Supplemental Material for "Far-field Super-resolution of Thermal Electromagnetic Sources at the Quantum Limit"

Ranjith Nair[1,*] and Mankei Tsang[2]

[1]*Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117583*

[2]*Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117551*

(Dated: September 6, 2016)

This document contains supporting calculations for "Far-field Super-resolution of Thermal Electromagnetic Sources at the Quantum Limit". Included are (1) the details of the derivation of the QCRB for separation estimation, and (2) the derivation of the lower bound on the Fisher information for direct imaging, Fin-SPADE, and Pix-SLIVER.

* Corresponding author: elernair@nus.edu.sg

CONTENTS

I. Fundamental quantum limit on transverse resolution 2
A. Transverse spatial modes 3
B. Density operators ρ_{d_1} and ρ_{d_2} 5
C. State fidelity 7
D. Quantum Cramér-Rao bound 8

II. Fisher Information lower bounds for concrete measurements 9
A. Lower bound on direct imaging 10
B. Lower bound on Fin-SPADE performance 12
C. Lower bound on Pix-SLIVER performance 14

References 17

I. FUNDAMENTAL QUANTUM LIMIT ON TRANSVERSE RESOLUTION

In this Section, we give the details of the derivation of the QCRB for separation estimation.

As in Eq. (4) of the main text, the quantum state of the electromagnetic field in the image plane is given by the coherent-state decomposition

$$\rho_d = \int_{\mathbb{C}^2} \mathrm{d}^2 A_+ \, \mathrm{d}^2 A_- \; P_{N_s}(A) \, |\psi_{A,d}\rangle \, \langle\psi_{A,d}|. \tag{1}$$

Here

$$P_{N_s}(A) = \left(\frac{1}{\pi N_s}\right)^2 \exp\left(-\frac{|A_+|^2 + |A_-|^2}{N_s}\right), \tag{2}$$

is the probability density of the source field amplitudes $A = (A_+, A_-)$ and the conditional state $|\psi_{A,d}\rangle$ is an eigenvector of the image-plane field operator $\hat{E}(\boldsymbol{\rho}, t)$ with eigenfunction

$$\psi_{A,d}(\boldsymbol{\rho}, t) = \left[A_+ \, \psi(\boldsymbol{\rho} - \mathbf{d}/2) + A_- \, \psi(\boldsymbol{\rho} + \mathbf{d}/2)\right] \xi(t), \tag{3}$$

where $\mathbf{d} = (d, 0)$. This eigenfunction is simply the semiclassical complex field amplitude that results from the superposition of the images of the two sources conditioned on the amplitude vector A.

In order to evaluate the fidelity $F(\rho_{d_1}, \rho_{d_2})$ in Eq. (6) of the main text, we need to first choose transverse spatial modes in which to express the quantum states ρ_{d_1} and ρ_{d_2}.

A. Transverse spatial modes

For an arbitrary vector $\mathbf{a} = (a_x, a_y)$ in the image plane, consider the overlap function

$$\delta(\mathbf{a}) := \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho}\, \psi^*(\boldsymbol{\rho})\, \psi(\boldsymbol{\rho} - \mathbf{a}). \tag{4}$$

The Cauchy-Schwarz inequality implies that $|\delta(\mathbf{a})| \leq \delta(\mathbf{0}) = 1$. We have

$$\delta^*(\mathbf{a}) = \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho}\, \psi^*(\boldsymbol{\rho} - \mathbf{a})\, \psi(\boldsymbol{\rho}) \tag{5}$$

$$= \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho}\, \psi^*(\boldsymbol{\rho})\, \psi(\boldsymbol{\rho} + \mathbf{a}) \tag{6}$$

$$= \delta(-\mathbf{a}). \tag{7}$$

For an inversion-symmetric PSF, we can say more. Changing variables to $\boldsymbol{\sigma} = -\boldsymbol{\rho}$ with $\mathrm{d}\boldsymbol{\sigma} = \mathrm{d}\boldsymbol{\rho}$, we have

$$\delta^*(\mathbf{a}) = \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\sigma}\, \psi^*(-\boldsymbol{\sigma})\, \psi(-\boldsymbol{\sigma} + \mathbf{a}) \tag{8}$$

$$= \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\sigma}\, \psi^*(\boldsymbol{\sigma})\, \psi(\boldsymbol{\sigma} - \mathbf{a}) \tag{9}$$

$$\equiv \delta(\mathbf{a}), \tag{10}$$

where we have used inversion-symmetry $\psi(-\boldsymbol{\rho}) = \psi(\boldsymbol{\rho})$ of the PSF in the last step. For such PSFs, the overlap function is thus real-valued for all $\mathbf{a} \in \mathcal{I}$. We make the inversion-symmetry assumption throughout this paper.

Since we are considering only the estimation of the x-component of the separation between the sources, we slightly abuse the above notation to define the overlap for a scalar argument as

$$\delta(d) := \delta((d, 0)). \tag{11}$$

We then have

$$\delta(d) = \delta^*(d) = \delta(-d) \leq 1 \tag{12}$$

for all values d of the x-separation.

Consider two different values d_1 and d_2 of the separation. For $\mathbf{d}_1 = (d_1, 0)$, the functions

$$\begin{aligned} \chi_1(\boldsymbol{\rho}) &= \frac{\psi(\boldsymbol{\rho} - \mathbf{d}_1/2) + \psi(\boldsymbol{\rho} + \mathbf{d}_1/2)}{\sqrt{2\mathcal{N}_1}} \\ \chi_3(\boldsymbol{\rho}) &= \frac{\psi(\boldsymbol{\rho} - \mathbf{d}_1/2) - \psi(\boldsymbol{\rho} + \mathbf{d}_1/2)}{\sqrt{2\mathcal{N}_3}} \end{aligned} \tag{13}$$

with normalization constants given by

$$\mathcal{N}_1 = 1 + \delta(d_1), \tag{14}$$

$$\mathcal{N}_3 = 1 - \delta(d_1), \tag{15}$$

are orthonormal over the image plane $\mathcal{I}$. The functions (13) will be two of our mode functions. In like manner, for $\mathbf{d}_2 = (d_2, 0)$, the functions

$$\widetilde{\chi}_2(\boldsymbol{\rho}) = \frac{\psi(\boldsymbol{\rho} - \mathbf{d}_2/2) + \psi(\boldsymbol{\rho} + \mathbf{d}_2/2)}{\sqrt{2\mathcal{N}_2}} \tag{16}$$

$$\widetilde{\chi}_4(\boldsymbol{\rho}) = \frac{\psi(\boldsymbol{\rho} - \mathbf{d}_2/2) - \psi(\boldsymbol{\rho} + \mathbf{d}_2/2)}{\sqrt{2\mathcal{N}_4}} \tag{17}$$

are orthonormal over the image plane with the normalization constants

$$\mathcal{N}_2 = 1 + \delta(d_2), \tag{18}$$

$$\mathcal{N}_4 = 1 - \delta(d_2). \tag{19}$$

Using (12), we can readily verify that $\widetilde{\chi}_2$ is orthogonal to χ_3 and $\widetilde{\chi}_4$ is orthogonal to χ_1. However $\widetilde{\chi}_2$ is not in general orthogonal to χ_1 and neither is $\widetilde{\chi}_4$ orthogonal to χ_3. In order to obtain an orthonormal set of transverse spatial modes, the Gram-Schmidt process can be used to define

$$\chi_2(\boldsymbol{\rho}) = \frac{\widetilde{\chi}_2(\boldsymbol{\rho}) - \mu_s\, \chi_1(\boldsymbol{\rho})}{\sqrt{1 - \mu_s^2}}, \tag{20}$$

$$\chi_4(\boldsymbol{\rho}) = \frac{\widetilde{\chi}_4(\boldsymbol{\rho}) - \mu_a\, \chi_3(\boldsymbol{\rho})}{\sqrt{1 - \mu_a^2}}, \tag{21}$$

with

$$\mu_s = \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho}\, \chi_1^*(\boldsymbol{\rho}) \widetilde{\chi}_2(\boldsymbol{\rho}) = \frac{\delta\left[(d_1 - d_2)/2\right] + \delta\left[(d_1 + d_2)/2\right]}{\sqrt{\mathcal{N}_1 \mathcal{N}_2}}, \tag{22}$$

$$\mu_a = \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho}\, \chi_3^*(\boldsymbol{\rho}) \widetilde{\chi}_4(\boldsymbol{\rho}) = \frac{\delta\left[(d_1 - d_2)/2\right] - \delta\left[(d_1 + d_2)/2\right]}{\sqrt{\mathcal{N}_3 \mathcal{N}_4}}. \tag{23}$$

The set $\{\chi_1, \chi_2, \chi_3, \chi_4\}$ is an orthonormal set of transverse spatial modes that span the same space as $\{\psi(\boldsymbol{\rho} \pm \mathbf{d}_1/2), \psi(\boldsymbol{\rho} \pm \mathbf{d}_2/2)\}$. Note that inversion symmetry of the PSF implies that χ_1 and χ_2 are symmetric with respect to inversion about $\boldsymbol{\rho} = 0$ while χ_3 and χ_4 are antisymmetric under inversion.

B. Density operators ρ_{d_1} and ρ_{d_2}

Equation (2) implies that the incoherent thermal source amplitudes A are circular-complex Gaussian random variables satisfying the relations:

$$\mathbb{E}[A_\mu] = 0 \tag{24}$$

$$\mathbb{E}[A_\mu A_\nu] = 0 \tag{25}$$

$$\mathbb{E}[A_\mu^* A_\mu] = N_s \tag{26}$$

$$\mathbb{E}[A_+^* A_-] = 0 \tag{27}$$

for $\mu, \nu \in \{+, -\}$ ranging over the two sources. Define the sum and difference amplitudes

$$S = A_+ + A_-, \tag{28}$$

$$D = A_+ - A_- \tag{29}$$

which satisfy the relations

$$\begin{aligned} &\mathbb{E}[S] = \mathbb{E}[D] = 0 \\ &\mathbb{E}[S^2] = \mathbb{E}[D^2] = \mathbb{E}[SD] = 0 \\ &\mathbb{E}[S^* S] = \mathbb{E}[D^* D] = 2N_s \\ &\mathbb{E}[S^* D] = 0 \end{aligned} \tag{30}$$

and are thus *statistically independent* circular-complex Gaussian random variables. Clearly, specifying the pair (S, D) is equivalent to specifying $A = (A_+, A_-)$. The random variables $|A_+|^2$ and $|A_-|^2$ are independent and are both distributed exponentially with mean N_s [1]. Analogously, the random variables $|S|^2$ and $|D|^2$ are also independent and are both distributed exponentially with mean $2N_s$.

Consider the coherent-state decomposition (1) for ρ_{d_1}. Conditioned on the source amplitudes, the eigenfunction (3) can be rewritten as

$$\psi_{A,d_1}(\boldsymbol{\rho}, t) = \left(S\sqrt{\frac{\mathcal{N}_1}{2}}\,\chi_1(\boldsymbol{\rho}) + D\sqrt{\frac{\mathcal{N}_3}{2}}\,\chi_3(\boldsymbol{\rho}) \right) \xi(t), \tag{31}$$

in terms of the spatial modes defined in the previous subsection. Since S and D are i.i.d. circular-Gaussian variables, we may write, given the P-representation (1) [2, 3]:-

$$\rho_{d_1} = \rho_{\mathsf{th}}(\mathcal{N}_1 N_s) \otimes |0\rangle\langle 0| \otimes \rho_{\mathsf{th}}(\mathcal{N}_3 N_s) \otimes |0\rangle\langle 0|, \tag{32}$$

where

$$\rho_{\mathsf{th}}(\overline{N}) = \sum_{n=0}^{\infty} \frac{\overline{N}^n}{(\overline{N}+1)^{n+1}} |n\rangle\langle n| \tag{33}$$

$$= \frac{1}{\pi\overline{N}} \int_{\mathbb{C}} \mathrm{d}^2\alpha \, \exp\left(-\frac{|\alpha|^2}{\overline{N}}\right) |\alpha\rangle\langle\alpha| \tag{34}$$

is the single-mode thermal state of $\overline{N}$ average photons (written above in its number-state and coherent-state decompositions). The four spatiotemporal modes in the above representation are respectively $\chi_1(\rho)\,\xi(t), \chi_2(\rho)\,\xi(t), \chi_3(\rho)\,\xi(t)$, and $\chi_4(\rho)\,\xi(t)$, and we have omitted including an infinity of other spatiotemporal modes which are in the vacuum state for all values of the separation and are not useful for estimating it.

Consider now the coherent-state decomposition (1) for ρ_{d_2}. Conditioned on the source amplitudes, the eigenfunction (3) can be rewritten as

$$\psi_{A,d_2}(\boldsymbol{\rho},t) = \left(S\sqrt{\frac{\mathcal{N}_2}{2}}\,\widetilde{\chi}_2(\boldsymbol{\rho}) + D\sqrt{\frac{\mathcal{N}_4}{2}}\,\widetilde{\chi}_4(\boldsymbol{\rho})\right)\xi(t), \tag{35}$$

$$= \left\{S\sqrt{\frac{\mathcal{N}_2}{2}}\left[\mu_s\chi_1(\boldsymbol{\rho}) + \sqrt{1-\mu_s^2}\,\chi_2(\boldsymbol{\rho})\right] + D\sqrt{\frac{\mathcal{N}_4}{2}}\left[\mu_a\chi_3(\boldsymbol{\rho}) + \sqrt{1-\mu_a^2}\,\chi_4(\boldsymbol{\rho})\right]\right\}\xi(t) \tag{36}$$

The unconditional density operator ρ_{d_2} can then be written in the same set of modes used for writing (32), as follows:-

$$\rho_{d_2} = \left\{U_s\left[\rho_{\mathsf{th}}\left(\mathcal{N}_2 N_s\right)\otimes|0\rangle\langle 0|\right]U_s^\dagger\right\}\otimes\left\{U_a\left[\rho_{\mathsf{th}}\left(\mathcal{N}_4 N_s\right)\otimes|0\rangle\langle 0|\right]U_a^\dagger\right\}, \tag{37}$$

where U_s is the two-mode beam-splitter unitary (see, e.g., ref. [4]) whose action on coherent states is

$$U_s\Big(|\alpha\rangle|\beta\rangle\Big) \mapsto \left|\mu_s\alpha - \sqrt{1-\mu_s^2}\beta\right\rangle\left|\mu_s\beta + \sqrt{1-\mu_s^2}\alpha\right\rangle \tag{38}$$

and on the number state-vacuum product is

$$U_s\Big(|n\rangle|0\rangle\Big) \mapsto \sum_{k=0}^{n}\sqrt{\binom{n}{k}}\,\mu_s^k\left(1-\mu_s^2\right)^{\frac{n-k}{2}}|k\rangle|n-k\rangle. \tag{39}$$

Similarly, U_a is the two-mode beamsplitter unitary whose action on coherent states is

$$U_a\Big(|\alpha\rangle|\beta\rangle\Big) \mapsto \left|\mu_a\alpha + \sqrt{1-\mu_a^2}\beta\right\rangle\left|\mu_a\beta - \sqrt{1-\mu_a^2}\alpha\right\rangle \tag{40}$$

and on the number state-vacuum product is

$$U_a\Big(\,|n\rangle\,|0\rangle\,\Big)\mapsto\sum_{k=0}^{n}\sqrt{\binom{n}{k}}\,\mu_a^k\,(1-\mu_a^2)^{\frac{n-k}{2}}\,|k\rangle\,|n-k\rangle\,. \tag{41}$$

C. State fidelity

The quantum fidelity between ρ_{d_1} and ρ_{d_2} is given by

$$F(\rho_{d_1},\rho_{d_2})=\mathrm{Tr}\,\sqrt{\sqrt{\rho_{d_1}}\rho_{d_2}\sqrt{\rho_{d_1}}}. \tag{42}$$

Since both density operators (32) and (37) factorize into a product of density operators on the symmetric (spanned by the modes $\chi_1(\rho)\,\xi(t)$ and $\chi_2(\rho)\,\xi(t)$) and the antisymmetric modes (spanned by the modes $\chi_3(\rho)\,\xi(t)$ and $\chi_4(\rho)\,\xi(t)$), we can mutliply the fidelities for each pair.

Considering the symmetric modes first, let

$$r_1:=\frac{\mathcal{N}_1N_s}{1+\mathcal{N}_1N_s}, \tag{43}$$
$$r_2:=\frac{\mathcal{N}_2N_s}{1+\mathcal{N}_2N_s}, \tag{44}$$

so that the symmetric components of the density operators under each hypothesis are

$$\rho_{d_1}^{(\mathrm{sym})}=(1-r_1)\sum_{n=0}^{\infty}r_1^n\,|n\rangle\,\langle n|\otimes|0\rangle\,\langle 0|, \tag{45}$$
$$\rho_{d_2}^{(\mathrm{sym})}=(1-r_2)\sum_{n=0}^{\infty}r_2^n\,U_s\Big(\,|n\rangle\,\langle n|\otimes|0\rangle\,\langle 0|\Big)U_s^\dagger. \tag{46}$$

Then

$$\sqrt{\rho_{d_1}^{(\mathrm{sym})}}\rho_{d_2}^{(\mathrm{sym})}\sqrt{\rho_{d_1}^{(\mathrm{sym})}} \tag{47}$$
$$=(1-r_1)(1-r_2)\sum_{n,n',n''=0}^{\infty}r_1^{\frac{n+n''}{2}}\,r_2^{n'}\,|n\,0\rangle\,\langle n\,0|\,U_s\,|n'\,0\rangle\,\langle n'\,0|\,U_s^\dagger\,|n''\,0\rangle\,\langle n''\,0|, \tag{48}$$
$$=(1-r_1)(1-r_2)\sum_{n,n',n''=0}^{\infty}r_1^{\frac{n+n''}{2}}\,r_2^{n'}\,|n\,0\rangle\,\mu_s^{n'}\,\delta_{n\,n'}\,\mu_s^{*n'}\,\delta_{n\,n''}\langle n''\,0| \tag{49}$$
$$=(1-r_1)(1-r_2)\sum_{n=0}^{\infty}r_1^n\,r_2^n\,|\mu_s|^{2n}\,|n\,0\rangle\,\langle n\,0|, \tag{50}$$

where we have used Eq. (39) to evaluate the matrix elements in Eq. (48). Consequently,

$$F\left(\rho_{d_1}^{(\mathrm{sym})},\rho_{d_2}^{(\mathrm{sym})}\right) = \mathrm{Tr}\,\sqrt{\sqrt{\rho_{d_1}^{(\mathrm{sym})}}\rho_{d_2}^{(\mathrm{sym})}\sqrt{\rho_{d_1}^{(\mathrm{sym})}}} \tag{51}$$

$$= \frac{(1-r_1)^{1/2}(1-r_2)^{1/2}}{1-|\mu_s|\sqrt{r_1 r_2}} \tag{52}$$

$$= \left[\sqrt{(1+\mathcal{N}_1 N_s)\,(1+\mathcal{N}_2 N_s)} - |\mu_s|\sqrt{\mathcal{N}_1\mathcal{N}_2}\,N_s\right]^{-1}. \tag{53}$$

In similar fashion, we find

$$F\left(\rho_{d_1}^{(\mathrm{asym})},\rho_{d_2}^{(\mathrm{asym})}\right) = \left[\sqrt{(1+\mathcal{N}_3 N_s)\,(1+\mathcal{N}_4 N_s)} - |\mu_a|\sqrt{\mathcal{N}_3\mathcal{N}_4}\,N_s\right]^{-1}, \tag{54}$$

resulting in the expression

$$\begin{aligned} F(\rho_{d_1},\rho_{d_2}) = &\left[\sqrt{(1+N_s\left[1+\delta(d_1)\right])\,(1+N_s[1+\delta(d_2)])} - N_s\Big|\delta\left[(d_1-d_2)/2\right]+\delta\left[(d_1+d_2)/2\right]\Big|\right]^{-1} \\ \times &\left[\sqrt{(1+N_s\left[1-\delta(d_1)\right])\,(1+N_s\left[1-\delta(d_2)\right])} - N_s\Big|\delta\left[(d_1-d_2)/2\right]-\delta\left[(d_1+d_2)/2\right]\Big|\right]^{-1} \end{aligned} \tag{55}$$

for the overall fidelity.

D. Quantum Cramér-Rao bound

Let $d_1 = d$ and $d_2 = d_1 + \Delta d$. The quantum Fisher information (QFI) $\mathcal{K}_d$ on d is given by [5, 6]

$$\mathcal{K}_d = 8 \times \lim_{\Delta d\to 0}\frac{1-F(\rho_d,\rho_{d+\Delta d})}{(\Delta d)^2} = -4\left.\frac{\partial^2 F(\rho_{d_1},\rho_{d_2})}{\partial d_2^2}\right|_{d_2=d_1}. \tag{56}$$

Since the symmetric and antisymmetric modes are in tensor-product states, $\mathcal{K}_d$ is the sum of the QFIs $\mathcal{K}_d^{\mathrm{sym}}$ and $\mathcal{K}_d^{\mathrm{asym}}$ from the respective subsystems [6]. Defining

$$\begin{aligned} \gamma(d) &= \delta'(d), \\ \beta(d) &= \gamma'(d), \end{aligned} \tag{57}$$

the QFI from the symmetric modes is found after some algebra to be:

$$\mathcal{K}_d^{\mathrm{sym}} = [\beta(d)-\beta(0)]N_s - \frac{N_s^2\,\gamma^2(d)}{1+N_s[1+\delta(d)]}. \tag{58}$$

Similarly, the QFI from the antisymmetric modes is found to be

$$\mathcal{K}_d^{\mathrm{asym}} = -[\beta(d)+\beta(0)]N_s - \frac{N_s^2\,\gamma^2(d)}{1+N_s[1-\delta(d)]}, \tag{59}$$

giving a total QFI

$$\mathcal{K}_d = \mathcal{K}_d^{\rm sym} + \mathcal{K}_d^{\rm asym} \tag{60}$$

$$= -2\beta(0)N_s - 2\gamma^2(d)\left[\frac{(1+N_s)N_s^2}{(1+N_s)^2 - N_s^2\delta^2(d)}\right], \tag{61}$$

which is Eq. (7) of the main text. Here

$$\beta(0) = -\int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho} \left|\frac{\partial\psi(\boldsymbol{\rho})}{\partial x}\right|^2 \equiv -(\Delta k_x^2), \tag{62}$$

For circularly symmetric PSFs, this quantity is independent of the direction of the x-axis and is the mean-squared spatial bandwidth of the PSF.

II. FISHER INFORMATION LOWER BOUNDS FOR CONCRETE MEASUREMENTS

In this Section, we give the derivation of the lower bound on the Fisher information for direct imaging, Fin-SPADE, and Pix-SLIVER.

Consider a vector random variable $\boldsymbol{Y} = (Y_1,\ldots,Y_M)^\mathsf{T} \in \mathbb{R}^M$ whose probability density $P_{\boldsymbol{Y}|X}(\boldsymbol{y}|x)$ depends on an unknown parameter x. The classical Fisher information (FI) $\mathcal{J}_x[\boldsymbol{Y}]$ of $\boldsymbol{Y}$ on x [7] is typically difficult to compute unless the components of $\boldsymbol{Y}$ are statistically independent. However, a general *lower bound*

$$\mathcal{J}_x[\boldsymbol{Y}] \geq \dot{\boldsymbol{\mu}}^\mathsf{T}\, \boldsymbol{C}^{-1}\, \dot{\boldsymbol{\mu}} \tag{63}$$

was recently derived in [8]. Here $\boldsymbol{\mu} = (\langle Y_1\rangle_x,\ldots,\langle Y_M\rangle_x)^\mathsf{T}$ is the mean observation vector, $\boldsymbol{C} = \langle(\boldsymbol{Y}-\boldsymbol{\mu})(\boldsymbol{Y}-\boldsymbol{\mu})^\mathsf{T}\rangle_x$ is the covariance matrix of $\boldsymbol{Y}$, and $\dot{\boldsymbol{\mu}} = \partial\boldsymbol{\mu}/\partial x$. All the above quantities are functions of x. The bound (63) is very convenient as it depends only on the first two moments of the observation vector, which are easier to compute. In contrast, the FI $\mathcal{J}_x[\boldsymbol{Y}]$ depends on the full joint probability density of Y (conditioned on x).

We compute this lower bound for various measurements below. Since all the measurements involve at most linear-optical processing prior to photodetection, the classicality (in the sense of having a non-negative P-representation [2, 3]) of the incoming state ρ_d is preserved. It is well known that, for such states, the quantum theory of photodetection gives the same quantitative statistics as the semiclassical theory of photodetection [2, 3]. Let the input field $E(\boldsymbol{\rho},t)$ be subjected to arbitrary linear-optics processing and the resulting output field $E_{\mathsf{det}}(\boldsymbol{\rho}, t)$ impinge on an ideal continuum photodetector surface. Semiclassical photodetection theory dictates that, conditioned on the source amplitudes A, the incident field generates a space-time Poisson random process at the photodetector output with the rate function (or intensity) $|E_{\mathsf{det}}(\boldsymbol{\rho}, t)|^2$. Unconditional statistics can then be obtained by averaging over the source distribution using (2). We will follow this approach in the sequel.

A. Lower bound on direct imaging

Consider first the case of direct detection in the image plane with a pixelated detector array centered at the origin and of width W in the x-direction. For simplicity, we assume it to be infinite in the y-direction, but pixelated in the x-direction with P_d pixels of width W/P_d. We assume ideal unity-quantum-efficiency and noiseless number-resolved photon counting in each pixel. Let $p \in \{1, \ldots, P_d\}$ be the pixel index and let pixel p be defined by the region

$$\mathcal{A}_p = \{(x, y) : l_p \le x \le r_p, -\infty \le y \le \infty\} \tag{64}$$

of the image plane. The observation consists of the vector $\boldsymbol{N} = (N_1, \ldots, N_{P_d})^{\mathsf{T}}$ of measured counts in each pixel.

Conditioned on A, the intensity function $I_A(\boldsymbol{\rho}, t)$ in the image plane is, using (3),

$$I_A(\boldsymbol{\rho}, t) = |\psi_{A,d}(\boldsymbol{\rho}, t)|^2 \tag{65}$$

$$= \left\{|A_+|^2 \, |\psi(\boldsymbol{\rho} - \mathbf{d}/2)|^2 + |A_-|^2 \, |\psi(\boldsymbol{\rho} - \mathbf{d}/2)|^2 + 2\mathrm{Re}\left[A_+^* \, A_- \, \psi^*(\boldsymbol{\rho} - \mathbf{d}/2) \, \psi(\boldsymbol{\rho} + \mathbf{d}/2)\right]\right\} |\xi(t)|^2 . \tag{66}$$

The conditional photocounts $N_{p|A}$ on the detectors $p \in \{1, \ldots, P_d\}$ integrated over the observation interval $[0, T]$ are then independent Poisson random variables with the means

$$\mu_{p|A} = \int_0^T \mathrm{d}t \int_{\mathcal{A}_p} \mathrm{d}\boldsymbol{\rho} \, I_A(\boldsymbol{\rho}, t). \tag{67}$$

We now suppose the PSF has the Gaussian form

$$\psi_G(\boldsymbol{\rho}) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{|\boldsymbol{\rho}|^2}{4\sigma^2}\right), \tag{68}$$

although the treatment is readily generalized to arbitrary PSFs. We obtain

$$\mu_{p|A} = |A_+|^2\, \alpha_p + 2\mathrm{Re}\,(A_+^*\, A_-)\, \beta_p + |A_-|^2\, \gamma_p, \tag{69}$$

where

$$\begin{aligned} \alpha_p &= Q\left(\frac{l_p + d/2}{\sigma}\right) - Q\left(\frac{r_p + d/2}{\sigma}\right), \\ \beta_p &= 2\exp\left(\frac{-d^2}{8\sigma^2}\right)\left[Q\left(\frac{l_p}{\sigma}\right) - Q\left(\frac{r_p}{\sigma}\right)\right], \\ \gamma_p &= Q\left(\frac{l_p - d/2}{\sigma}\right) - Q\left(\frac{r_p - d/2}{\sigma}\right), \end{aligned} \tag{70}$$

and

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \mathrm{d}t\, \exp\left(\frac{-t^2}{2}\right) \tag{71}$$

is the Q-function.

The mean photocount $\mu_p = \mathbb{E}[N_p] = \mathbb{E}_A[\mu_{p|A}]$ is then

$$\mu_p = N_s(\alpha_p + \gamma_p), \tag{72}$$

where we have used eqs. (24)-(27). We then have

$$\begin{aligned} \dot{\mu}_p &= \frac{\partial \mu_p}{\partial d} \\ &= \frac{N_s}{2\sqrt{2\pi}\sigma}\left\{\exp\left[\frac{-(l_p - d/2)^2}{2\sigma^2}\right] - \exp\left[\frac{-(r_p - d/2)^2}{2\sigma^2}\right] + \exp\left[\frac{-(r_p + d/2)^2}{2\sigma^2}\right] - \exp\left[\frac{-(l_p + d/2)^2}{2\sigma^2}\right]\right\}. \end{aligned} \tag{73}$$

The (p, p')-th element of the covariance matrix of $\boldsymbol{N}$ equals $\mathbb{E}[N_p N_{p'}] - \mu_p\, \mu_{p'}$. Now

$$\mathbb{E}[N_p N_{p'}] = \mathbb{E}_A[\mu_{p|A}\, \mu_{p'|A}] \tag{74}$$

$$= \begin{cases} \mathbb{E}_A[\mu_{p|A}]\mathbb{E}_A[\mu_{p'|A}] & \text{if } p \neq p' \\ \mathbb{E}_A[\mu_{p|A}^2] & \text{if } p = p'. \end{cases} \tag{75}$$

Straightforward computations using the relations (24)-(27) and (69) give the matrix elements

$$C_{pp'} = \begin{cases} N_s^2(\alpha_p^2 + 2\beta_p^2 + \gamma_p^2) + N_s(\alpha_p + \gamma_p) & \text{if } p = p', \\ N_s^2(\alpha_p \alpha_{p'} + 2\beta_p\, \beta_{p'} + \gamma_p\, \gamma_{p'}) & \text{if } p \neq p'. \end{cases} \tag{76}$$

In obtaiing the elements of the covariance matrix, we have also used the fact that $\mathbb{E}[|A_+|^4] = \mathbb{E}[|A_-|^4] = 2N_s^2$, which follows from the exponential statistics of $|A_+|^2$ and $|A_-|^2$ (see Sec. I B). Using eqs. (73) and (76), the lower bound (63) can be evaluated numerically for any given system parameters – see Figs. 2, 4, and 6 of the main text. The limit of continuum image-plane photodetection is achieved for $P_d \to \infty$, but it was observed that the the FI lower bound did not change discernibly for $P_d \gtrsim 50$, so $P_d = 50$ was used in plotting the direct imaging curves in Figs. 2, 4, and 6 of the main text.

B. Lower bound on Fin-SPADE performance

Suppose the PSF has the Gaussian form

$$\psi_G(\boldsymbol{\rho}) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left(-\frac{|\boldsymbol{\rho}|^2}{4\sigma^2}\right). \tag{77}$$

As discussed in the main text, the Fin-SPADE measurement measures the photon number in each Hermite-Gaussian mode TEM_{q0} (with profile $\psi_{q0}(\boldsymbol{\rho})$) of the image-plane field for $0 \le q \le Q$ over the interval $[0, T]$. This results in a $(Q+1)$-vector $\boldsymbol{N} = (N_0, \ldots, N_Q)^\mathsf{T}$ of the number of counts in each mode. The moments of $\boldsymbol{N}$ can be found using the semiclassical photodetection theory as follows.

Conditioned on A, the amplitude $B_{q|A}$ in the q-th channel can be written (cf. Eq. (11) of the main text):-

$$B_{q|A} = \int_0^T \mathrm{d}t \int_{\mathcal{I}} \mathrm{d}\boldsymbol{\rho}\, \psi_{A,d}(\boldsymbol{\rho}, t)\, \psi_{q0}^*(\boldsymbol{\rho})\, \xi^*(t). \tag{78}$$

As shown in [9], the integrals may be associated to the probability amplitudes of a coherent state in the Fock basis so that

$$B_{q|A} = \frac{\kappa^{q/2} \exp(-\kappa/2)}{\sqrt{q!}}\, R_q, \tag{79}$$

where

$$R_q = \begin{cases} S & \text{(if } q \text{ even)} \\ D & \text{(if } q \text{ odd),} \end{cases} \tag{80}$$

and

$$\kappa = \frac{d^2}{16\sigma^2}. \tag{81}$$

Conditioned on A, the photocounts $N_{q|A}$ in each q-channel are independent Poisson random variables with the means

$$\mu_{q|A} = \left|B_{q|A}\right|^2 = \frac{\kappa^q \exp(-\kappa)}{q!} \; |R_q|^2 \tag{82}$$

$$\equiv f_q \, |R_q|^2 , \tag{83}$$

where f_q is the Poisson probability of mean κ. For the unconditional mean, we have

$$\mu_q := \langle N_q \rangle = \mathbb{E}_A \left[\mu_{q|A}\right] \tag{84}$$

$$= \mathbb{E}_A[f_q \, |R_q|^2] \tag{85}$$

$$= 2N_s f_q, \tag{86}$$

since $|S|^2$ and $|D|^2$ are i.i.d. random variables distributed exponentially with mean $2N_s$. We also need

$$\frac{\partial \mu_q}{\partial d} = \frac{N_s d}{4\sigma^2} \, \frac{\kappa^{q-1}[q-\kappa] \exp(-\kappa)}{q!} \tag{87}$$

$$= \frac{N_s d}{4\sigma^2} (f_{q-1} - f_q), \tag{88}$$

where we define $f_{-1} = 0$.

For the second moments, three cases arise. First, for $q = q'$, we have

$$\mathbb{E}[N_q^2] = \mathbb{E}_A \left[\mathbb{E}[N_{q|A}^2]\right] \tag{89}$$

$$= \mathbb{E}_A \left[f_q^2 \, |R_q|^4 + f_q \, |R_q|^2\right] \tag{90}$$

$$= 8N_s^2 \, f_q^2 + 2N_s f_q, \tag{91}$$

where we have used the fact that $N_{q|A}$ is Poisson-distributed. If $q \neq q'$ but $q - q'$ is even, $R_q = R_{q'}$, so we get

$$\mathbb{E}[N_q \, N_{q'}] = \mathbb{E}_A \left[\mathbb{E}[N_{q|A} N_{q'|A}]\right] \tag{92}$$

$$= \mathbb{E}_A \left[\mu_{q|A} \, \mu_{q'|A}\right] \tag{93}$$

$$= \mathbb{E}_A \left[f_q \, f_{q'} \, |R_q|^4\right] \tag{94}$$

$$= 8N_s^2 \, f_q \, f_{q'}. \tag{95}$$

If $q \neq q'$ and $q - q'$ is odd, $\mathbb{E}_A[|R_q|^2 \, |R_{q'}|^2] = \mathbb{E}_A[|R_q|^2]\,\mathbb{E}_A[|R_{q'}|^2]$, so that

$$\mathbb{E}[N_q \, N_{q'}] = \mathbb{E}_A\left[\mathbb{E}[N_{q|A} N_{q'|A}]\right] \tag{96}$$

$$= \mathbb{E}_A\left[\mu_{q|A}\, \mu_{q'|A}\right] \tag{97}$$

$$= \mathbb{E}_A\left[f_q \, f_{q'} \, |R_q|^2 \, |R_{q'}|^2\right] \tag{98}$$

$$= 4N_s^2 \, f_q \, f_{q'}. \tag{99}$$

Thus, the covariance matrix $\boldsymbol{C}$ of $\boldsymbol{N}$ has the $qq'-$th entry

$$C_{qq'} = \begin{cases} 4N_s^2 \, f_q^2 + 2N_s f_q & \text{if } q = q', \\ 4N_s^2 \, f_q \, f_{q'} & \text{if } q \neq q' \text{ and } q - q' \text{ is even,} \\ 0 & \text{if } q \neq q' \text{ and } q - q' \text{ is odd.} \end{cases} \tag{100}$$

From Eqs. (87) and (100), the lower bound (63) can be numerically evaluated, as displayed in Fig. 4 of the main text.

C. Lower bound on Pix-SLIVER performance

Consider the Pix-SLIVER setup of Fig. 5 of the main text with identical detector arrays in the symmetric (s) and antisymmetric (a) output ports. The overall dimensions of the arrays are as in Sec. II A, except that we consider P pixels in each array. For a conservative comparison, we take $P < P_d$. In addition, we also assume on-off (Geiger mode) detection in each pixel, so that each component of the observation $\boldsymbol{K} = (K_1^{(s)}, \ldots, K_P^{(s)}, K_1^{(a)}, \ldots, K_P^{(a)})$ is 0 (if the corresponding pixel did not fire) or 1 (if it did). In contrast, we allowed number-resolved detection in direct imaging (see Sec. II A).

We now assume that the PSF is symmetric relative to *reflection* about the $y-$axis, i.e., $\psi(-x,y) = \psi(x,y)$ for all x and y – circular symmetry of the PSF is clearly a sufficient condition for this to hold. Conditioned on A, the semiclassical field amplitude in the two interferometer outputs is given by (cf. Eq. (13) of the main text):-

$$E_A^{(s(a))}(x,y,t) = \left[\psi_{A,d}(x,y,t) \pm \psi_{A,d}(-x,y,t)\right]/2. \tag{101}$$

Since the field $\hat{E}_v(\boldsymbol{\rho}, t)$ is in vacuum, the open input port of the first beam splitter does not contribute to the field amplitude. We can rewrite the above as

$$E_A^{(s)}(x,y,t) = \frac{S}{2}\left[\psi(x+d/2,y,t)+\psi(x-d/2,y,t)\right], \tag{102}$$

$$E_A^{(a)}(x,y,t) = \frac{D}{2}\left[\psi(x-d/2,y,t)-\psi(x+d/2,y,t)\right]. \tag{103}$$

where we have used the reflection symmetry of the PSF. The resulting conditional intensity patterns on the two detectors are

$$I_A^{(s)}(x,y,t) = \frac{|S|^2}{4}\left[|\psi(x-d/2,y,t)|^2+|\psi(x+d/2,y,t)|^2\right] + \frac{|S|^2}{2}\mathrm{Re}\left[\psi^*(x-d/2,y,t)\,\psi(x+d/2,y,t)\right], \tag{104}$$

$$I_A^{(a)}(x,y,t) = \frac{|D|^2}{4}\left[|\psi(x-d/2,y,t)|^2+|\psi(x+d/2,y,t)|^2\right] - \frac{|D|^2}{2}\mathrm{Re}\left[\psi^*(x-d/2,y,t)\,\psi(x+d/2,y,t)\right]. \tag{105}$$

The integrated intensity $I_{p|A}^{(\alpha)}$ on pixel $p \in \{1,\dots,P\}$ of the $\alpha \in \{s,a\}$ detector array over the observation interval $[0,T]$ is then

$$I_{p|A}^{(\alpha)} = \int_0^T \mathrm{d}t \int_{\mathcal{A}_p} \mathrm{d}\boldsymbol{\rho}\, I_A^{(\alpha)}(x,y,t). \tag{106}$$

Specializing to the Gaussian PSF (77), these integrals evaluate to

$$I_{p|A}^{(s)} = \frac{|S|^2}{4}\left[\alpha_p+\gamma_p+\beta_p\right] \equiv \frac{|S|^2}{4} f_p^{(s)}, \tag{107}$$

$$I_{p|A}^{(a)} = \frac{|D|^2}{4}\left[\alpha_p+\gamma_p-\beta_p\right] \equiv \frac{|D|^2}{4} f_p^{(a)}, \tag{108}$$

where α_p, γ_p, and β_p are defined in Eq. (70) and the above equations serve to define the quantities $\{f_p^{(\alpha)}\}$.

Conditioned on A, the probability of a detector click in the (α,p)-th pixel is simply the probability that one or more photons impinge on the pixel:

$$\mathbb{E}[K_{p|A}^{\alpha}] \equiv \mu_{p|A}^{(\alpha)} = 1-\exp(-I_{p|A}^{(\alpha)}). \tag{109}$$

Consequently,

$$\mu_p^{(\alpha)} \equiv \mathbb{E}[K_p^{\alpha}] \tag{110}$$

$$= \mathbb{E}_A\left[K_{p|A}^{(\alpha)}\right] \tag{111}$$

$$= 1-\mathbb{E}_A\left[\exp(-I_{p|A}^{(\alpha)})\right] \tag{112}$$

$$= \frac{f_p^{(\alpha)} N_s}{2+f_p^{(\alpha)} N_s}, \tag{113}$$

where we have used the fact that $|S|^2$ and $|D|^2$ are exponentially distributed with mean $2N_s$ to evaluate the expectation over A. It follows that

$$\dot{\mu}_p^{(\alpha)} = \frac{2\dot{f}_p^{(\alpha)} N_s}{\left(2 + f_p^{(\alpha)} N_s\right)^2}, \tag{114}$$

for

$$\begin{aligned}\dot{f}_p^{(s(a))} &= \frac{1}{2\sqrt{2\pi}\sigma}\left\{\exp\left[\frac{-(l_p - d/2)^2}{2\sigma^2}\right] - \exp\left[\frac{-(r_p - d/2)^2}{2\sigma^2}\right] + \exp\left[\frac{-(r_p + d/2)^2}{2\sigma^2}\right] - \exp\left[\frac{-(l_p + d/2)^2}{2\sigma^2}\right]\right\} \\ &\mp \left(\frac{d}{2\sqrt{2\pi}\sigma^2}\right)\exp\left(\frac{-d^2}{8\sigma^2}\right)\left[Q\left(\frac{l_p}{\sigma}\right) - Q\left(\frac{r_p}{\sigma}\right)\right].\end{aligned} \tag{115}$$

For the second moments $\mathbb{E}\left[K_p^{(\alpha)} K_{p'}^{(\alpha')}\right]$, three cases arise. If $p = p'$ and $\alpha = \alpha'$,

$$\mathbb{E}\left[K_p^{(\alpha)} K_{p'}^{(\alpha')}\right] = \mathbb{E}\left[K_p^{(\alpha)}\right] \tag{116}$$

$$= \mathbb{E}_A[\mu_{p|A}^{(\alpha)}] \tag{117}$$

$$= \mu_p^{(\alpha)}. \tag{118}$$

If $\alpha \neq \alpha'$ (so that the pixels are in different detector arrays), the independence of S and D ensures that $K_p^{(\alpha)}$ and $K_{p'}^{(\alpha')}$ are independent also so that

$$\mathbb{E}\left[K_p^{(\alpha)} K_{p'}^{(\alpha')}\right] = \mu_p^{(\alpha)}\mu_{p'}^{(\alpha')}. \tag{119}$$

Finally, if $\alpha = \alpha'$ but $p \neq p'$,

$$\mathbb{E}\left[K_p^{(\alpha)} K_{p'}^{(\alpha')}\right] = \mathbb{E}_A\left[\mathbb{E}\left[K_{p|A}^{(\alpha)} K_{p'|A}^{(\alpha)}\right]\right] \tag{120}$$

$$= \mathbb{E}_A\left[\mu_{p|A}^{(\alpha)}\mu_{p'|A}^{(\alpha)}\right] \tag{121}$$

$$= \mathbb{E}_A\left[\left(1 - \exp(-I_{p|A}^{(\alpha)})\right)\left(1 - \exp(-I_{p'|A}^{(\alpha)})\right)\right] \tag{122}$$

$$= 1 - \frac{2}{2 + f_p^{(\alpha)} N_s} - \frac{2}{2 + f_{p'}^{(\alpha)} N_s} + \frac{2}{2 + \left(f_p^{(\alpha)} + f_{p'}^{(\alpha)}\right) N_s}, \tag{123}$$

where we again use the exponential distribution of $|S|^2$ and $|D|^2$ to evaluate the expectation over A. From these second moments, means (113), and (114), the lower bound (63) can be numerically evaluated, with the results displayed in Fig. 6 of the main text. Note that Eq. (119) implies that the covariance matrix $\boldsymbol{C}$ is a direct sum of matrices for the symmetric and antisymmetric outputs, so that the lower bound (63) is also the sum of corresponding terms – these are shown separately in Fig. 6 of the main text for the case of $P = 40$.

---

[1] J. W. Goodman, *Statistical Optics* (John Wiley & Sons, 1985).

[2] L. Mandel and E. Wolf, *Optical Coherence and Quantum Optics* (Cambridge University, 1995).

[3] J. H. Shapiro, IEEE J. Sel. Top. Quantum Electron. **15**, 1547 (2009).

[4] C. Gerry and P. Knight, *Introductory Quantum Optics* (Cambridge University Press, 2005).

[5] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. **72**, 3439 (1994).

[6] M. Hayashi, *Quantum Information* (Springer, 2006).

[7] H. L. Van Trees, *Detection, Estimation, and Modulation Theory : Part I*, 1st ed. (Wiley-Interscience 1st Ed, 2001).

[8] M. Stein, A. Mezghani, and J. A. Nossek, IEEE Sig. Proc. Lett. **21**, 796 (2014).

[9] M. Tsang, R. Nair, and X.-M. Lu, Phys. Rev. X **6**, 031033 (2016).