Time-Symmetric Quantum Smoothing: An Optimal Estimation Theory for Quantum Sensing

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Outline

- Bayesian Estimation
- Classical and Quantum Filtering
- “Hybrid” Classical-Quantum Filtering
- Classical Smoothing
- Time-Symmetric Quantum Smoothing
  - Optical Phase-Locked Loop Design
  - Force Sensing
- References
  - Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008).
  - Tsang, Shapiro, and Lloyd, PRA 79, 053843 (2009).
  - Tsang, PRL 102, 250403 (2009).
  - Tsang, PRA 80, 033840 (2009).
  - Tsang, PRA 81, 013824 (2010).
Tracking an Aircraft

- or submarine, terrorist, criminal, mosquito, cancer cell, nanoparticle, . . .

Use Bayes theorem:

\[ P(x_t|y_t) = \frac{P(y_t|x_t)P(x_t)}{\int dx_t \text{ (numerator)}}, \]

- \( P(y_t|x_t) \) from observation noise, \( P(x_t) \) from a priori information.
Assume $x_t$ is a Markov process, use Chapman-Kolmogorov equation:

$$P(x_{t+\delta t}|y_t) = \int dx_t P(x_{t+\delta t}|x_t) P(x_t|y_t)$$  \hspace{1cm} (2)
Filtering: Real-Time Estimation

Applying Bayes theorem and Chapman-Kolmogorov equation repeatedly, we can obtain

\[ P(x_t | y_{t-\delta t}, \ldots, y_{t_0+\delta t}, y_{t_0}) \]  

Useful for control, weather and finance forecast, etc.

Wiener, Stratonovich, Kalman, Kushner, etc.
Quantum Filtering

Use "quantum Bayes theorem,"

\[ \hat{\rho}_t(\mid y_t) = \frac{\hat{M}(y_t) \hat{\rho}_t \hat{M}^\dagger(y_t)}{\text{tr}(\text{numerator})} \]  (4)

Use a completely positive map to evolve the system state (analogous to Chapman-Kolmogorov),

\[ \hat{\rho}_{t+\delta t}(\mid y_t) = \sum_{\mu} \hat{K}_{\mu} \hat{\rho}_t(\mid y_t) \hat{K}_{\mu}^\dagger \]  (5)

Belavkin, Barchielli, Carmichael, Caves, Milburn, Wiseman, Mabuchi, etc.

Useful for cavity QED, squeezing by QND measurement and feedback, etc.
Hybrid Classical-Quantum Filtering

Define hybrid density operator \( \hat{\rho}(x_t) \), \( P(x_t) = \text{tr}[\hat{\rho}(x_t)] \), \( \hat{\rho}_t = \int dx_t \hat{\rho}(x_t) \)

Use generalized quantum Bayes theorem and positive map + Chapman-Kolmogorov:

\[
\hat{\rho}(x_t \mid y_t) = \frac{\hat{M}(y_t \mid x_t) \hat{\rho}(x_t) \hat{M}^\dagger(y_t \mid x_t)}{\int dx_t \text{tr}(\text{numerator})},
\]

\( (6) \)

\[
\hat{\rho}(x_{t+\delta t} \mid y_t) = \int dx_t P(x_{t+\delta t} \mid x_t) \sum_\mu \hat{K}_\mu(x_t) \hat{\rho}(x_t \mid y_t) \hat{K}_\mu^\dagger(x_t)
\]

\( (7) \)
Hybrid Filtering Equations

Hybrid Belavkin equation (analogous to Kushner-Stratonovich):

\[
\begin{align*}
\frac{d\hat{\rho}(x,t)}{dt} &= \left[\mathcal{L}_0 + \mathcal{L}(x) - \frac{\partial}{\partial x_\mu} A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu} B_{\mu\nu}\right] \hat{\rho}(x,t) \\
&\quad + \frac{dt}{8} \left[2\hat{C}^T R^{-1} \hat{\rho}(x,t) \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{\rho}(x,t) - \hat{\rho}(x,t) \hat{C}^\dagger T R^{-1} \hat{C}\right] \\
&\quad + \frac{1}{2} \left(dy_t - \frac{dt}{2} \left\langle \hat{C} \hat{C}^\dagger \right\rangle\right)^T R^{-1} \left[\left(\hat{C} - \langle\hat{C}\rangle\right) \hat{\rho}(x,t) + \text{H.c.}\right] \\
&\quad + \frac{dt}{8} \left[2\hat{C}^T R^{-1} \hat{f}(x,t) \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{\rho}(x,t) - \hat{f}(x,t) \hat{C}^\dagger T R^{-1} \hat{C}\right] \\
&\quad + \frac{1}{2} dy_t^T R^{-1} \left[\hat{C} \hat{f}(x,t) + \text{H.c.}\right], \\
\hat{\rho}(x,t) &= \frac{\hat{f}(x,t)}{\int dx \text{ tr}[\hat{f}(x,t)]}.
\end{align*}
\]
Phase-Locked Loop Design


Smoothing: Estimation with Delay

More accurate than filtering

Sensing, analog communication, astronomy, crime investigation, ...

Delay
Conventional quantum theory is a predictive theory.
Quantum state described by $|\Psi_t\rangle$ or $\hat{\rho}_t$ can only be conditioned only upon past observations.
“Weak values” by Aharonov, Vaidman, et al.
“Quantum retrodiction” by Barnett, Pegg, Yanagisawa, etc.
Hybrid Time-Symmetric Smoothing

Use two operators to describe system: density operator $\hat{\rho}(x_t|y_{\text{past}})$ and a retrodictive effect operator $\hat{E}(y_{\text{future}}|x_t)$

$$P(x_t|y_{\text{past}},y_{\text{future}}) = \frac{\text{tr} \left[ \hat{E}(y_{\text{future}}|x_t) \hat{\rho}(x_t|y_{\text{past}}) \right]}{\int dx \text{(numerator)}}$$

(10)
Solve the predictive equation from $t_0$ to $t$ using \textit{a priori} $\hat{f}$ as initial condition, and solve the retrodictive equation from $T$ to $t$ using $\hat{g}(x, T) \propto \hat{1}$ as the final condition

\[
    df = dt \mathcal{L}(x) \hat{f} + \frac{dt}{8} \left(2 \hat{C}^T R^{-1} \hat{f} \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{f} - \hat{f} \hat{C}^\dagger T R^{-1} \hat{C} \right) + \frac{1}{2} dy_t^T R^{-1} \left( \hat{C} \hat{f} + \hat{f} \hat{C}^\dagger \right)
\]

\[
    -dg = dt \mathcal{L}^*(x) \hat{g} + \frac{dt}{8} \left(2 \hat{C}^\dagger T R^{-1} \hat{g} \hat{C} - \hat{C}^\dagger T R^{-1} \hat{C} \hat{g} - \hat{g} \hat{C}^\dagger T R^{-1} \hat{C} \right) + \frac{1}{2} dy_t^T R^{-1} \left( \hat{C}^\dagger \hat{g} + \hat{g} \hat{C} \right)
\]

\[
    h(x, t) = P(x_t = x | y_{\text{past}}, y_{\text{future}}) = \frac{\text{tr} \left[ \hat{g}(x, t) \hat{f}(x, t) \right]}{\int dx (\text{numerator})}
\]

Tsang, PRL 102, 250403 (2009).

Phase-Space Smoothing

Convert to Wigner distributions:

\[
\text{tr}[\hat{g}(x, t)\hat{f}(x, t)] \propto \int dqdp \ g(q, p, x, t)f(q, p, x, t)
\]

\[
h(x, t) = \frac{\int dqdp \ g(q, p, x, t)f(q, p, x, t)}{\int dx \ (\text{numerator})}
\]

If \( f(q, p, x, t) \) and \( g(q, p, x, t) \) are non-negative, equivalent to classical smoothing:

\[
h(q, p, x, t) = \frac{g(q, p, x, t)f(q, p, x, t)}{\int dx dq dp \ (\text{numerator})}, \quad h(x, t) = \int dq dp \ h(q, p, x, t)
\]

\( q \) and \( p \) can be regarded as classical, with \( h(q, p, x, t) \) the classical smoothing probability distribution.

If \( f(q, p, x, t) \) and \( g(q, p, x, t) \) are Gaussian, equivalent to linear smoothing, use two Kalman filters (Mayne-Fraser-Potter smoother).
Phase-Locked Loop: Post-Loop Smoother

Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008); 79, 053843 (2009); Tsang, PRA 80, 033840 (2009).

Assume force is an Ornstein-Uhlenbeck process:

\[
dx_t = -ax_t dt + bdW_t
\]  

(14)

Hamiltonian:

\[
\hat{H}(x_t) = \frac{\hat{p}^2}{2m} - x_t \hat{q}
\]  

(15)

Caves et al., RMP 52, 341 (1980); Braginsky and Khalili, Quantum Measurements (Cambridge University Press, Cambridge, 1992); Mabuchi, PRA 58, 123 (1998); Verstraete et al., PRA 64, 032111 (2001).
Filtering equation:

\[
d\hat{f}(x,t) = -\frac{i}{\hbar}dt \left[ \hat{H}(x), \hat{f}(x,t) \right] + dt \left( a \frac{\partial}{\partial x} + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} \right) \hat{f}(x,t) \\
+ \frac{\gamma}{8} dt \left[ 2\hat{q}\hat{f}(x,t)\hat{q} - \hat{q}^2 \hat{f}(x,t) - \hat{f}(x,t)\hat{q}^2 \right] + \frac{\gamma}{2} dy_t \left[ \hat{q}\hat{f}(x,t) + \hat{f}(x,t)\hat{q} \right],
\]

In terms of \( f(q,p,x,t) \):

\[
df = dt \left( -\frac{p}{m} \frac{\partial}{\partial q} - x \frac{\partial}{\partial p} + a \frac{\partial}{\partial x} + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} + \frac{\hbar^2\gamma}{8} \frac{\partial^2}{\partial p^2} \right) f + \gamma dy_t qf.
\]

Kalman filter:

\[
d\mu = A\mu dt + \Sigma C^T \gamma d\eta, \quad \frac{d\Sigma}{dt} = A\Sigma + \Sigma A^T - \Sigma C^T \gamma C \Sigma + Q,
\]

\[
A = \begin{pmatrix}
0 & 1/m & 0 \\
0 & 0 & 1 \\
0 & 0 & -a
\end{pmatrix}, \quad C = \begin{pmatrix}
1 & 0 & 0
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & 0 & 0 \\
0 & \hbar^2\gamma/4 & 0 \\
0 & 0 & b^2
\end{pmatrix}.
\]
Steady-State Filtering

- \( \mu_q, \mu_p, \Sigma_{qq}, \Sigma_{qp}, \) and \( \Sigma_{pp} \) determine conditional sensor quantum state \( \hat{\rho} \), \( \Sigma_{xx} \) is mean-square force estimation error.

- At steady state, let \( d\Sigma/dt = 0 \) (solve numerically).

- \( x_t = 0 \) limit (analytic):

\[
\Sigma_{qq} = \sqrt{\frac{\hbar}{m\gamma}}, \quad \Sigma_{qp} = \frac{\hbar}{2}, \quad \Sigma_{pp} = \frac{\hbar\sqrt{\hbar m\gamma}}{2}, \quad \Sigma_{qq}\Sigma_{pp} - \Sigma_{qp}^2 = \frac{1}{4} \quad (16)
\]

Smoothing

- Write retrodictive equation for $\hat{g}(x, t)$
- Convert $\hat{g}(x, t)$ to $g(q, p, x, t)$

solve for mean vector $\nu$ and covariance matrix $\Xi$ of $g(q, p, x, t)$ using a retrodictive Kalman filter

Define

$$h(q, p, x, t) = \frac{g(q, p, x, t)f(q, p, x, t)}{\int dqdpdx g(q, p, x, t)f(q, p, x, t)}$$

Mean $\xi$ and covariance $\Pi$ of $h(q, p, x, t)$:

$$\xi = \begin{pmatrix} \xi_q \\ \xi_p \\ \xi_x \end{pmatrix} = \Pi \left( \Sigma^{-1} \mu + \Xi^{-1} \nu \right)$$

$$\Pi = \begin{pmatrix} \Pi_{qq} & \Pi_{qp} & \Pi_{qx} \\ \Pi_{pq} & \Pi_{pp} & \Pi_{px} \\ \Pi_{xq} & \Pi_{xp} & \Pi_{xx} \end{pmatrix} = \left( \Sigma^{-1} + \Xi^{-1} \right)^{-1}$$

- $\xi_x$ is smoothing estimate of force, $\Pi_{xx}$ is smoothing error
- but what are $\xi_q, \xi_p, \Pi_{qq}, \Pi_{qp}, \Pi_{pp},$ and $h(q, p, x, t)$ in general?
Filtering vs Smoothing at Steady State

\[ a/\sqrt{b} = 0.01/\left(\hbar m\right)^{1/4}, \quad G = \hbar^{3/2}/m^{1/2}b, \quad s_{qq} = \sqrt{m\gamma/\hbar \Sigma_{qq}}, \quad s_{qp} = \Sigma_{qp}/\hbar, \quad s_{pp} = \Sigma_{pp}/\sqrt{\hbar^{3}m\gamma}, \]

blue: filtering, green: \( x_t = 0 \), red: smoothing

Signal-to-Noise Ratio can be further improved by frequency-dependent squeezing and coherent quantum filtering [Kimble et al., PRD 65, 022002 (2001)].
Summary

- Bayesian Quantum Estimation
- Hybrid Classical-Quantum Filtering
- Time-Symmetric Quantum Smoothing
  - Phase-Locked Loop Design
  - Force Sensing
- Other applications:
  - Atomic Magnetometry
  - Hardy’s Paradox
  - Tsang, PRA 81, 013824 (2010).
  - Epistemology: Einstein vs Bohr
  - Novel way of doing quantum mechanics? Aharonov, Vaidman et al.; Wheeler and Feynman; Gell-Mann and Hartle

http://sites.google.com/site/mankeitsang/ or Google “Mankei Tsang”

Life can only be understood backwards; but it must be lived forwards. – Soren Kierkegaard
Time-Symmetric Smoothing

\[
P(x_t|y_{t_0}, \ldots, y_{t-\delta t}) = \frac{P(y_{\text{future}}|x_t)P(x_t|y_{\text{past}})}{\int dx_t (\text{numerator})}
\]  

(18)

Mayne, Fraser, Potter, Pardoux

Unnormalized versions of \( P(x_t|\delta y_{\text{past}}) \) and \( P(\delta y_{\text{future}}|x_t) \) (retrodictive likelihood function) obey a pair of adjoint equations, one to be solved forward in time and one backward in time.
\[
df = dt \left\{ -\sum_{\mu} \frac{\partial}{\partial x_{\mu}} (A_{\mu} f) + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial x_{\mu} \partial x_{\nu}} \left[ (BQB^T)_{\mu\nu} f \right] 
- \left[ (\chi - \frac{\gamma}{2}) \frac{\partial}{\partial q} (q f) + \left(-\chi - \frac{\gamma}{2}\right) \frac{\partial}{\partial p} (p f) \right] + \frac{\gamma}{4} \left( \frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial p^2} \right) \right\} 
+ dy_t \left[ \sin(\phi - \phi'_t) \left( 2b + \sqrt{2\gamma q} + \sqrt{\gamma^2 \frac{\partial}{\partial q}} \right) + \cos(\phi - \phi'_t) \left( \sqrt{2\gamma p} + \sqrt{\gamma^2 \frac{\partial}{\partial p}} \right) \right] f. \tag{19}
\]

\[
dg = dt \left\{ \sum_{\mu} A_{\mu} \frac{\partial g}{\partial x_{\mu}} + \frac{1}{2} \sum_{\mu,\nu} (BQB^T)_{\mu\nu} \frac{\partial^2 g}{\partial x_{\mu} \partial x_{\nu}} 
+ \left[ (\chi - \frac{\gamma}{2}) q \frac{\partial g}{\partial q} + \left(-\chi - \frac{\gamma}{2}\right) p \frac{\partial g}{\partial p} \right] + \frac{\gamma}{4} \left( \frac{\partial^2 g}{\partial q^2} + \frac{\partial^2 g}{\partial p^2} \right) \right\} 
+ dy_t \left[ \sin(\phi - \phi'_t) \left( 2b + \sqrt{2\gamma q} - \sqrt{\gamma^2 \frac{\partial}{\partial q}} \right) + \cos(\phi - \phi'_t) \left( \sqrt{2\gamma p} - \sqrt{\gamma^2 \frac{\partial}{\partial p}} \right) \right] g. \tag{20}
\]
Quantum Smoothing

Can we use

\[ h(q, p, x, t) = \frac{g(q, p, x, t) f(q, p, x, t)}{\int dqdpdx} \]  \hspace{1cm} (21)

to estimate \( q \) and \( p \)?

- when \( g \) and \( f \) are non-negative, problem becomes classical

\[ h(x, t) = \int dqdp h(q, p, x, t) \]  \hspace{1cm} (22)

\( \xi_q, \xi_p \equiv \text{real part of weak values} \)

\( h(q, p, x, t) \) arises from statistics of weak measurements

\( h(q, p, x, t) \) can go negative, many versions of Wigner distributions for discrete degrees of freedom
Beyond Heisenberg?

Bayesian view: shouldn’t we be able to learn more about a system, whether classical or quantum, in retrospect?

Everett view: Need two wavefunctions or two density operators to solve problems.
**Weak Measurements**

The smoothing estimates $\xi_q$ and $\xi_p$ are equivalent to "weak values":

$$\xi_q = \text{Re} \frac{\int dx \text{tr}(\hat{g}\hat{q}\hat{f})}{\int dx \text{tr}(\hat{f})}, \quad \xi_p = \text{Re} \frac{\int dx \text{tr}(\hat{g}\hat{p}\hat{f})}{\int dx \text{tr}(\hat{f})}. \quad (23)$$


Make weak Gaussian and backaction evading measurements of $q$ and $p$:

$$\hat{M}(y_q) \propto \int dq \exp \left[-\frac{\epsilon (y_q - q)^2}{4}\right] |q\rangle\langle q|, \text{ same for } y_p, \quad (24)$$

$$P(y_q, y_p | y_{\text{past}}, y_{\text{future}}) \propto \int dq dp \exp \left\{ -\frac{\epsilon}{2} \left[ (y_q - q)^2 + (y_p - p)^2 \right] \right\} h(q, p, t) \quad (25)$$

in the limit of $\epsilon \to 0$ (opposite limit to von Neumann measurements)

$h(q, p, t)$ describes the apparent statistics of $q$ and $p$ in the weak measurement limit.