# Quantum information for semiclassical optics

Mankei Tsang[a,b], Ranjith Nair[a], and Xiao-Ming Lu[a]

[a]Department of Electrical and Computer Engineering, National University of Singapore, 4 Engineering Drive 3, Singapore 117583

[b]Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117551

## ABSTRACT

Applying the mathematics of quantum information to a Poisson semiclassical photodetection model, we derive fundamental limits to parameter estimation and hypothesis testing with any measurement of weak incoherent optical sources via linear optics and photon counting. Connections with our recent work on superresolution imaging are highlighted.

**Keywords:** Statistical optics, Parameter estimation, Hypothesis testing, Quantum information, Quantum metrology

## 1. INTRODUCTION

Modern optical imaging research recognizes that both the wave nature and the particle nature of light play equally important roles in determining the fundamental resolution of incoherent optical imaging. Diffraction blurs the image, while the random arrival of photons introduces shot noise;[1] the combination of the two effects contribute to difficulties in extracting information from the blurred and noisy imaging data.[2] As photon shot noise is becoming the dominant noise source in fluorescence microscopy[2] as well as astronomical imaging,[3] a quantum formalism that fully accounts for the wave-particle duality of light can offer novel and timely insights into the age-old problem of optical resolution.[4–7] The theoretical machinery of quantum information and quantum metrology,[4,8–10] in particular, enabled our recent discoveries on superresolution incoherent imaging,[11–17] which promise substantial improvements beyond previously established limits[18–20] and have generated significant interest in the quantum optics community.[21–26]

The goal of this paper is to introduce a semiclassical formalism that can reproduce most of our recent results, which are focused on thermal sources, passive linear optics, and photon counting. Within such restrictions, it is known that a semiclassical photodetection model suffices,[27] but here we assume a simpler Poisson model that has been widely employed in fluorescence microscopy[2] as well as astronomical imaging[28] to make connections with the modern literature. We also apply the mathematics of quantum information[4,8,9] to our model to investigate fundamental limits to statistical inference problems in optical sensing and imaging, without resorting to the full quantum formalism. Although this semiclassical formalism is more restrictive, it offers a more pedagogical treatment for readers less familiar with quantum mechanics and may be applied to other problems, such as LIDAR and electron microscopy,[19] that do not satisfy the assumptions needed by the quantum formalism.

## 2. MUTUAL COHERENCE MATRIX

Define $\alpha$ as a column vector of the complex amplitudes of the optical fields in $J$ optical modes, which can be spatial, frequency, or polarization modes. Explicitly, it can be written as

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_J \end{pmatrix}. \tag{1}$$

Further author information: (Send correspondence to M. T.)
M. T.: E-mail: mankei@nus.edu.sg

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Define also its complex transpose as

$$\alpha^\dagger = \begin{pmatrix} \alpha_1^* & \alpha_2^* & \dots & \alpha_J^* \end{pmatrix}. \tag{2}$$

Assuming $\alpha$ to be random variables, the central quantity in statistical optics is the mutual coherence matrix,[1,27] defined as

$$\Gamma \equiv \mathbb{E}\left(\alpha\alpha^\dagger\right), \tag{3}$$

where $\mathbb{E}$ denotes the statistical expectation. Mathematically, $\Gamma$ must be positive-semidefinite. The total energy in the fields is given by

$$N \equiv \sum_j \Gamma_{jj} = \operatorname{tr}\Gamma, \tag{4}$$

where tr denotes the matrix trace operation. A normalized mutual coherence can also be defined as

$$g \equiv \frac{\Gamma}{\operatorname{tr}\Gamma}, \tag{5}$$

such that $\operatorname{tr} g = 1$ and $\Gamma = Ng$. The mathematical similarities and physical connections of $g$ with the quantum density matrix have been noticed by many.[4–7,11,29,30]

## 3. MEASUREMENTS

Consider a linear optical network that takes the $J$ optical modes as part of its inputs and $K$ more ancillary inputs with zero fields, as depicted in Fig. 1. Mathematically, the mutual coherence matrix of the input fields can be written as

$$\Gamma \oplus 0 = \begin{pmatrix} \Gamma & 0 \\ 0 & 0 \end{pmatrix}, \tag{6}$$

where 0 denotes a matrix with all zero elements and $\Gamma \oplus 0$ becomes the mutual coherence of the combined $J+K$ inputs. A passive linear optical network that processes the fields gives an output mutual coherence matrix given by $U(\Gamma \oplus 0)U^\dagger$, where $U$ is the field scattering matrix.[1,27] If the network is lossless, the total energy must be conserved, and $U$ must be a unitary matrix. The average energy in each output mode is then a diagonal component of $U(\Gamma \oplus 0)U^\dagger$, or

$$\bar{n}_k = e_k^\dagger U(\Gamma \oplus 0)U^\dagger e_k = N p_k, \tag{7}$$

$$p_k = e_k^\dagger U(g \oplus 0)U^\dagger e_k, \tag{8}$$

where $e_k$ is the unit column vector, with the $k$th element being 1 and other elements being zero, and $p_k$ is normalized as $\sum_k p_k = 1$. It is known mathematically that $p_k$ can be equivalently expressed as[31]

$$p_k = \operatorname{tr}\left(E_k g\right), \tag{9}$$

where the set of $E_k$'s are called a positive operator-valued measure (POVM) in quantum measurement theory.[31] Conversely, given a POVM, it is known that there always exists a realization in the form of Eq. (8).[31] This means that any given POVM can in principle be realized via a linear optical network with vacuum ancilla inputs and photon counting. Note that the physics here is still classical, and only the mathematics of quantum measurement is employed.

Consider now a measurement of the energies of the output modes and assume that the photodetection statistics of the outputs $n = (n_1, n_2, \dots, n_{J+K})$ are Poisson, viz.,

$$P(n) = \prod_k \exp\left(-\bar{n}_k\right) \frac{\bar{n}_k^{n_k}}{n_k!}, \tag{10}$$

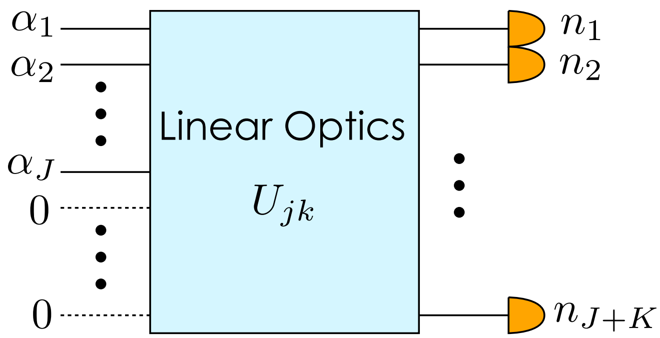

Figure 1. A schematic of an optical measurement with vacuum ancilla inputs, passive lossless linear optics, and photon-counting measurements.

which is a standard model in fluorescence microscopy[2] and also stellar imaging[28] at optical frequencies and beyond. It ignores bunching or antibunching effects but can be reproduced from ab-initio considerations in quantum optics by assuming thermal statistics with low occupancy numbers.[1,11,16,27] Another illuminating way of writing Eq. (10) is

$$P(n) = \sum_{L=0}^{\infty} \mathcal{M}(n|L)\Pi(L), \tag{11}$$

where

$$\mathcal{M}(n|L) = \begin{cases} L! \prod_k p_k^{n_k}/n_k!, & \sum_k n_k = L, \\ 0, & \text{otherwise} \end{cases} \tag{12}$$

is the multinomial distribution conditioned on $L$ photoelectrons[32] and

$$\Pi(L) = \exp(-N)\frac{N^L}{L!} \tag{13}$$

is the Poisson distribution for $L$ with $N$ being the average number of photoelectrons. Physically, $p_k$ can be regarded as the probability distribution of each photoelectron, as determined by the normalized mutual coherence matrix $g$ of the optical fields and the measurement $\{E_k\}$ according to Eq. (9).

## 4. PARAMETER ESTIMATION

The Fisher information is a standard precision measure for parameter estimation[32,33] and is recently gaining popularity in incoherent imaging.[2,3,28] For a family of probability distributions parameterized as $P(n|\theta)$ with $\theta$ being a vector of parameters, the Fisher information matrix is defined as

$$J_{\mu\nu}(\theta) \equiv \sum_n P(n|\theta)\left[\frac{\partial}{\partial\theta_\mu}\ln P(n|\theta)\right]\left[\frac{\partial}{\partial\theta_\nu}\ln P(n|\theta)\right]. \tag{14}$$

It is often used to lower-bound the mean-square estimation error of any unbiased estimator via the Cramér-Rao bound.[32,33] Bayesian and minimax generalizations of the bound valid for any biased or unbiased estimator are also possible.[15,33,34]

Assume that $N$ is given and $g$ depends on the unknown parameters $\theta$, such as the locations of fluorophores or stars. For a given $n$, $L = \sum_k n_k$ is also given, and the log-likelihood function can be expressed as

$$\ln P(n|\theta) = \ln \mathcal{M}(n|L,\theta) + \ln \Pi(L). \tag{15}$$

Since only $g(\theta)$ and thus $p_k(\theta)$ depend on the parameters, we obtain

$$\frac{\partial}{\partial\theta_\mu}\ln P(n|\theta) = \frac{\partial}{\partial\theta_\mu}\ln\mathcal{M}(n|L,\theta), \tag{16}$$

$$J(\theta) = \sum_L \Pi(L)J(\theta|L), \tag{17}$$

$$J_{\mu\nu}(\theta|L) = \sum_n \mathcal{M}(n|L,\theta)\left[\frac{\partial}{\partial\theta_\mu}\ln\mathcal{M}(n|L,\theta)\right]\left[\frac{\partial}{\partial\theta_\nu}\ln\mathcal{M}(n|L,\theta)\right]. \tag{18}$$

In other words, the Fisher information for the Poisson model is the average of the information for the multinomial model. They can be easily evaluated to give

$$J(\theta|L) = L\mathcal{J}(\theta), \tag{19}$$

$$J_{\mu\nu}(\theta) = N\mathcal{J}(\theta), \tag{20}$$

$$\mathcal{J}(\theta) \equiv \sum_k p_k(\theta)\left[\frac{\partial}{\partial\theta_\mu}\ln p_k(\theta)\right]\left[\frac{\partial}{\partial\theta_\nu}\ln p_k(\theta)\right], \tag{21}$$

where $\mathcal{J}(\theta)$ is the Fisher information per photoelectron.

Equation (21) has the same form as Eq. (14), with $p_k(\theta)$ now playing the role of a probability distribution, which can be expressed in terms of a unit-trace positive-semidefinite matrix $g(\theta)$ and a POVM $\{E_k\}$ according to Eq. (9). These mathematical facts lead to an upper bound on $\mathcal{J}(\theta)$ given by[35]

$$\mathcal{J}(\theta) \le \mathcal{K}(\theta), \tag{22}$$

which means that $\mathcal{K} - \mathcal{J}$ is positive-semidefinite. $\mathcal{K}$ is the Helstrom-Fisher information matrix[4] defined as

$$\mathcal{K}_{\mu\nu}(\theta) \equiv \operatorname{Re}\operatorname{tr}\left[\mathcal{L}_\mu(\theta)\mathcal{L}_\nu(\theta)g(\theta)\right], \tag{23}$$

and the matrices $\mathcal{L}_\mu(\theta)$ are given implicitly by

$$\frac{\partial g(\theta)}{\partial\theta_\mu} = \frac{1}{2}\left[\mathcal{L}_\mu(\theta)g(\theta) + g(\theta)\mathcal{L}_\mu(\theta)\right]. \tag{24}$$

$\mathcal{K}$ depends on the mutual coherence matrix $g(\theta)$ only and not the measurement $\{E_k\}$. In other words, Eq. (22) is a limit on the Fisher information that can be extracted from the light using any linear optics and photon counting. This is a more specific result than the quantum formalism,[4,11] which is valid for any quantum measurement, although the semiclassical formalism here involves only basic statistics optics concepts and does not presume any knowledge of quantum mechanics.

Consider, for example, the problem of estimating the separation between two incoherent point sources.[11,14,18–20] Using the multinomial or Poisson model, it was previously shown that, for direct imaging, the information $\mathcal{J}(\theta)$ decreases to zero for decreasing separation, especially when Rayleigh's criterion is violated;[18–20] this vanishing of Fisher information is called Rayleigh's curse in our work.[11,14] Our computation of the Helstrom-Fisher information,[11,14] on the other hand, shows that $\mathcal{K}(\theta)$ is constant regardless of the separation and can be much higher than the direct-imaging information. We have further shown that the methods of spatial-mode demultiplexing and image-inversion interferometry, both of which involve only linear optics and photon counting, offer information that approaches the Helstrom-Fisher value and substantially improves upon direct imaging.[11,14]

## 5. BINARY HYPOTHESIS TESTING

For detection and hypothesis-testing problems, Chernoff and Bhattacharyya distance measures are more useful.[33] The detection of binary stars in astronomy[36] and protein multimers in fluorescence microscopy[37] are notable applications. These measures are also relevant to the Ziv-Zakai and Weiss-Weinstein error bounds for parameter estimation.[34] Consider two probability distributions $P(n|\theta_0)$ and $P(n|\theta_1)$ and the quantity

$$B(s) \equiv \sum_n P(n|\theta_0) \left[\frac{P(n|\theta_1)}{P(n|\theta_0)}\right]^s, \quad 0 \le s \le 1. \tag{25}$$

$-\inf_{0\le s\le 1} \ln B(s)$ is called the Chernoff information[38] and $-\ln B(1/2)$ is called the Bhattacharyya distance.[33]

For a given $n$, $L$ is uniquely determined, and if we assume further that $\Pi(L)$ does not depend on $\theta$, the likelihood ratio becomes

$$\frac{P(n|\theta_1)}{P(n|\theta_0)} = \frac{\mathcal{M}(n|L,\theta_1)}{\mathcal{M}(n|L,\theta_0)}, \tag{26}$$

which gives

$$B(s) = \sum_L \Pi(L) B(s|L), \qquad B(s|L) = \sum_n \mathcal{M}(n|L,\theta_0)\left[\frac{\mathcal{M}(n|L,\theta_1)}{\mathcal{M}(n|L,\theta_0)}\right]^s. \tag{27}$$

Again, this implies that $B(s)$ for the Poisson model is the average of $B(s|L)$ for the multinomial model. It is straightforward to show that

$$B(s|L) = b^L(s), \qquad b(s) = \exp\left\{N\left[b(s)-1\right]\right\}, \qquad b(s) \equiv \sum_k p_k(\theta_0)\left[\frac{p_k(\theta_1)}{p_k(\theta_0)}\right]^s. \tag{28}$$

Notice that $b(s)$ has the same form as Eq. (25) with $p_k(\theta_0)$ and $p_k(\theta_1)$ playing the role of probability distributions. Together with Eq. (9), these facts imply[39,40]

$$b(s) \ge \inf_{0\le r\le 1} \operatorname{tr}\left[g^r(\theta_1) g^{1-r}(\theta_0)\right], \tag{29}$$

$$b(1/2) \ge \operatorname{tr}\sqrt{\sqrt{g(\theta_1)}g(\theta_0)\sqrt{g(\theta_1)}}. \tag{30}$$

The lower bounds are again independent of the measurement and quantify the fundamental indistinguishability of the two mutual coherence matrices. In particular, we have recently used the right-hand sides of Eqs. (29) and (30) to investigate the fundamental limits to the resolution of one versus two incoherent sources;[17] Krovi, Guha, and Shapiro have similar results.[41]

Another fundamental quantity is the relative entropy, defined as

$$D \equiv \sum_n P(n|\theta_1) \ln \frac{P(n|\theta_1)}{P(n|\theta_0)}, \tag{31}$$

which is useful not only for hypothesis testing but also for communications.[38] The assumption of $\Pi(L)$ being independent of $\theta$ again leads to

$$D = \sum_L \Pi(L) D(L) = Nd, \quad D(L) = \sum_n \mathcal{M}(n|L,\theta_1)\ln\frac{\mathcal{M}(n|L,\theta_1)}{\mathcal{M}(n|L,\theta_0)} = Ld, \quad d \equiv \sum_k p_k(\theta_1)\ln\frac{p_k(\theta_1)}{p_k(\theta_0)}, \tag{32}$$

and a measurement-independent bound given by[9]

$$d \le \operatorname{tr}\left\{g(\theta_1)\left[\ln g(\theta_1) - \ln g(\theta_0)\right]\right\}. \tag{33}$$

## 6. CONCLUSION

The measurement-independent bounds presented here are powerful results, as they quantify the fundamental limits to information extraction from weak incoherent optical sources through linear optics and photon counting. Furthermore, given the POVM model of the optical measurement presented here, the attainability of the bounds can be investigated by borrowing results from quantum information, which has identified POVMs that can attain the bounds in many special cases.[4,35,42,43] Compared with the full photodetection model,[12,13,41] the Poisson model is less general but leads to much simpler mathematics. Our recent work on superresolution imaging[11,15–17] showcases the utility of such a mathematical formalism, and it is hoped that the semiclassical treatment here will bring it to a wider audience.

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