Seize the Moments for Subdiffraction Incoherent Imaging

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Resolved:

Not resolved:
Telescopes, Fluorescence Microscopy

(images from the internet)
Random arrival of photons
Each photon arrives at camera with position $x$ and probability density $f(x|\theta)$

Noisy data $Y$: e.g., positions $(x_1, x_2, \ldots)$ or histogram of photon count $(n_1, n_2, \ldots)$

$f$ depends on some unknown parameters $\theta$

**Estimator** $\hat{\theta}(Y)$: guess $\theta$ from noisy data $Y$

**Mean-square error:**

$$MSE = \mathbb{E} \left[ \hat{\theta}(Y) - \theta \right]^2.$$  \hspace{1cm} (1)
Cramér-Rao bound:

\[ \text{MSE}(\theta) \geq J(\theta)^{-1}, \quad J(\theta) = N \int_{-\infty}^{\infty} dx \frac{1}{f(x|\theta)} \left[ \frac{\partial f(x|\theta)}{\partial \theta} \right]^2. \] 

\( J \): Fisher information

Conventional **direct imaging** (photon counting on image plane):

\[ J(0) = 0, \quad J(\infty) = \frac{N}{4\sigma^2}, \quad \sigma = \frac{\lambda}{\text{NA}} \]

\[ \frac{1}{J(0)} = \infty, \quad \frac{1}{J(\infty)} = \frac{4\sigma^2}{N} \]

"Rayleigh's curse"

PALM, STORM, etc.: make **sparse subsets** of fluorophores emit

Avoid violating Rayleigh

Need **controllable** fluorophores

**slow, cumbersome**

doesn’t work for stars, passive imaging

https://cam.facilities.northwestern.edu/588-2/single-molecule-localization-microscopy/
Quantum: Direct imaging/photon counting is just one of the infinitely many possible measurements.

Helstrom (1967), etc.: For any measurement,

\[ \text{MSE} \geq J^{-1} \geq K^{-1}, \tag{3} \]

\[ K(\rho{\otimes^M}) = M \text{ tr } L^2 \rho, \tag{4} \]

\[ \frac{\partial \rho}{\partial \theta} = \frac{1}{2} (L \rho + \rho L). \tag{5} \]

\( K(\rho) \) is the quantum Fisher information, the ultimate amount of information in the photons.
Quantum Optics for Incoherent Imaging

- Thermal optical source: **average photon number per mode** $\epsilon \ll 1$

- Quantum state in $M$ spectral modes $= \rho^{\otimes M}$,

\[
\rho = (1 - \epsilon) \vert \text{vac} \rangle \langle \text{vac} \vert + \frac{\epsilon}{2} (\vert \psi_1 \rangle \langle \psi_1 \vert + \vert \psi_2 \rangle \langle \psi_2 \vert) + O(\epsilon^2),
\]

\[
\vert \psi_s \rangle \equiv \int_{-\infty}^{\infty} dx \psi(x - X_s) \vert x \rangle.
\]

- derived from zero-mean Gaussian Glauber-Sudarshan function

- see, e.g., Tsang, PRL **107**, 270402 (2011); Tsang, Nair, Lu, PRX (2016).
Quantum and classical Fisher information

\[
\frac{\mathcal{K}_{22}}{N(4\sigma^2)} \quad \text{vs} \quad \theta_2/\sigma
\]

Mean-square error / \((4\sigma^2/N)\)

Cramér-Rao bounds on separation error

- **Quantum** \((1/\mathcal{K}_{22})\)
- **Direct imaging** \((1/J_{22}^{(\text{direct})})\)

Tsang, Nair, Lu, PRX 6, 031033 (2016).
Select the photons in **Hermite-Gaussian (TEM)** modes first, then do photon counting

- Tsang, Nair, Lu, PRX (2016); Rehacek et al., OL 42, 231 (2017).
Spatial-Mode Demultiplexing (SPADE)

- Tsang, Nair, Lu, PRX (2016); Nair and Tsang, OE 24, 3684 (2016)
- Many other ways (in optical comm., photonic circuits, etc.)
- **Classical sources**
- **Far-field linear optics/photon counting**
- **Important applications** (astronomy, fluorescence microscopy, etc.)
□ **Incoherent sources**: energy in 1st-order mode is

\[ \propto \left( \frac{d}{2} \right)^2 + \left( -\frac{d}{2} \right)^2 = \frac{d^2}{2}. \]  

□ **0th-order mode** is just background noise; filtering it improves SNR.
1. Tham, Ferretti, Steinberg (Toronto), PRL 118, 070801 (2017).
   \[\sim 2 \times \text{QCRB}\]


6. Donohue et al. (Paderborn, Germany), PRL 121, 090501 (2018).


SUPER-RESOLUTION FOR ASTRONOMICAL APPLICATIONS

Due to the diffraction of light, an image of a point will never be a perfect point, but rather a widened dot. For this reason, every imaging system has a resolution limit, which is defined as the smallest separation that the object points can have, in order to still be distinguishable in the recorded image [1]. In the case of a conventional telescope system, the resolution limit is determined by the size of the primary mirror and to achieve better resolutions, ever larger telescopes have therefore been designed, making them extremely expensive and technologically challenging to build.

Using advanced strategies, it is possible to go beyond these limitations of conventional systems, which is called super-resolution. One way of enhancing resolution of an
Direct imaging of arbitrary source distribution $= F(X|\theta)$:

$$f(x|\theta) = \int dX |\psi(x - X)|^2 F(X|\theta),$$

$$J_{\mu\nu} = \int_{-\infty}^{\infty} dx \frac{1}{f} \frac{\partial f}{\partial \theta_\mu} \frac{\partial f}{\partial \theta_\nu}.$$  \hspace{1cm} (7)

Infinite number of sources: Infinite number of parameters
Defining Subdiffraction Regime

Sparse (Good Regime, PALM, STED, compressed sensing, etc.)

Subdiffraction (Bad Regime)

\[ \text{object width } \equiv \Delta \ll 1. \] (8)
Define parameters as object moments:

\[ \theta_{\mu} = \int dX X^\mu F(X|\theta) \] (9)

CRB for \( \Delta \ll 1 \):

\[ (J^{-1})_{\mu\nu} = \frac{O(1)}{N}. \] (10)

Special case: for 2 point sources with separation \( d \), \( \theta_2 = d^2/4 \),

\[ J^{(d)} = \left( \frac{\partial \theta_2}{\partial d} \right)^2 J_{22} = NO(d^2). \] (11)

Tsang, NJP 19, 023054 (2017); PRA 97, 023830 (2018).
Quantum Limit

- One-photon density operator:

\[ \rho_1(\theta) = \int dXF(X|\theta) \psi_X \langle \psi_X |, \quad |\psi_X \rangle = \int_{-\infty}^{\infty} dx \psi(x - X) |x\rangle. \quad (12) \]

- mixed state
- infinite number of spatial modes
- infinite number of parameters


\[
(J^{-1})_{\mu\mu} \geq (K^{-1})_{\mu\mu} \geq O(\Delta^2[\mu/2]) \frac{N}{N}.
\]

(13)

See also Zhou and Jiang (Yale), arXiv:1801.02917 (2018).

- Big enhancements possible when
  - **Subdiffraction:** \( \Delta \ll 1 \)
  - **Second or higher moments:** \( \mu \geq 2 \)
Gaussian PSF [Tsang, NJP (2017)]:
- For moments with even $\mu_1$ & even $\mu_2$: TEM basis
  - See also Yang et al., Optica (2016)
- For other moments: interference of pairs of TEM modes
Any centrosymmetric and separable PSF [Tsang, PRA (2018)]:

- For moments with even $\mu_1$, even $\mu_2$: “PSF-adapted” (PAD) basis
  (Rehacek et al. OL (2017), generalizes TEM)
- For other moments: interference of pairs of PAD modes
Performance of SPADE

- **MSE of SPADE:**

\[
\text{MSE}_{\mu\mu} = \frac{O\left(\Delta^2\lceil\mu/2\rceil\right)}{N} + O\left(\Delta^{2\mu+4}\right).
\]

\(\sim\) quantum limit, big enhancement over direct imaging when

- \(\Delta \ll 1\) (subdiffraction)
- \(\mu = \mu_X + \mu_Y \geq 2\)
- bias is negligible.

- **Caveat:** \(\theta_\mu^2 = O(\Delta^{2\mu})\), fractional error:

\[
\frac{\text{MSE}}{\theta_\mu^2} = \frac{O(\Delta^{-2\lceil\mu/2\rceil})}{N} + O(\Delta^4).
\]

Need many photons, especially for large \(\mu\).
Elementary Explanation

- Wavefunction from each point source:
  \[ \psi(x - X) \approx +X \times \]

- For one point source,
  \[
  \text{Energy in first-order mode } \propto X^2. \quad (14)
  \]

- A distribution of incoherent sources:
  \[
  \text{Total energy in first-order mode } \propto \int dX F(X|\theta)X^2. \quad (15)
  \]
1. Tsang, Nair, Lu, PRX 6, 031033 (2016).
5. Ang, Nair, Tsang, PRA 95, 063847 (2017).
Future Directions

☐ 3D?
   - Backlund, Shechtman, Walsworth (Harvard/Technion), PRL 121, 023904 (2018);
   - Yu and Prasad (New Mexico, USA), arXiv:1805.09227 (2018);

☐ Experiments: the rest is engineering.

☐ FAQ:
   - https://sites.google.com/site/mankeitsang/news/rayleigh/faq
   - Mirror: https://www.ece.nus.edu.sg/stfpage/tmk/faq.html

☐ Thank you.
Quantum Technology 1.5

v1.0
- Classical Source
  - Quantum Processing
    - Classical Measurement
      - Solid-state Electronics

v1.5
- Classical Source
  - Quantum Processing
    - Quantum Measurement
      - Sensing Imaging

v2.0
- Quantum Source
  - Quantum Processing
    - Quantum Measurement
      - Quantum-Enhanced Metrology Quantum Computer Quantum Internet
**Diffraction-limited** $(\lambda/\text{NA})$

1. Fluorescence microscopy
2. Space telescopes (Webb, $10$ billion)
3. Ground-based telescopes (corrected by adative optics):
   (a) Large Binocular Telescope (LBT) (Strehl ratio $> 80\%$, $120$ million)
   (b) Giant Magellan Telescope (GMT)
   (c) Thirty Meter Telescope (TMT)
   (d) European Extremely Large Telescope (E-ELT) ($>1$ billion each)

Esposito et al., SPIE 8149, 814902 (2011).
Photon Shot Noise

- **Thermal sources** (stars, etc.)
  - Poisson, bunching negligible at optical

- **Fluorophores** (GFP, dye molecules, quantum dots, etc.)
  - Poisson, negligible anti-bunching
Two Point Sources

(a) 

\[ \psi_1(x) \quad \psi_2(x) \]

(b) 

\[ f(x | \theta) \]
- **Astrophotonics**: photonic circuits for stellar interferometry
- **Conventional wisdom**: less sensitive to atmospheric turbulence
- **Our work**: Fundamental advantage with diffraction + photon shot noise
- **Singapore**: fluorescence microscopy

\[
\Delta = \text{centroid displacement} + \text{object size} \ll 1
\]