag{E10}$$

in order to obtain

$$S_\nu(\xi) = 0 \text{ for } \nu < 2\left\lfloor \frac{\mu}{2} \right\rfloor, \quad \pi(\xi|\theta) = O(\Delta^{2\lfloor \mu/2 \rfloor}). \tag{E11}$$

This leads to

$$\max_{E_1} J_{\mu\mu}(\theta) \overset{?}{=} N O(\Delta^{-2\lfloor \mu/2 \rfloor}), \tag{E12}$$

which is essentially their Theorem 3. This argument seems to be flawed, however: it is not clear that Eq. (E10) is a necessary condition for the optimal POVM. Although it leads to a scaling that is close to the one suggested by Eq. (E7), the scaling is not the only concern when evaluating $\max_{E_1} J_{\mu\mu}(\theta)$ at a specific θ; the prefactor also matters. There may exist a POVM that violates Eq. (E10) and obeys a worse overall scaling but gives a prefactor large enough to make the information higher at that specific θ. This would imply that the optimal POVM does not satisfy Eq. (E10), and Eq. (E12) does not follow from Eq. (E10).

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