Continuous Quantum Hypothesis Testing: Supplementary Material

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This document contains a derivation of the likelihood-ratio formula for continuous quantum measurements with Poissonian noise in Sec. I and an example of quantum optomechanical stochastic force detection in Sec. II.

I. LIKELIHOOD-RATIO FORMULA FOR CONTINUOUS POISSONIAN MEASUREMENTS

The completely positive map for a weak Poissonian measurement is given by

\[ J_j(\delta y)\rho = \sum_{\delta z=0,1} P(\delta y|\delta z) \left\{ \delta z c_j^{\dagger} \rho c_j^{\dagger} \delta t + (1 - \delta z) \left[ \rho - \frac{\delta t}{2} (c_j^{\dagger} c_j \rho + \rho c_j^{\dagger} c_j) \right] \right\}, \]

(1)

where \( \delta y, \delta z \in \{0, 1\} \) and

\[ P(\delta y|\delta z) = (1 - \delta y)(1 - \eta_j \delta z) + \eta_j \delta y \delta z \]

(2)

models the effect of imperfect quantum efficiency \( 0 < \eta_j \leq 1 \).

Rearranging terms [1],

\[ J_j(\delta y)\rho = \tilde{P}(\delta y) \left[ \rho + \frac{\delta t}{2} \left( 2 c_j^{\dagger} \rho c_j^{\dagger} - c_j^{\dagger} c_j \rho - \rho c_j^{\dagger} c_j \right) \right] + (\delta y - \alpha \delta t) \left( \frac{\eta_j}{\alpha} c_j^{\dagger} \rho c_j^{\dagger} - \rho \right), \]

(3)

\[ \tilde{P}(\delta y) \equiv (1 - \delta y)(1 - \alpha \delta t) + \delta y \alpha \delta t, \]

(4)

where \( \tilde{P}(\delta y) \) is a reference probability distribution and \( \alpha \) is an arbitrary positive number.

This gives

\[ \Lambda = \frac{\text{tr} f_1}{\text{tr} f_0}, \]

(5)

\[ df_j = dt \mathcal{L}_j f_j + \frac{dt}{2} \left( 2 c_j^{\dagger} f_j c_j^{\dagger} - c_j^{\dagger} c_j f_j - f_j c_j^{\dagger} c_j \right) + (dy - \alpha dt) \left( \frac{\eta_j}{\alpha} c_j^{\dagger} f_j c_j^{\dagger} - f_j \right). \]

(6)

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Equation (6) coincides with the quantum filtering equation for an unnormalized posterior density operator $f_j$ given Poissonian observations $dy$ [1]. Next, consider

\begin{align}
    d \text{tr} f_j &= \text{tr} df_j = (dy - \alpha dt) \left( \frac{\mu_j}{\alpha} - 1 \right) \text{tr} f_j, \\
    \mu_j &\equiv \frac{1}{\text{tr} f_j} \text{tr} \left( \eta_j c_j^\dagger c_j f_j \right),
\end{align}

where $\mu_j$ is the filtering estimate of the observable $\eta_j c_j^\dagger c_j$ assuming that the hypothesis $\mathcal{H}_j$ is true. Expanding $d \ln \text{tr} f_j$ in Taylor series,

\begin{align}
    d \ln \text{tr} f_j &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(\text{tr} f_j)^n} (d \text{tr} f_j)^n \\
    &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (dy - \alpha dt)^n \left( \frac{\mu_j}{\alpha} - 1 \right)^n. 
\end{align}

With $(dy - \alpha dt)^n = dy^n + o(dt)$ for $n \geq 2$ and $dy^n = dy$ for Poissonian observations,

\begin{align}
    d \ln \text{tr} f_j &= dy \ln \frac{\mu_j}{\alpha} - dt (\mu_j - \alpha), \\
    \ln \text{tr} f_j(T) &= \int_{t_0}^{T} dy \ln \frac{\mu_j}{\alpha} - \int_{t_0}^{T} dt (\mu_j - \alpha), \\
    \Lambda(T) &= \exp \left[ \int_{t_0}^{T} dy \ln \frac{\mu_1}{\mu_0} - \int_{t_0}^{T} dt (\mu_1 - \mu_0) \right].
\end{align}

II. QUANTUM OPTOMECHANICAL DETECTION OF A GAUSSIAN STOCHASTIC FORCE

Let $F = Cx$ be a classical force acting on a moving mirror with position operator $q$ and momentum $p$, and $x$ be a vectorial classical Gaussian stochastic process $x$ described by the Ito equation

\begin{align}
    dx = Ax dt + dW, \quad dW dW^T = B dt.
\end{align}

The mirror is assumed to be a harmonic oscillator with mass $m$ and frequency $\omega$ and part of an optical cavity pumped by a near-resonant continuous-wave laser. The phase quadrature of the cavity output is measured continuously by homodyne detection, with an observation process given by $dy$. For simplicity, I assume that the optical intracavity dynamics can be adiabatically eliminated, the phase modulation by the mirror motion is much smaller than $\pi/2$ radians, such that the homodyne detection is effectively measuring the mirror
position, and there is no excess decoherence. Under hypothesis $H_1$, the force is present and the quantum filtering equation for the unnormalized hybrid density operator $f_1(x,t)$ is then given by [2]

$$
d f_1 = dt L_1(x) f_1 + dt (2qf_1 q - q^2 f_1 - f_1 q^2), \tag{15}
$$

$$
L_1(x) f_1 \equiv -\frac{i}{\hbar} [H_1(x), f_1] + L_c(x) f_1, \tag{16}
$$

$$
H_1(x) \equiv p^2 + \frac{m\omega^2}{2} q^2 - q C x, \tag{17}
$$

$$
L_c(x) f_1 \equiv -\sum_{\mu} \frac{\partial}{\partial x_{\mu}} [(Ax_{\mu}) f_1] + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} (B_{\mu\nu} f_1), \tag{18}
$$

where $R$ is the measurement noise variance that depends on the laser intensity and the cavity properties and $L_c(x)$ is the forward Kolmogorov generator for the classical process $x$.

Under the null hypothesis $H_0$, the force is absent and the filtering equation for the oscillator density operator $f_0$ is

$$
d f_0 = dt L_0 f_0 + dt (2qf_0 q - q^2 f_0 - f_0 q^2), \tag{19}
$$

$$
L_0 f_0 \equiv -\frac{i}{\hbar} [H_0, f_0], \tag{20}
$$

$$
H_0 \equiv \frac{p^2}{2m} + \frac{m\omega^2}{2} q^2. \tag{21}
$$

These filtering equations can be transformed to equations for the Wigner functions of $f_j$:

$$
g_1(q,p,x,t) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} du (q - u/2 | f_1(x,t) | q + u/2) \exp(ipu/\hbar), \tag{22}
$$

$$
g_0(q,p,t) \equiv \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} du (q - u/2 | f_0(t) | q + u/2) \exp(ipu/\hbar), \tag{23}
$$

$$
d g_j = dt L'_j g_j + dt R q g_j, \tag{24}
$$

$$
L'_1 g_1 \equiv L_c(x) g_1 + dt \left[ -\frac{p}{m} \frac{\partial g_1}{\partial q} + (m\omega^2 q - C x) \frac{\partial g_1}{\partial p} + \frac{h^2}{8R} \frac{\partial^2 g_1}{\partial p^2} \right], \tag{25}
$$

$$
L'_0 g_0 \equiv -\frac{p}{m} \frac{\partial g_0}{\partial q} + m\omega^2 q \frac{\partial g_0}{\partial p} + \frac{h^2}{8R} \frac{\partial^2 g_0}{\partial p^2}, \tag{26}
$$

where $q$ and $p$ are now phase-space variables, $p$ is seen to suffer from measurement-back-action-induced diffusion, and $L'_1$ has the form of a forward Kolmogorov generator for a new
Gaussian process \( z = (q, p, x^T)^T \):

\[
\mathcal{L}_1' g_1(z, t) = -\sum_\mu \frac{\partial}{\partial z_\mu} [(J_1 z)_\mu g_1] + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial z_\mu \partial z_\nu} (S_{1\mu\nu} g_1),
\]

\[
J_1 \equiv \begin{pmatrix} 0 & 1/m & 0 \\ -m\omega^2 & 0 & C \\ 0 & 0 & A \end{pmatrix}, \quad S_1 \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & h^2/4R & 0 \\ 0 & 0 & B \end{pmatrix},
\]

with 0 denoting zero matrices. Similarly, under \( \mathcal{H}_0 \), we have \( z = (q, p)^T \) and

\[
\mathcal{L}_0' g_0(z, t) = -\sum_\mu \frac{\partial}{\partial z_\mu} [(J_0 z)_\mu g_0] + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial z_\mu \partial z_\nu} (S_{0\mu\nu} g_0),
\]

\[
J_0 \equiv \begin{pmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{pmatrix}, \quad S_0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & h^2/4R \end{pmatrix}.
\]

The Gaussian statistics mean that we can use Kalman-Bucy filters to compute the filtering estimates of the mirror position \( \mu_j \) given \( dy[2] \):

\[
dz_j' = J_j z_j' dt + \Gamma_j (dy - K_j z_j' dt), \quad K_j \equiv (1, 0, \ldots, 0),
\]

\[
\Gamma_j \equiv \Sigma_j K_j^T R^{-1},
\]

\[
\frac{d\Sigma_j}{dt} = J_j \Sigma_j + \Sigma_j J_j^T - \Sigma_j K_j^T R^{-1} K_j \Sigma_j + S_j,
\]

\[
\mu_j = K_j z_j',
\]

and the likelihood ratio becomes

\[
\Lambda(T) = \exp \left[ \int_{t_0}^{T} dy R (\mu_1 - \mu_0) - \int_{t_0}^{T} dt \frac{\mu_1^2}{2R} - \frac{\mu_2^2}{2R} \right].
\]

Given the Gaussian structure of the problem under each hypothesis, we can use known results about the Chernoff upper bounds for classical waveform estimation to bound the error probabilities [3]:

\[
P_{10} \leq \exp \left[ \mu(s) - s\gamma \right],
\]

\[
P_{01} \leq \exp \left[ \mu(s) + (1-s)\gamma \right],
\]

\[
\mu(s) = \frac{1}{2R} \int_{t_0}^{T} dt \left[ (1-s)\Sigma_1q(t) + s\Sigma_0q(t) - \tilde{\Sigma}_q(s,t) \right],
\]

where \( 0 \leq s \leq 1 \), \( \gamma \) is the threshold of the likelihood-ratio test, \( \Sigma_j q(t) \) is the \( q \) variance component of \( \Sigma_j \), which obeys Eq. (33), and \( \tilde{\Sigma}_q(s,t) \) is the variance of \( \sqrt{s}q_0 + \sqrt{1-s}q_1 \) for
a different filtering problem, in which observations of $\sqrt{s_0} + \sqrt{1-s_1}$ are made with noise variance $R$ and $q_j$ has the statistics of $q$ under $\mathcal{H}_j$, viz.,

$$\frac{d\tilde{\Sigma}}{dt} = \tilde{J}\tilde{\Sigma} + \tilde{\Sigma}\tilde{J}^T - \tilde{\Sigma}\tilde{K}^T R^{-1} \tilde{K}\tilde{\Sigma}^T + \tilde{S}, \quad (39)$$

$$\tilde{J} \equiv \begin{pmatrix} J_0 & 0 \\ 0 & J_1 \end{pmatrix}, \quad \tilde{S} \equiv \begin{pmatrix} S_0 & 0 \\ 0 & S_1 \end{pmatrix}, \quad \tilde{K} \equiv \begin{pmatrix} \sqrt{s} & 0 \\ 0 & \sqrt{1-s} \end{pmatrix}, \quad \tilde{\Sigma}_q \equiv \tilde{K}\tilde{\Sigma}\tilde{K}^T. \quad (40)$$

The tightest upper bounds are obtained by minimizing the bounds with respect to $s$. If $x$ and therefore $q$ are stationary, $\Sigma_j$ and $\tilde{\Sigma}$ will converge to steady states in the long-time limit, and the Chernoff bounds will decay exponentially with time.

