Quantum Theory of Superresolution for Incoherent Optical Imaging *

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Generalized “Energetic” Quantum Limits

- Decision-theoretic, valid for any POVM
- Hypothesis testing: Tsang, Nair, PRA 86, 042115 (2012).
- Spectrum parameter (e.g., stochastic background), Theory+Experiment: Ng et al., PRA 93, 042121 (2016).

Quantum-Mechanics-Free Subsystems

- a.k.a. Backaction Evasion, Quantum Noise Cancellation, beat backaction-enforced SQL
- Tsang, Caves, PRL 105, 123601 (2010); PRX 2, 031016 (2012).

Time-Series Analysis

- Quantum Smoothing: Tsang, PRL 102, 250403 (2009); PRA 80, 033840 (2009); 81, 013824 (2010).

Cavity Quantum Electro-Optics

- Tsang, PRA 81, 063837 (2010); 84, 043845 (2011).
Tsang, Nair, and Lu, Physical Review X 6, 031033 (2016).

Experiments

Imaging of One Point Source

1. Fluorescence Microscopy ($X$ billion/year)
2. Space telescopes (Webb, $8.8$ billion)
3. Ground-based telescopes:
   - Large Binocular Telescope (LBT) ($120$ million)
   - Giant Magellan Telescope (GMT) (∼$1$ billion)
   - Thirty Meter Telescope (TMT) (∼$1.2$ billion)
   - European Extremely Large Telescope (E-ELT) (∼$1.2$ billion)
- Strehl ratio > 80% at infrared
Inferring Position of One Point Source

**Classical source**

**Given** $N$ detected photons, **mean-square error:**

$$\Delta X_1^2 = \frac{\sigma^2}{N}, \quad \sigma \sim \frac{\lambda}{\text{N.A.}}. \quad (1)$$

- Farrell (1966), Helstrom (1970), Lindegren (1978), Bobroff (1986), ... 
Superresolution Microscopy

- PALM, STORM, etc.: isolate emitters. Locate centroids.
  
  ![Diagram showing PALM, STORM, etc. process]

  [Link: https://cam.facilities.northwestern.edu/588-2/single-molecule-localization-microscopy/]

- Special fluorophores
- slow
- doesn’t work for stars

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The Nobel Prize in Chemistry 2014
Eric Betzig, Stefan W. Hell, William E. Moerner

The Nobel Prize in Chemistry 2014 was awarded jointly to Eric Betzig, Stefan W. Hell and William E. Moerner "for the development of super-resolved fluorescence microscopy".
Two Point Sources

(a) \( \Lambda(x) = \frac{1}{2} \left[ |\psi_1(x)|^2 + |\psi_2(x)|^2 \right] \)

(b) Rayleigh's criterion (1879): requires \( \theta_2 \gtrsim \sigma \) (heuristic)
Centroid and Separation Estimation

- Incoherent sources, Poisson statistics
- \( X_1 = \theta_1 - \theta_2 / 2, \quad X_2 = \theta_1 + \theta_2 / 2. \)
- Cramér-Rao bound for centroid:
  \[
  \Delta \theta_1^2 \geq \frac{\sigma^2}{N}.
  \]  \hspace{1cm} (2)

- CRB for separation estimation: two regimes
  - \( \theta_2 \gg \sigma: \)
    \[
    \Delta \theta_2^2 \geq \frac{4\sigma^2}{N},
    \]  \hspace{1cm} (3)
  - \( \theta_2 \ll \sigma: \)
    \[
    \Delta \theta_2^2 \to \frac{4\sigma^2}{N} \times \infty
    \]  \hspace{1cm} (4)
Cramér-Rao bound (standard in single-molecule imaging):

\[
\Delta \theta_1^2 \geq \frac{1}{\mathcal{J}_{11}^{(\text{direct})}}
\]

\[
\Delta \theta_2^2 \geq \frac{1}{\mathcal{J}_{22}^{(\text{direct})}}
\]  

(5)

\(\mathcal{J}^{(\text{direct})}\) is Fisher information for direct imaging

- Gaussian PSF, similar behavior for other PSF
- Rayleigh’s curse
- PALM/STED/STORM: avoid violating Rayleigh
Quantum Metrology

- Measuring the spatial intensity (direct imaging, CCD) is just one measurement method. **Quantum mechanics allows infinite possibilities.**
- Helstrom (1967): For any measurement

\[
\Sigma \geq J^{-1} \geq K^{-1},
\]  

(6)

\[
K_{\mu\nu} = M \text{Re}(\text{tr} \, L_{\mu} L_{\nu} \rho),
\]  

(7)

\[
\frac{\partial \rho}{\partial \theta_{\mu}} = \frac{1}{2} (L_{\mu} \rho + \rho L_{\mu}).
\]  

(8)

- \(K(\rho)\) is the quantum Fisher information, the ultimate amount of information in the photons.
- Mixed states:

\[
\rho = \sum_{n} D_{n} |e_{n}\rangle \langle e_{n}|,
\]  

(9)

\[
L_{\mu} = 2 \sum_{n,m; D_{n} + D_{m} \neq 0} \frac{|e_{n}\rangle \langle e_{m}|}{D_{n} + D_{m}} \left| \frac{\partial \rho}{\partial \theta_{\mu}} \right| e_{m} \rangle \langle e_{m}|.
\]  

(10)
- Mandel and Wolf, *Optical Coherence and Quantum Optics*; Goodman, *Statistical Optics*
- Thermal sources, e.g., stars, fluorescent particles.
- **Average photon number per mode** $\epsilon \ll 1$ at optical frequencies (visible, UV, X-ray, etc.).
- $\epsilon \sim 0.01$ for the sun at visible, $\epsilon \sim 10^{-6}$ for fluorophores.

Quantum state in $M$ temporal modes on image plane is $\rho^\otimes M$, where

$$
\rho = (1 - \epsilon) \langle \text{vac} | \langle \text{vac} | + \frac{\epsilon}{2} (|\psi_1\rangle \langle \psi_1 | + |\psi_2\rangle \langle \psi_2 |) + O(\epsilon^2)
$$

$|\psi_1\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_1) |x\rangle$, $|\psi_2\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_2) |x\rangle$.  \hspace{1cm} (11)

- derive from zero-mean Gaussian P function, mutual coherence
- Multiphoton coincidence: rare, little info as $\epsilon \ll 1$ (homeopathy)
- Similar model for stellar interferometry in Gottesman, Jennewein, Croke, PRL 109, 070503 (2012); Tsang, PRL 107, 270402 (2011).
Tsang, Nair, and Lu, Physical Review X 6, 031033 (2016)

\[ \Delta \theta_2^2 \geq \frac{1}{\mathcal{K}_{22}} = \frac{1}{N \Delta k^2}. \]  

**thermal sources with arbitrary** \( \epsilon \): Nair and Tsang, PRL (Editors’ Suggestion) 117, 190801 (2016); Lupo and Pirandola, *ibid.* 117, 190802 (2016).

Hayashi *ed.*, *Asymptotic Theory of Quantum Statistical Inference*; Fujiwara JPA 39, 12489 (2006): there exists a POVM such that \( \Delta \theta_\mu^2 \rightarrow 1/\mathcal{K}_{\mu\mu}, N \rightarrow \infty. \)
project in Hermite-Gaussian basis:

\[ E_1(q) = |\phi_q\rangle \langle \phi_q|, \quad (14) \]
\[ |\phi_q\rangle = \int_{-\infty}^{\infty} dx \phi_q(x) |x\rangle, \quad (15) \]
\[ \phi_q(x) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} H_q \left( \frac{x}{\sqrt{2}\sigma} \right) \exp \left( -\frac{x^2}{4\sigma^2} \right). \quad (16) \]

Assume PSF \( \psi(x) \) is Gaussian (common).

\[ \frac{1}{J^{(HG)}_{22}} = \frac{1}{K_{22}} = \frac{4\sigma^2}{N}. \quad (17) \]

Maximum-likelihood estimator can saturate the classical bound asymptotically for large \( N \).

arXiv:1605.03799v2: Using SPADE and Max-Like, it is proven that

\[ \Sigma \leq \frac{16\sigma^2}{N} \text{ for any detected } N. \quad (18) \]
Spatial-Mode Demultiplexing (SPADE)

- Many other ways (optical comm.), e.g.,
**Elementary Explanation**

- **Incoherent sources**: energy in **first-order mode** is $\propto (d/2)^2 + (-d/2)^2 = d^2/2$
- **Zeroth-order mode** is just **background noise**, removing it improves SNR.
- Why quantum formalism?
  - Fundamental quantum limit
  - Ensures measurement is physical
  - Discover new possibilities
**SLIVER (SuperLocalization via Image-inVERsion interferometry)**

- Laser Focus World, Feb 2016 issue.
- Prior classical theory/experiment of image-inversion interferometer:
  - *No statistical analysis, predicted* &lt;2X resolution enhancement
## Theoretical Follow-up

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<td>N/A</td>
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<tr>
<td>4.</td>
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<tr>
<td>5.</td>
<td>1D</td>
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<td>Quantum, Bayesian, Minimax</td>
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### Other groups:

- Lupo and Pirandola, PRL (Editors’ Suggestion) 117, 190801 (2016).
Experiments

  - SLIVER
  - Laser, classical noise

  - Mode heterodyne
  - Laser

  - variation of SPADE
  - two independent SPDC sources, \( \sim 2 \times \text{quantum limit} \)

  - variation of SPADE
  - laser, close to quantum limit (caveat)
Seth Lloyd of the Massachusetts Institute of Technology in the US is impressed. ‘This is awesome work and I am amazed that it hasn’t been done before,’ he says. ‘Perhaps everyone thought it was too good to be true.’
Arbitrary Source Distributions

- Yang et al., Optica 3, 1148 (2016): even moments
Quantum Metrology Kills Rayleigh’s Criterion

Cramér-Rao bounds on separation error

- Quantum \(\frac{1}{\mathcal{K}_{22}}\)
- Direct imaging \(\frac{1}{\mathcal{J}_{22}^{\text{(direct)}}}\)

Mean-square error / \((4\sigma^2/N)\)

FAQ: https://sites.google.com/site/mankeitsang/news/rayleigh/faq
email: mankei@nus.edu.sg
If the count degeneracy parameter is much less than 1, it is highly probable that there will be either zero or one counts in each separate coherence interval of the incident classical wave. In such a case the classical intensity fluctuations have a negligible "bunching" effect on the photo-events, for (with high probability) the light is simply too weak to generate multiple events in a single coherence cell.

Zmuidzinas (https://pma.caltech.edu/content/jonas-zmuidzinas), JOSA A 20, 218 (2003):

"It is well established that the photon counts registered by the detectors in an optical instrument follow statistically independent Poisson distributions, so that the fluctuations of the counts in different detectors are uncorrelated. To be more precise, this situation holds for the case of thermal emission (from the source, the atmosphere, the telescope, etc.) in which the mean photon occupation numbers of the modes incident on the detectors are low, \( n \ll 1 \). In the high occupancy limit, \( n \gg 1 \), photon bunching becomes important in that it changes the counting statistics and can introduce correlations among the detectors. We will discuss only the first case, \( n \ll 1 \), which applies to most astronomical observations at optical and infrared wavelengths."


See also Labeyrie et al., An Introduction to Optical Stellar Interferometry, etc.

Fluorescent particles: Pawley ed., Handbook of Biological Confocal Microscopy, Ram, Ober, Ward (2006), etc., may have antibunching, but Poisson model is fine and standard because of \( \epsilon \ll 1 \).
Binary SPADE

Classical Fisher information

\[ J_{22}^{(HG)} = K_{22} \]

\[ J_{22}^{(direct)} \]

\[ J_{22}^{(b)} \]

Fisher information for sinc PSF

\[ K_{22} \]

\[ J_{22}^{(direct)} \]

\[ J_{22}^{(b)} \]
Numerical Performance of Maximum-Likelihood Estimators

Simulated errors for SPADE

Simulated errors for binary SPADE

- $L = \text{number of detected photons}$
- **biased** (violate CRB), $< 2 \times \text{CRB}$.
Bayesian CRB for any biased/unbiased estimator (e-print arXiv:1605.03799v2)

Quantum/SPADE: \( \sup_{\theta} \Sigma_{22}(\theta) \geq \frac{4\sigma^2}{N} \), \quad \text{Direct imaging: } \sup_{\theta} \Sigma_{22}^{(\text{direct})}(\theta) \geq \frac{\sqrt{2}\sigma^2}{3\sqrt{N}}. \tag{19} \
2D SPADE and SLIVER

Ang, Nair, Tsang, e-print arXiv:1606.00603
Misalignment

- $\xi \equiv |\hat{\theta}_1 - \theta|/\sigma \ll 1$
- Overhead photons $N_1 \sim 1/\xi^2$
- $\xi = 0.1, \ N_1 \sim 100$
- CRB for $X_s = \theta_1 \pm \theta_2/2$
Design quantum computer to
- Maximize information extraction
- Reduce classical computational complexity