Continuous Quantum Hypothesis Testing: Supplementary Material

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This document contains a derivation of the likelihood-ratio formula for continuous quantum measurements with Poissonian noise in Sec. I and an example of quantum optomechanical stochastic force detection in Sec. II.

I. LIKELIHOOD-RATIO FORMULA FOR CONTINUOUS POISSONIAN MEASUREMENTS

The completely positive map for a weak Poissonian measurement is given by

$$\mathcal{J}_j(\delta y)\rho = \sum_{\delta z=0,1} P(\delta y|\delta z)\left\{\delta z c_j \rho c_j^\dagger \delta t + (1-\delta z)\left[\rho - \frac{\delta t}{2}(c_j^\dagger c_j \rho + \rho c_j^\dagger c_j)\right]\right\}, \tag{1}$$

where $\delta y, \delta z \in \{0,1\}$ and

$$P(\delta y|\delta z) = (1-\delta y)(1-\eta_j \delta z) + \eta_j \delta y \delta z \tag{2}$$

models the effect of imperfect quantum efficiency $0 < \eta_j \leq 1$. Rearranging terms [1],

$$\mathcal{J}_j(\delta y)\rho = \tilde{P}(\delta y)\left[\rho + \frac{\delta t}{2}\left(2c_j\rho c_j^\dagger - c_j^\dagger c_j \rho - \rho c_j^\dagger c_j\right) + (\delta y - \alpha\delta t)\left(\frac{\eta_j}{\alpha}c_j \rho c_j^\dagger - \rho\right)\right], \tag{3}$$

$$\tilde{P}(\delta y) \equiv (1-\delta y)(1-\alpha\delta t) + \delta y \alpha \delta t, \tag{4}$$

where $\tilde{P}(\delta y)$ is a reference probability distribution and α is an arbitrary positive number. This gives

$$\Lambda = \frac{\operatorname{tr} f_1}{\operatorname{tr} f_0}, \tag{5}$$

$$df_j = dt\mathcal{L}_j f_j + \frac{dt}{2}\left(2c_j f_j c_j^\dagger - c_j^\dagger c_j f_j - f_j c_j^\dagger c_j\right) + (dy - \alpha dt)\left(\frac{\eta_j}{\alpha}c_j f_j c_j^\dagger - f_j\right). \tag{6}$$

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Equation (6) coincides with the quantum filtering equation for an unnormalized posterior density operator f_j given Poissonian observations dy [1]. Next, consider

$$d\,\mathrm{tr}\, f_j = \mathrm{tr}\, df_j = (dy - \alpha dt)\left(\frac{\mu_j}{\alpha} - 1\right)\mathrm{tr}\, f_j, \tag{7}$$

$$\mu_j \equiv \frac{1}{\mathrm{tr}\, f_j}\,\mathrm{tr}\left(\eta_j c_j^\dagger c_j f_j\right), \tag{8}$$

where μ_j is the filtering estimate of the observable $\eta_j c_j^\dagger c_j$ assuming that the hypothesis $\mathcal{H}_j$ is true. Expanding $d\ln\mathrm{tr}\, f_j$ in Taylor series,

$$d\ln\mathrm{tr}\, f_j = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(\mathrm{tr}\, f_j)^n}(d\,\mathrm{tr}\, f_j)^n \tag{9}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(dy - \alpha dt)^n\left(\frac{\mu_j}{\alpha} - 1\right)^n. \tag{10}$$

With $(dy - \alpha dt)^n = dy^n + o(dt)$ for $n \geq 2$ and $dy^n = dy$ for Poissonian observations,

$$d\ln\mathrm{tr}\, f_j = dy\ln\frac{\mu_j}{\alpha} - dt\,(\mu_j - \alpha), \tag{11}$$

$$\ln\mathrm{tr}\, f_j(T) = \int_{t_0}^{T} dy\ln\frac{\mu_j}{\alpha} - \int_{t_0}^{T} dt\,(\mu_j - \alpha), \tag{12}$$

$$\Lambda(T) = \exp\left[\int_{t_0}^{T} dy\ln\frac{\mu_1}{\mu_0} - \int_{t_0}^{T} dt\,(\mu_1 - \mu_0)\right]. \tag{13}$$

II. QUANTUM OPTOMECHANICAL DETECTION OF A GAUSSIAN STOCHASTIC FORCE

Let $F = Cx$ be a classical force acting on a moving mirror with position operator q and momentum p, and x be a vectoral classical Gaussian stochastic process x described by the Ito equation

$$dx = Axdt + dW, \quad dWdW^T = Bdt. \tag{14}$$

The mirror is assumed to be a harmonic oscillator with mass m and frequency ω and part of an optical cavity pumped by a near-resonant continuous-wave laser. The phase quadrature of the cavity output is measured continuously by homodyne detection, with an observation process given by dy. For simplicity, I assume that the optical intracavity dynamics can be adiabatically eliminated, the phase modulation by the mirror motion is much smaller than $\pi/2$ radians, such that the homodyne detection is effectively measuring the mirror position, and there is no excess decoherence. Under hypothesis $\mathcal{H}_1$, the force is present and the quantum filtering equation for the unnormalized hybrid density operator $f_1(x,t)$ is then given by [2]

$$df_1 = dt\mathcal{L}_1(x)f_1 + \frac{dy}{2R}\left(qf_1 + f_1q\right) + \frac{dt}{8R}\left(2qf_1q - q^2f_1 - f_1q^2\right), \tag{15}$$

$$\mathcal{L}_1(x)f_1 \equiv -\frac{i}{\hbar}[H_1(x), f_1] + \mathcal{L}_c(x)f_1, \tag{16}$$

$$H_1(x) \equiv \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2 - qCx, \tag{17}$$

$$\mathcal{L}_c(x)f_1 \equiv -\sum_{\mu}\frac{\partial}{\partial x_\mu}\left[(Ax)_\mu f_1\right] + \frac{1}{2}\sum_{\mu,\nu}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\left(B_{\mu\nu}f_1\right), \tag{18}$$

where R is the measurement noise variance that depends on the laser intensity and the cavity properties and $\mathcal{L}_c(x)$ is the forward Kolmogorov generator for the classical process x. Under the null hypothesis $\mathcal{H}_0$, the force is absent and the filtering equation for the oscillator density operator f_0 is

$$df_0 = dt\mathcal{L}_0f_0 + \frac{dy}{2R}\left(qf_0 + f_0q\right) + \frac{dt}{8R}\left(2qf_0q - q^2f_0 - f_0q^2\right), \tag{19}$$

$$\mathcal{L}_0f_0 \equiv -\frac{i}{\hbar}[H_0, f_0], \tag{20}$$

$$H_0 \equiv \frac{p^2}{2m} + \frac{m\omega^2}{2}q^2. \tag{21}$$

These filtering equations can be transformed to equations for the Wigner functions of f_j:

$$g_1(q,p,x,t) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} du\langle q - u/2|f_1(x,t)|q + u/2\rangle \exp(ipu/\hbar), \tag{22}$$

$$g_0(q,p,t) \equiv \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} du\langle q - u/2|f_0(t)|q + u/2\rangle \exp(ipu/\hbar), \tag{23}$$

$$dg_j = dt\mathcal{L}'_jg_j + \frac{dy}{R}qg_j, \tag{24}$$

$$\mathcal{L}'_1g_1 \equiv \mathcal{L}_c(x)g_1 + dt\left[-\frac{p}{m}\frac{\partial g_1}{\partial q} + (m\omega_m^2q - Cx)\frac{\partial g_1}{\partial p} + \frac{\hbar^2}{8R}\frac{\partial^2 g_1}{\partial p^2}\right], \tag{25}$$

$$\mathcal{L}'_0g_0 \equiv -\frac{p}{m}\frac{\partial g_0}{\partial q} + m\omega^2q\frac{\partial g_0}{\partial p} + \frac{\hbar^2}{8R}\frac{\partial^2 g_0}{\partial p^2}, \tag{26}$$

where q and p are now phase-space variables, p is seen to suffer from measurement-back-action-induced diffusion, and $\mathcal{L}'_1$ has the form of a forward Kolmogorov generator for a new

Gaussian process $z = (q, p, x^T)^T$:

$$\mathcal{L}_1' g_1(z,t) = -\sum_\mu \frac{\partial}{\partial z_\mu}\left[(J_1 z)_\mu g_1\right] + \frac{1}{2}\sum_{\mu,\nu}\frac{\partial^2}{\partial z_\mu \partial z_\nu}\left(S_{1\mu\nu} g_1\right), \tag{27}$$

$$J_1 \equiv \left(\begin{array}{cc|c} 0 & 1/m & \mathbf{0} \\ -m\omega^2 & 0 & C \\ \hline \mathbf{0} & \mathbf{0} & A \end{array}\right), \quad S_1 \equiv \left(\begin{array}{cc|c} 0 & 0 & \mathbf{0} \\ 0 & \hbar^2/4R & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & B \end{array}\right), \tag{28}$$

with $\mathbf{0}$ denoting zero matrices. Similarly, under $\mathcal{H}_0$, we have $z = (q, p)^T$ and

$$\mathcal{L}_0' g_0(z,t) = -\sum_\mu \frac{\partial}{\partial z_\mu}\left[(J_0 z)_\mu g_0\right] + \frac{1}{2}\sum_{\mu,\nu}\frac{\partial^2}{\partial z_\mu \partial z_\nu}\left(S_{0\mu\nu} g_0\right), \tag{29}$$

$$J_0 \equiv \begin{pmatrix} 0 & 1/m \\ -m\omega^2 & 0 \end{pmatrix}, \quad S_0 \equiv \begin{pmatrix} 0 & 0 \\ 0 & \hbar^2/4R \end{pmatrix}. \tag{30}$$

The Gaussian statistics mean that we can use Kalman-Bucy filters to compute the filtering estimates of the mirror position μ_j given dy [2]:

$$dz_j' = J_j z_j' dt + \Gamma_j (dy - K_j z_j' dt), \quad K_j \equiv (1, 0, \ldots, 0), \tag{31}$$

$$\Gamma_j \equiv \Sigma_j K_j^T R^{-1}, \tag{32}$$

$$\frac{d\Sigma_j}{dt} = J_j \Sigma_j + \Sigma_j J_j^T - \Sigma_j K_j^T R^{-1} K_j \Sigma_j^T + S_j, \tag{33}$$

$$\mu_j = K_j z_j', \tag{34}$$

and the likelihood ratio becomes

$$\Lambda(T) = \exp\left[\int_{t_0}^T \frac{dy}{R}(\mu_1 - \mu_0) - \int_{t_0}^T \frac{dt}{2R}(\mu_1^2 - \mu_0^2)\right]. \tag{35}$$

Given the Gaussian structure of the problem under each hypothesis, we can use known results about the Chernoff upper bounds for classical waveform estimation to bound the error probabilities [3]:

$$P_{10} \le \exp\left[\mu(s) - s\gamma\right], \tag{36}$$

$$P_{01} \le \exp\left[\mu(s) + (1-s)\gamma\right], \tag{37}$$

$$\mu(s) = \frac{1}{2R}\int_{t_0}^T dt\left[(1-s)\Sigma_{1q}(t) + s\Sigma_{0q}(t) - \tilde{\Sigma}_q(s,t)\right], \tag{38}$$

where $0 \le s \le 1$, γ is the threshold of the likelihood-ratio test, $\Sigma_{jq}(t)$ is the q variance component of Σ_j, which obeys Eq. (33), and $\tilde{\Sigma}_q(s,t)$ is the variance of $\sqrt{s}q_0 + \sqrt{1-s}q_1$ for a different filtering problem, in which observations of $\sqrt{s}q_0 + \sqrt{1-s}q_1$ are made with noise variance R and q_j has the statistics of q under $\mathcal{H}_j$, viz.,

$$\frac{d\tilde{\Sigma}}{dt} = \tilde{J}\tilde{\Sigma} + \tilde{\Sigma}\tilde{J}^T - \tilde{\Sigma}\tilde{K}^T R^{-1}\tilde{K}\tilde{\Sigma}^T + \tilde{S}, \tag{39}$$

$$\tilde{J} \equiv \left(\begin{array}{c|c} J_0 & \mathbf{0} \\ \hline \mathbf{0} & J_1 \end{array}\right), \quad \tilde{S} \equiv \left(\begin{array}{c|c} S_0 & \mathbf{0} \\ \hline \mathbf{0} & S_1 \end{array}\right), \quad \tilde{K} \equiv \left(\begin{array}{cc|cc} \sqrt{s} & \mathbf{0} & \sqrt{1-s} & \mathbf{0} \end{array}\right), \quad \tilde{\Sigma}_q \equiv \tilde{K}\tilde{\Sigma}\tilde{K}^T. \tag{40}$$

The tightest upper bounds are obtained by minimizing the bounds with respect to s. If x and therefore q are stationary, Σ_j and $\tilde{\Sigma}$ will converge to steady states in the long-time limit, and the Chernoff bounds will decay exponentially with time.

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[2] M. Tsang, Phys. Rev. A **80**, 033840 (2009).

[3] H. L. Van Trees, *Detection, Estimation, and Modulation Theory. Part III* (Wiley, New York, 2001).