Fundamental Quantum Limit to Waveform Estimation

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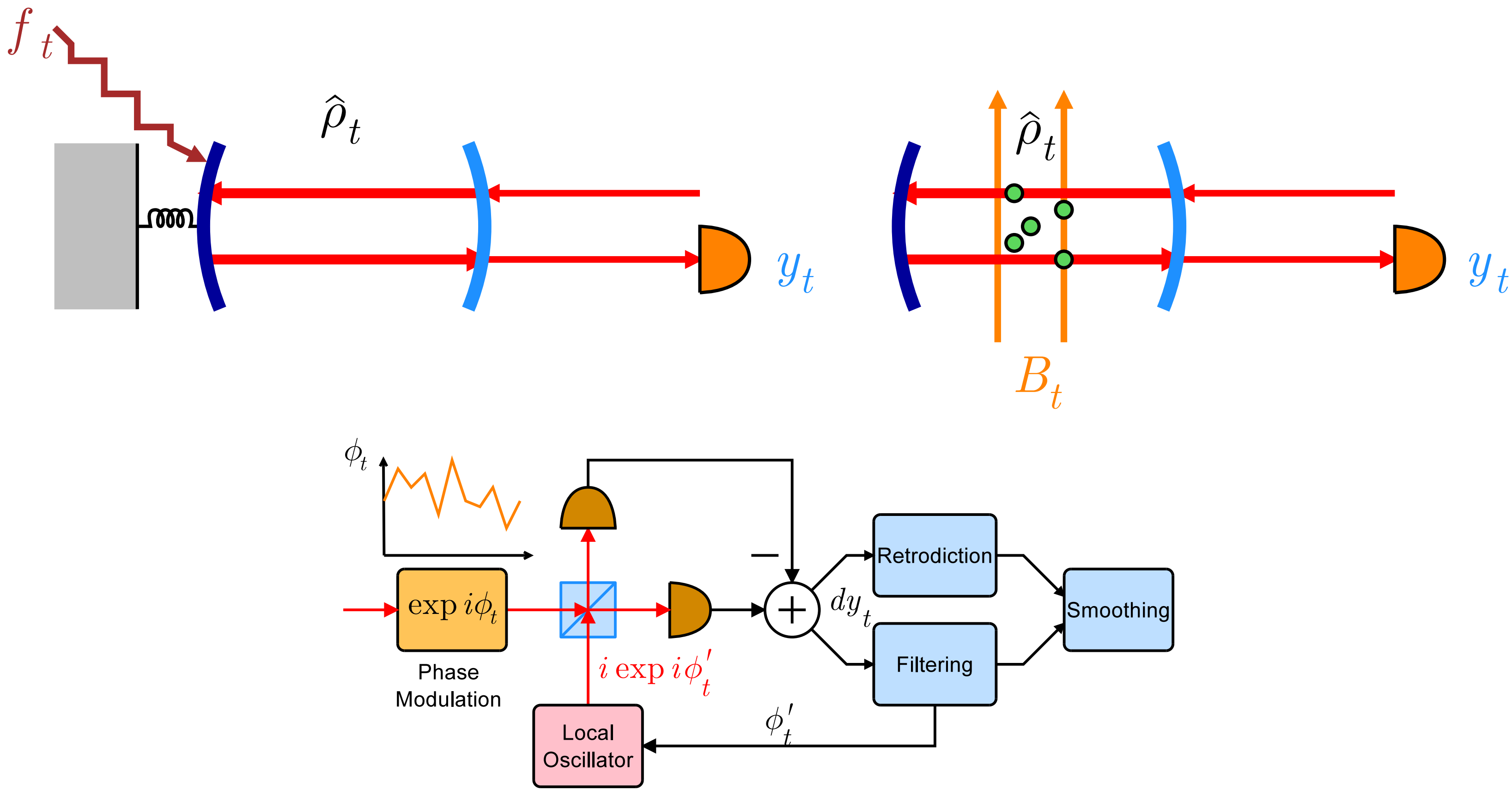
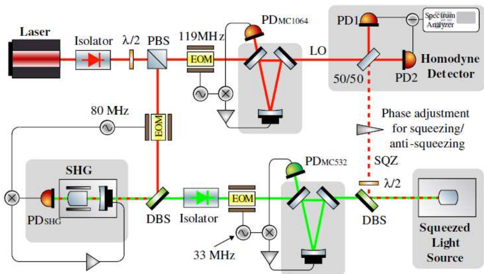
Basic Problem

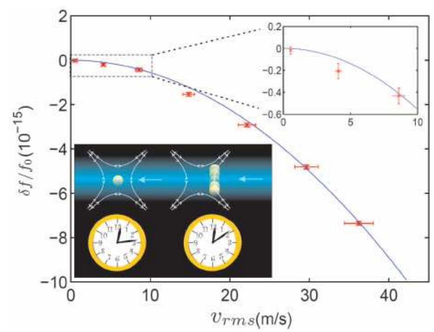


- **Quantum Lower Bound** on mean-square estimation error for continuous sensing
- How to achieve the limit?
  - Quantum Estimation: Optimal Data Processing
  - Quantum Control: Optimal Experiment
- Examples: optical phase estimation, force sensing

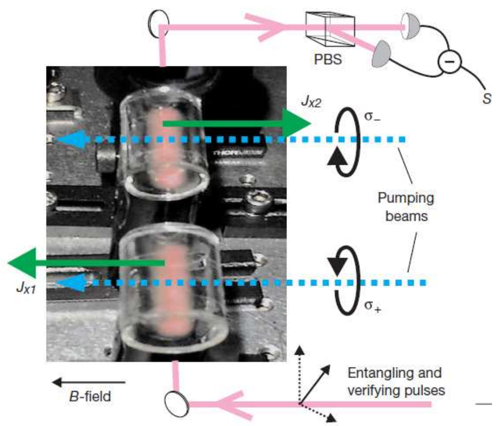
Experimental Advances



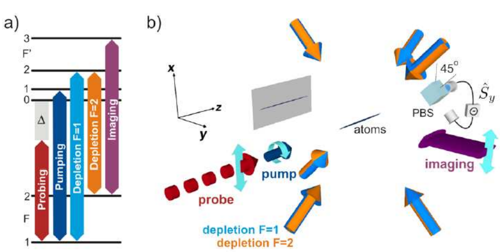
10dB squeezing, Vahlbruch *et al.*, PRL **100**, 033602 (2008).



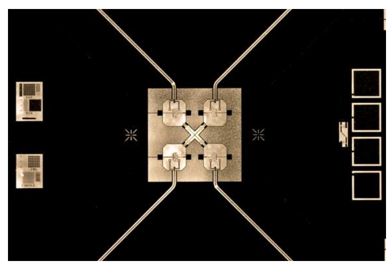
Chou *et al.*, Science **329**, 1630 (2010).



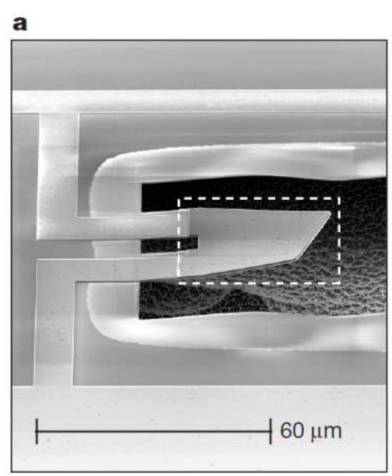
Julsgaard, Kozhekin, and Polzik, Nature **413**, 400 (2001).



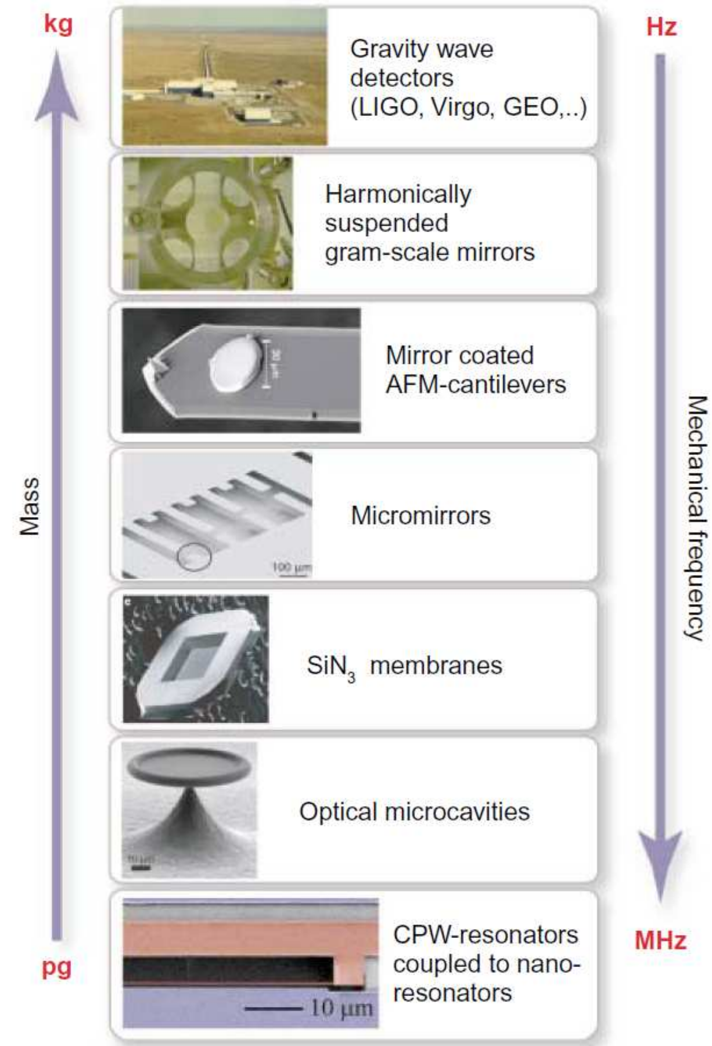
Koschorreck *et al.*, PRL **104**, 093602 (2010).



Neeley et al., Nature **467**, 570 (2010)

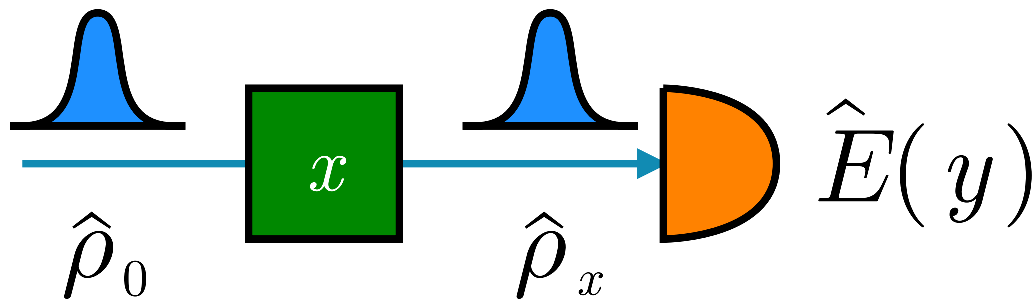


O'Connell *et al.*, Nature **464**, 697 (2010).



Kippenberg and Vahala, Science **321**, 1172 (2008), and references therein.

Parameter-Based Uncertainty Relation



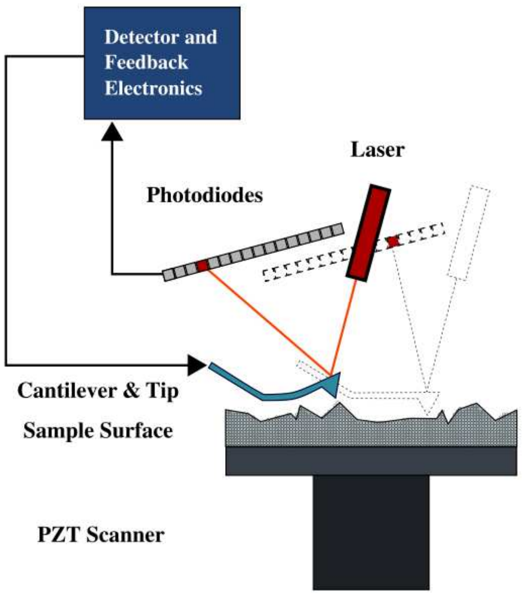
- Evolution depending on unknown parameter x: $\hat{\rho}_x = \exp(-i\hat{H}_x T/\hbar)\hat{\rho}_0 \exp(i\hat{H}_x T/\hbar)$
- Quantum Cramér-Rao bound (QCRB) ($\hat{h} \equiv \partial(\hat{H}_x T)/\partial x$):

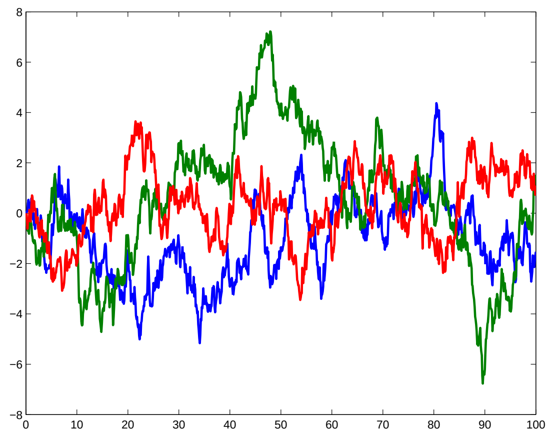
$$\langle \delta x^2 \rangle \left\langle \Delta \hat{h}^2 \right\rangle \geq \frac{\hbar^2}{4}, \tag{1}$$

$$\text{e.g. } \langle \delta \phi^2 \rangle \left\langle \Delta \hat{N}^2 \right\rangle \geq \frac{1}{4}. \tag{2}$$

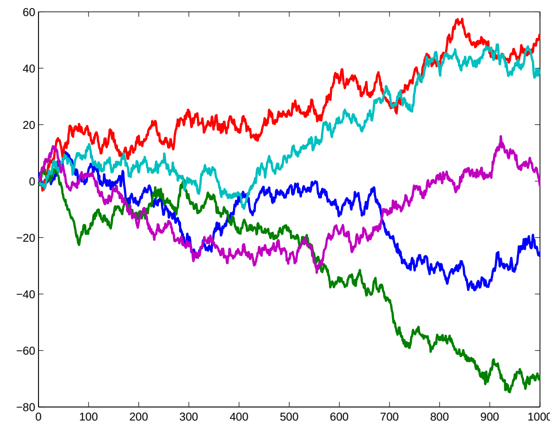
- C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, New York, 1976)
- V. Giovannetti, S. Lloyd, and L. Maccone, Science **306**, 1330 (2004); e-print arXiv:1102.2318 (2011).

Problem 1: $x(t)$ Changes in Time





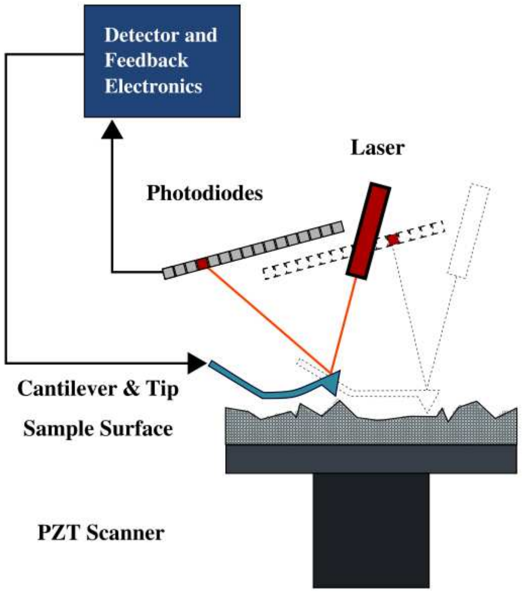
3 Realizations of Ornstein-Uhlenbeck Process (Brownian velocity)

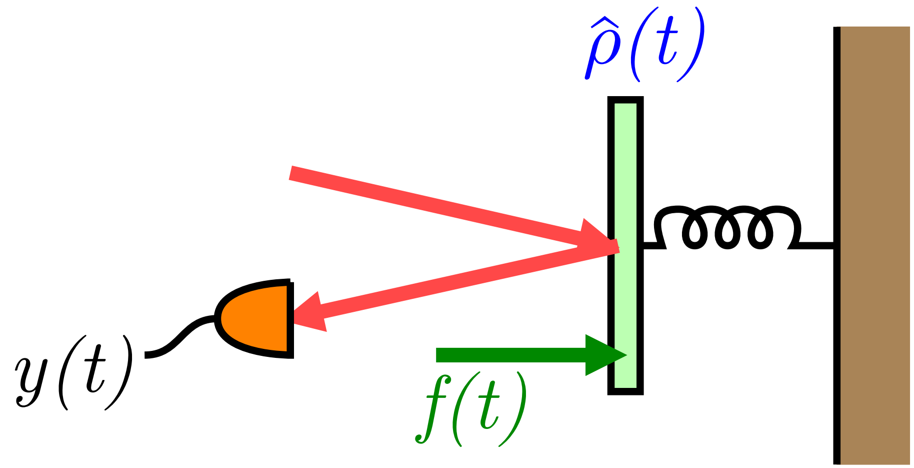


5 Realizations of Wiener Process (Brownian position in coarse-grained time)

- Gravitational waves [Buonanno *et al.*, PRD **55**, 3330 (1997); Apreda *et al.*, Nuclear Phys. B **631**, 342 (2002)], magnetic fields [Hall *et al.*, PRL **103**, 220802 (2009)] and clock signal relative to local oscillator phase can all be stochastic
- Stochastic process characterized by *a priori* probability density functional $P[x(t)]$

Problem 2: Continuous Dynamics





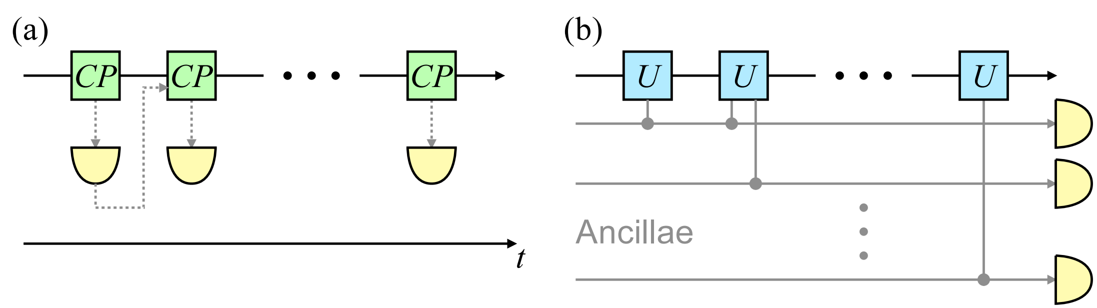
- Continuous coupling of time-varying signal to sensor
- Continuous quantum dynamics of sensor
- Continuous non-destructive measurements: must include measurement back-action
- Braginsky and Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992)

Quantum Information Theory to the Rescue

- Discretize time: finite number of measurement results and parameters
- Measurements and dynamics described by a sequence of completely positive (CP) maps:

$$P[y|x] = \operatorname{tr}\left[\mathcal{K}(y_N|x_N)\dots\mathcal{K}(y_1|x_1)\hat{\rho}_0\right] \tag{3}$$

- Equivalent quantum circuit model (Kraus representation theorem, principle of deferred measurements):



$$P[y|x] = \operatorname{tr}\left[\hat{E}(y)e^{-\frac{i}{\hbar}\hat{H}(x_N)\delta t}\dots e^{-\frac{i}{\hbar}\hat{H}(x_1)\delta t}\hat{\rho}_0 e^{\frac{i}{\hbar}\hat{H}(x_1)\delta t}\dots e^{\frac{i}{\hbar}\hat{H}(x_N)\delta t}\right] \tag{4}$$

- K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory* (Springer, Berlin, 1983).
- M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, Cambridge, 2010).

Dynamical Quantum Cramér-Rao Bound

$$\boxed{\langle \delta x_t^2 \rangle \geq F^{-1}(t,t)} \tag{5}$$

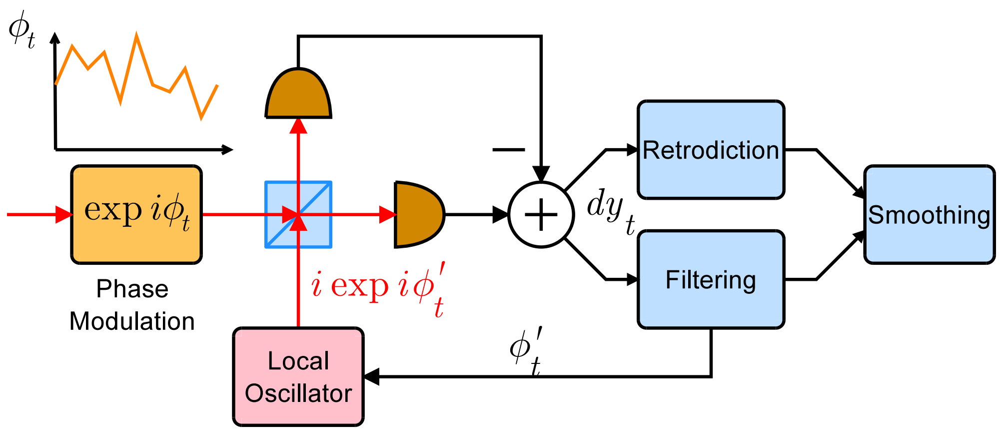
- $F^{-1}(t,t')$ is the inverse of Fisher information matrix $F(t,t')$, defined as $\int dt' F(t,t') F^{-1}(t',\tau) = \delta(t-\tau)$.
- Fisher information function has two components: $F(t,t') = F^{(Q)}(t,t') + F^{(C)}(t,t')$.
- $F^{(Q)}$ is quantum:

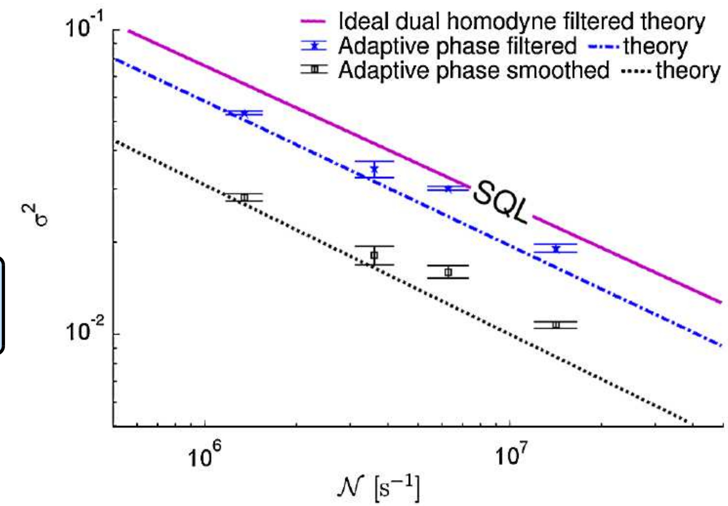
$$F^{(Q)}(t,t') = \frac{4}{\hbar^2}\left\langle \Delta\hat{h}(t)\Delta\hat{h}(t')\right\rangle, \quad \hat{h}(t) \equiv \int_{t_0}^{t_J} d\tau \hat{U}^\dagger(\tau,t_0)\frac{\delta \hat{H}[x(t),\tau]}{\delta x(t)}\hat{U}(\tau,t_0). \tag{6}$$

- $F^{(C)}$ incorporates *a priori* waveform information

$$F^{(C)}(t,t') = \int Dx P[x] \frac{\delta \ln P[x]}{\delta x(t)} \frac{\delta \ln P[x]}{\delta x(t')}. \tag{7}$$

Example 1: Optical Phase Estimation



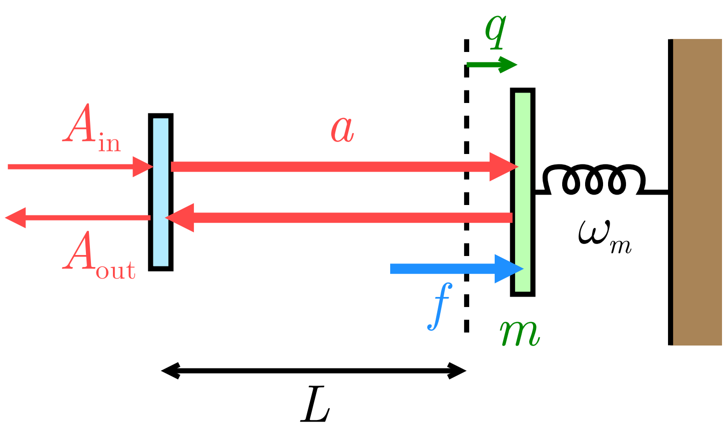


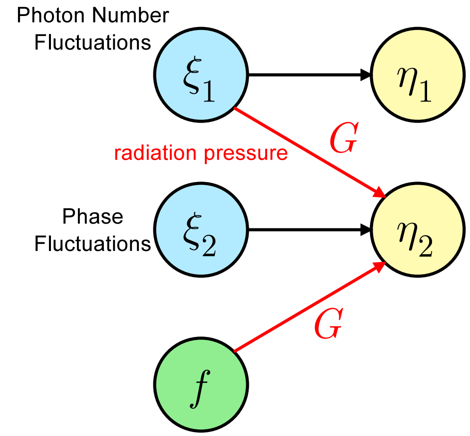
$$\langle \delta\phi^2 \rangle_{\mathrm{QCRB}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{4S_{\Delta \hat{I}}(\omega) + 1/S_\phi(\omega)}, \text{ e.g. } S^{\mathrm{coh}}_{\Delta \hat{I}}(\omega) = \frac{\bar{P}}{\hbar\omega_0}, \quad S^{\mathrm{OU}}_{\phi}(\omega) = \frac{\kappa}{\omega^2 + \epsilon^2}$$

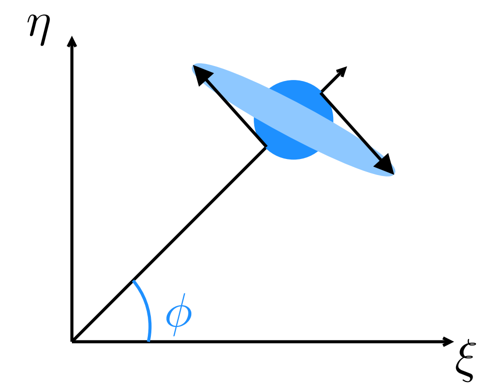
$$\langle \delta\phi^2 \rangle_{\mathrm{smoothing}} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{1/S_{\mathrm{noise}}(\omega) + 1/S_\phi(\omega)}, \quad \boxed{S_{\mathrm{noise}}(\omega) \geq \frac{1}{4S_{\Delta \hat{I}}(\omega)} = \frac{\hbar\omega_0}{4\bar{P}}}$$

- Filtering: Berry and Wiseman, PRA **65**, 043803 (2002); **73**, 063824 (2006), Smoothing: M. Tsang, J. H. Shapiro, and S. Lloyd, PRA **78**, 053820 (2008); **79**, 053843 (2009); M. Tsang, PRL **102**, 250403 (2009); PRA **80**, 033840 (2009); **81**, 013824 (2010).
- Wheatley *et al.*, PRL (*Editors' Suggestion*) **104**, 093601 (2010): very close to QCRB for coherent state

Example 2: Optomechanical Force Sensor







- QCRB:

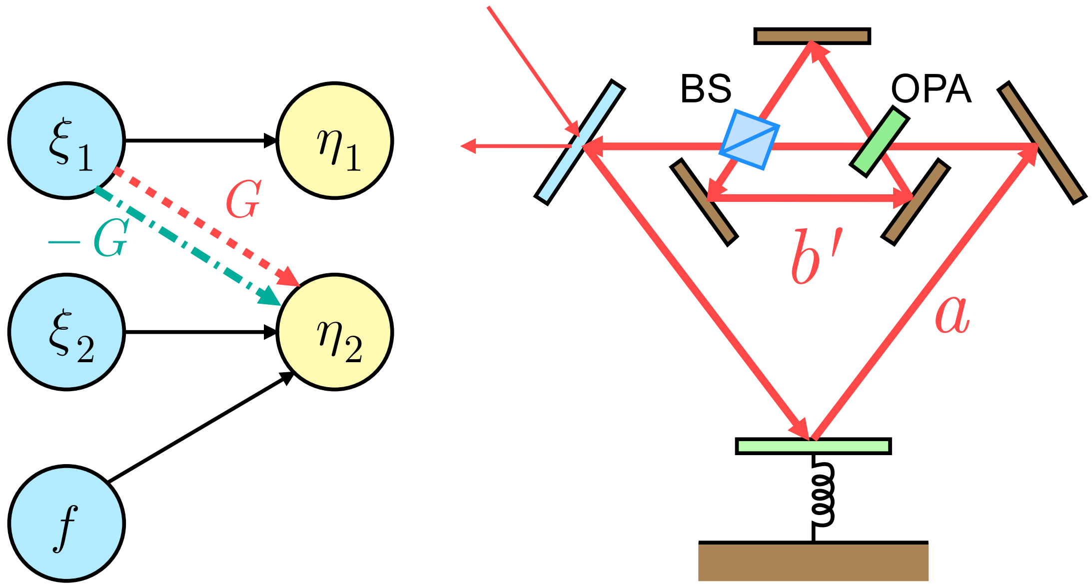
$$\langle \delta f^2 \rangle_{\rm QCRB} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(4/\hbar^2) S_{\Delta \hat{q}}(\omega) + 1/S_f(\omega)} \tag{8}$$

- Apply smoothing to $y(\omega) = \eta_2(\omega) = G(\omega)[f(\omega) + \mathrm{noise}(\omega)]$, SQL noise floor: $S_{\rm noise}(\omega) = S_2(\omega)/|G(\omega)|^2 + S_1(\omega) \geq \hbar/|G(\omega)| > \hbar^2/4S_{\Delta \hat{q}}(\omega)$.

$$\langle \delta f^2 \rangle_{\rm smoothing} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{1/S_{\rm noise}(\omega) + 1/S_f(\omega)} > \langle \delta f^2 \rangle_{\rm QCRB} . \tag{9}$$

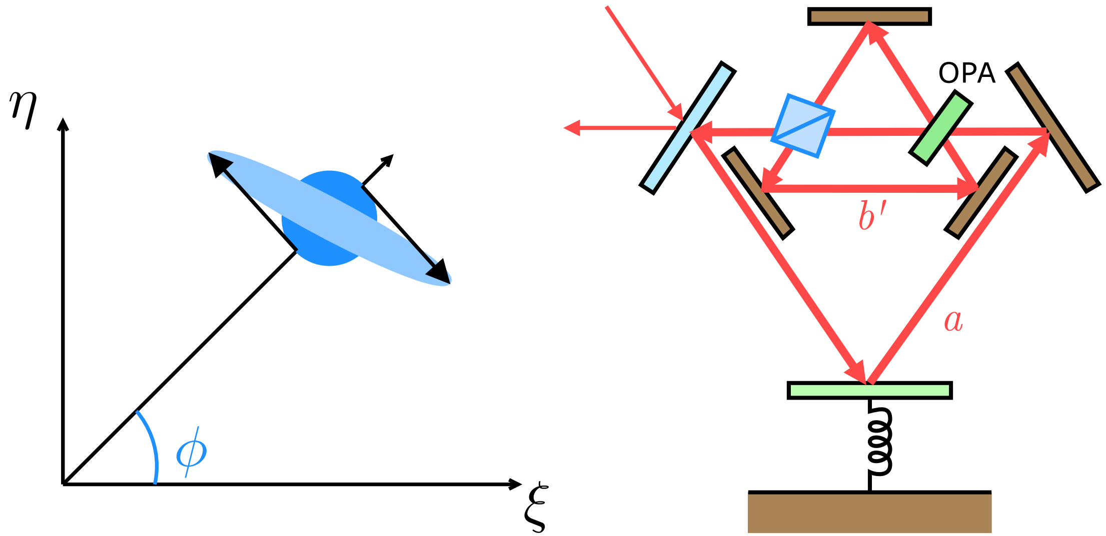
- $S_2(\omega)$: noise in optical phase, $S_1(\omega)$: measurement back-action noise

Quantum Noise Cancellation



- Coherent Feedforward Quantum Control
- M. Tsang and C. M. Caves, PRL **105**, 123601 (2010).

QCRB-Optimal Force Sensing



$$S_{\text{noise}}(\omega) = \frac{1}{|G(\omega)|^2} S_2(\omega) + \cancel{S_1(\omega)} = \frac{\hbar^2}{4 S_{\Delta \hat{q}}(\omega)}, \tag{10}$$

$$\langle \delta f^2 \rangle_{\text{smoothing,QNC}} = \langle \delta f^2 \rangle_{\text{QCRB}} \tag{11}$$

Summary

- **Quantum Cramér-Rao Bound for Waveform Estimation**
  - M. Tsang, H. M. Wiseman, and C. M. Caves, “Fundamental Quantum Limit to Waveform Estimation,” to appear in PRL [e-print arXiv:1006.5407].
- Quantum Smoothing
  - M. Tsang, J. H. Shapiro, and S. Lloyd, PRA **78**, 053820 (2008); **79**, 053843 (2009); M. Tsang, PRL **102**, 250403 (2009); PRA **80**, 033840 (2009); **81**, 013824 (2010).
- Quantum Noise Cancellation
  - M. Tsang and C. M. Caves, PRL **105**, 123601 (2010).
- Future Work
  - Applications: Optical phase estimation, force sensing, magnetometry, atomic clocks, ...
  - SQL, shot-noise limit, decoherence, “Heisenberg” limit, ...
  - Collaboration with Huntington’s group at ADFA@UNSW
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