Quantum Theory of Optical Sensing: Estimation, Control, and Fundamental Limits *

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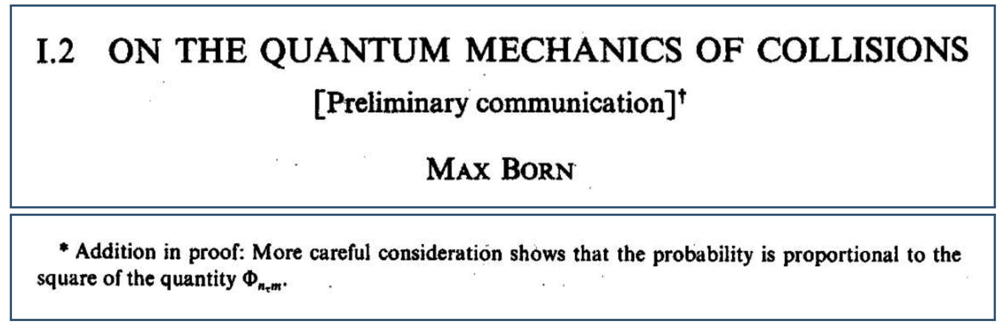
- Two foundations of quantum mechanics:

Schrödinger's equation

(https://www.flickr.com/photos/sababa-dan/396970817/)

Born's rule


I.2 ON THE QUANTUM MECHANICS OF COLLISIONS

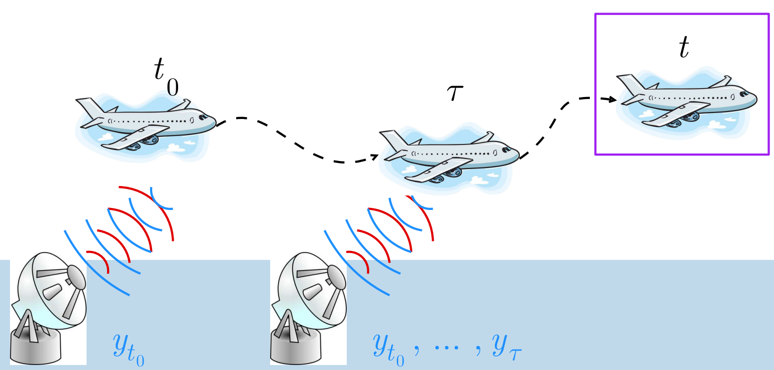
[Preliminary communication]†

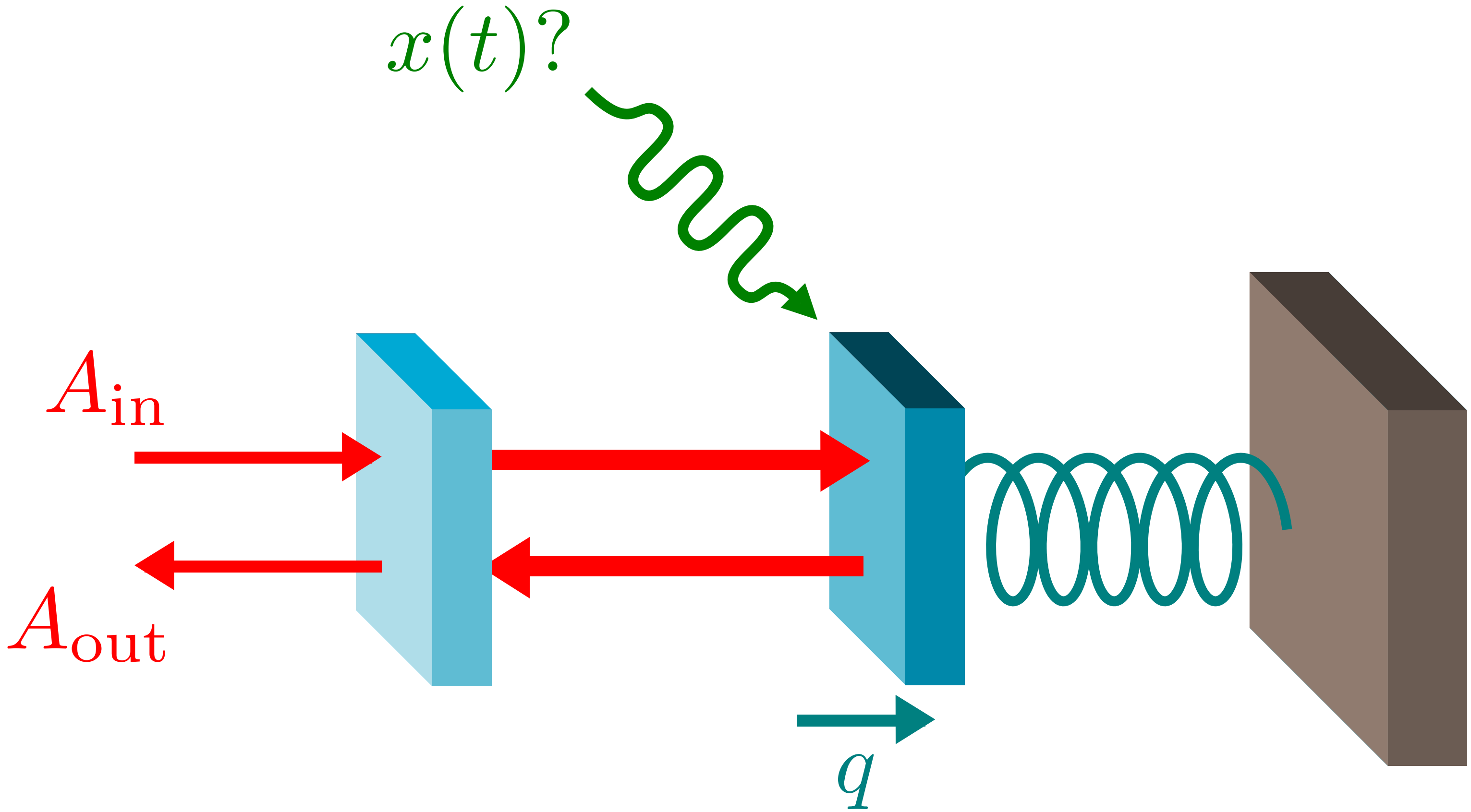
Max Born

* Addition in proof: More careful consideration shows that the probability is proportional to the square of the quantity $\Phi_{n_\tau m}$.

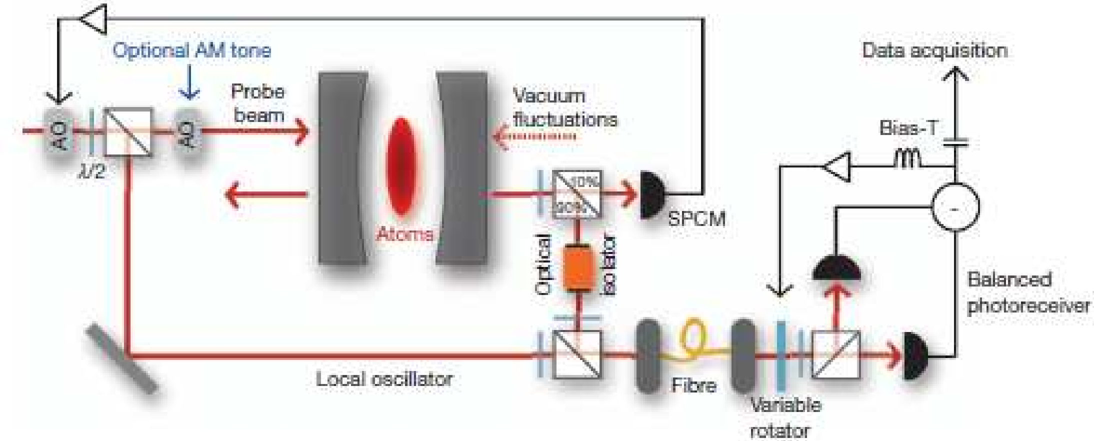
$$\Pr(x) = |\psi(x)|^2$$

- Impose **fundamental uncertainties** to information processing, communication, computing, and **sensing**.
- **Experimental technology** is catching up (see 2012 Nobel Prize), especially in **quantum optics**, **optomechanics**.
- Use **engineering techniques** (Bayesian inference, information theory, control theory, etc.) to address the challenges.

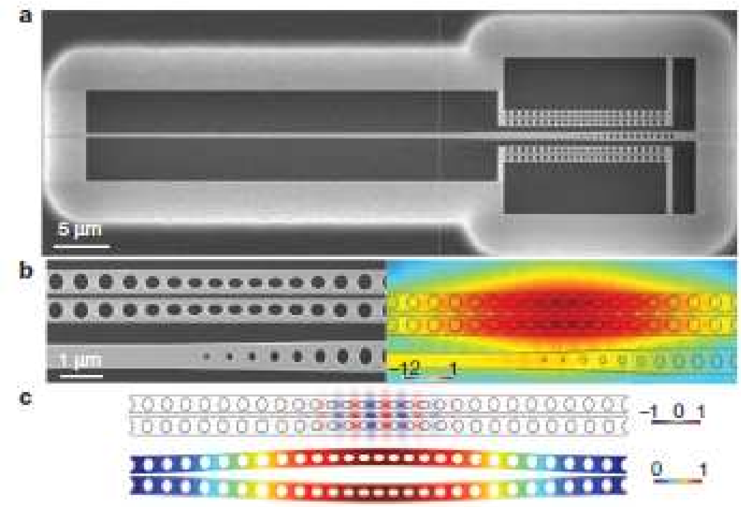




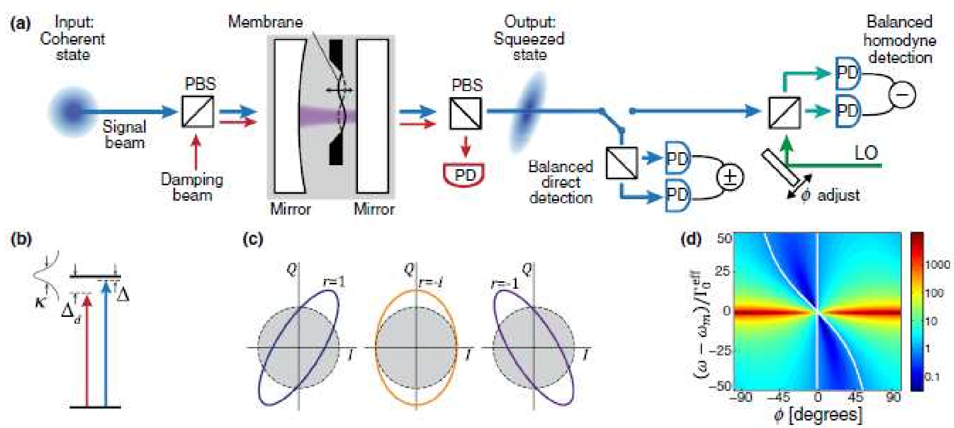
- Caves *et al.*, "On the measurement of a weak classical force coupled to a quantum-mechanical oscillator. I. Issues of principle," Rev. Mod. Phys. **52**, 341 (1980).
- Braginsky and Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).



Brooks *et al.*, Nature **488**, 476 (2012)

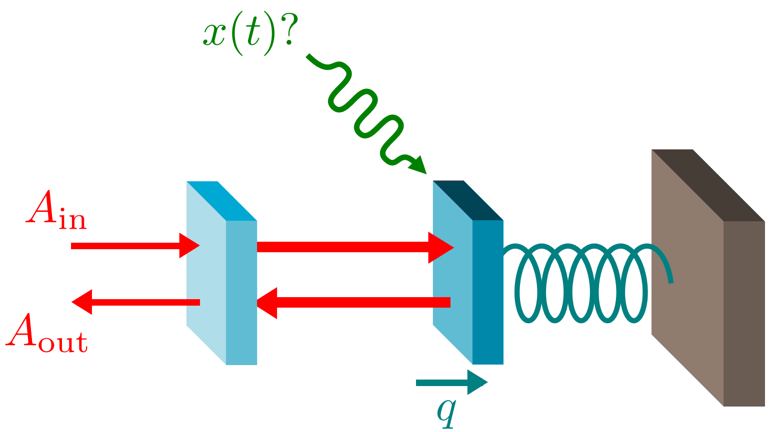


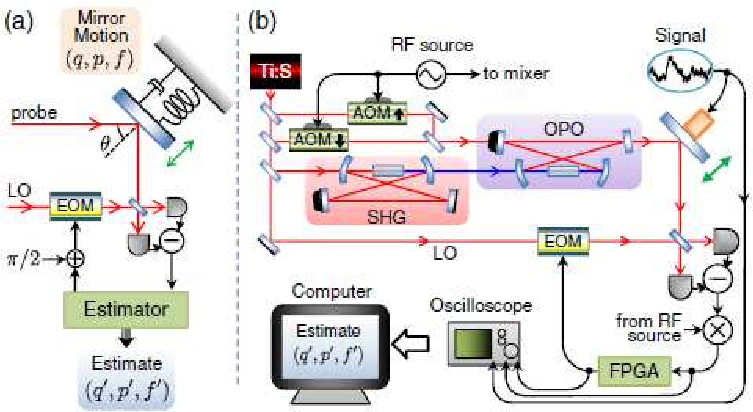
Safavi-Naeini *et al.*, Nature **500**, 185 (2013)

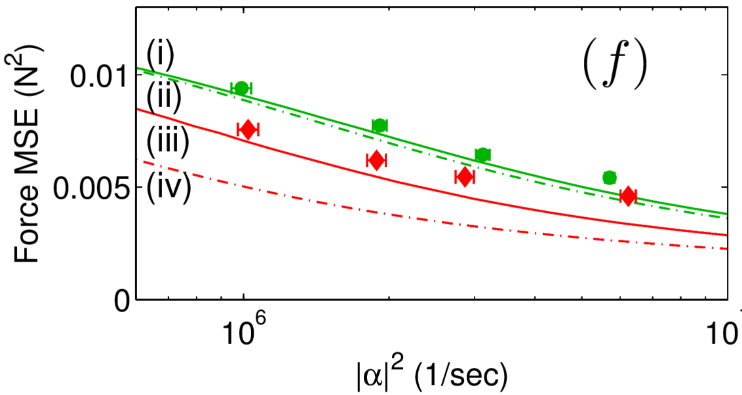


Purdy *et al.*, PRX **3**, 031012 (2013)

Quantum Waveform Estimation

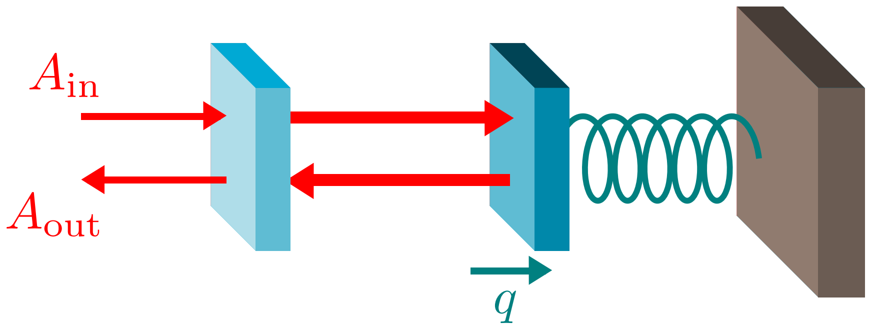




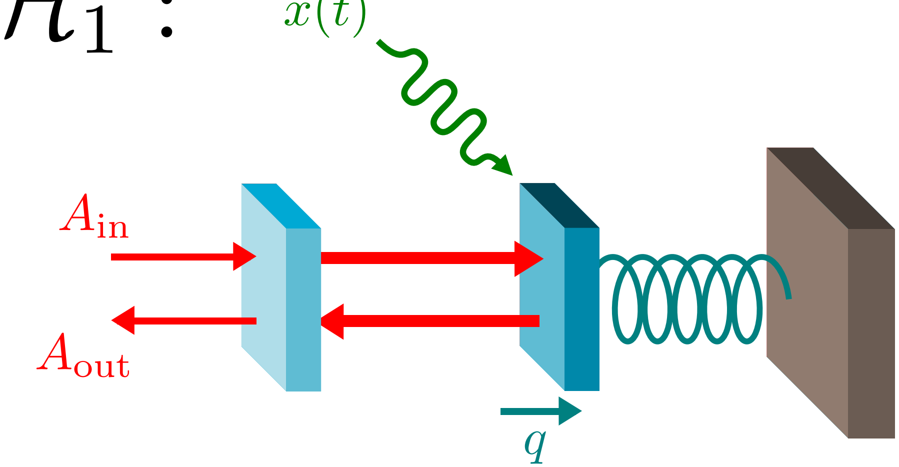


- **Quantum smoothing**: Tsang, PRL **102**, 250403 (2009); PRA **80**, 033840 (2009); **81**, 013824 (2010); Tsang, arXiv:1403.3353 (2014).
- **Quantum limits**: Tsang, Wiseman, Caves, PRL **106**, 090401 (2011); Tsang, NJP **15**, 073005 (2013).
- **Experiment** (with Furusawa's group at U. Tokyo, Huntington's group at UNSW@ADFA, Australia): Iwasawa, Tsang *et al.*, PRL **111**, 163602 (2013).
- **Optomechanical parameter estimation** (with Bowen's group at U. Queensland, Australia): Ang, Harris, Tsang, Bowen, NJP **15**, 103028 (2013).

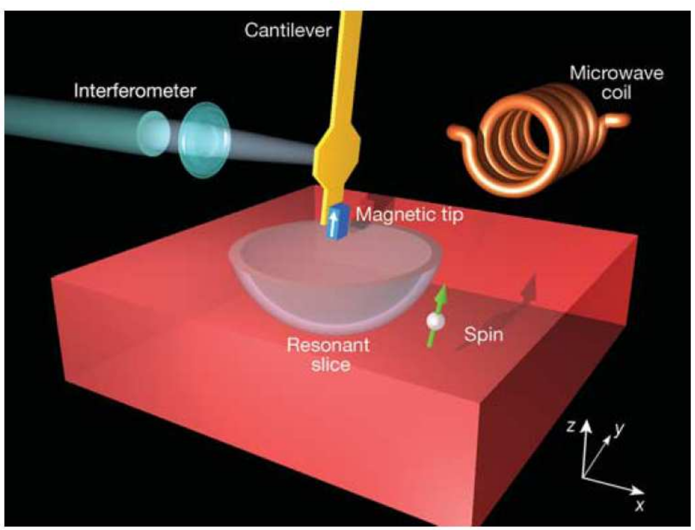
$\mathcal{H}_0$:



$\mathcal{H}_1$:



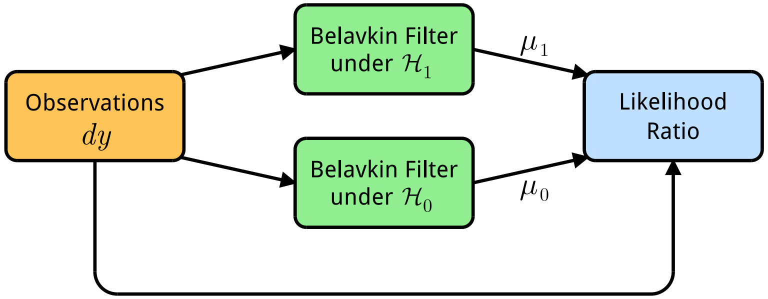
LIGO, Hanford



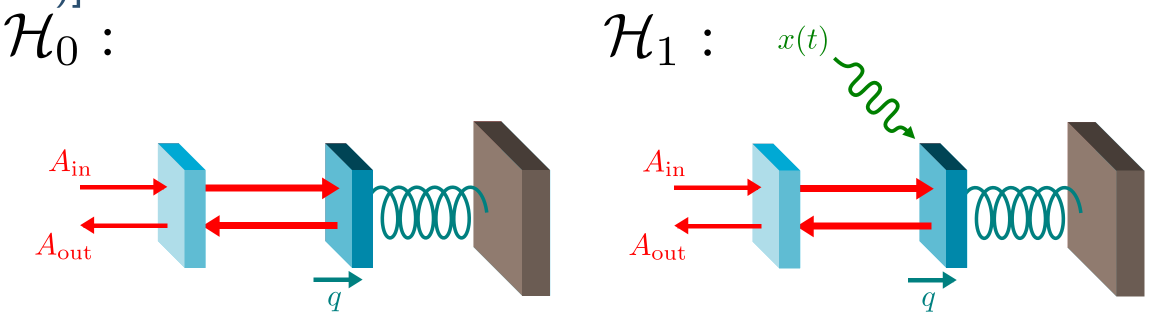
Rugar *et al.*, Nature **430**, 329 (2004).

Continuous Quantum Hypothesis Testing

- **Signal processing**: likelihood-ratio test [Tsang, PRL **108**, 230401 (2012); Ng and Tsang, PRA **90**, 022325 (2014)]



- **Fundamental quantum limit**: force detection [Tsang and Nair, PRA **86**, 042115 (2012); Tsang, NJP **15**, 073005 (2013)]



Statistical Binary Hypothesis Testing

- Null hypothesis: $P(Y|\mathcal{H}_0)$, alternative: $P(Y|\mathcal{H}_1)$.
- Type-I error probability (false-alarm probability):

$$P_{10} = \Pr\left(\tilde{\mathcal{H}}(Y) = \mathcal{H}_1 \middle| \mathcal{H}_0\right) \tag{1}$$

- Type-II error probability (miss probability):

$$P_{01} = \Pr\left(\tilde{\mathcal{H}}(Y) = \mathcal{H}_0 \middle| \mathcal{H}_1\right) \tag{2}$$

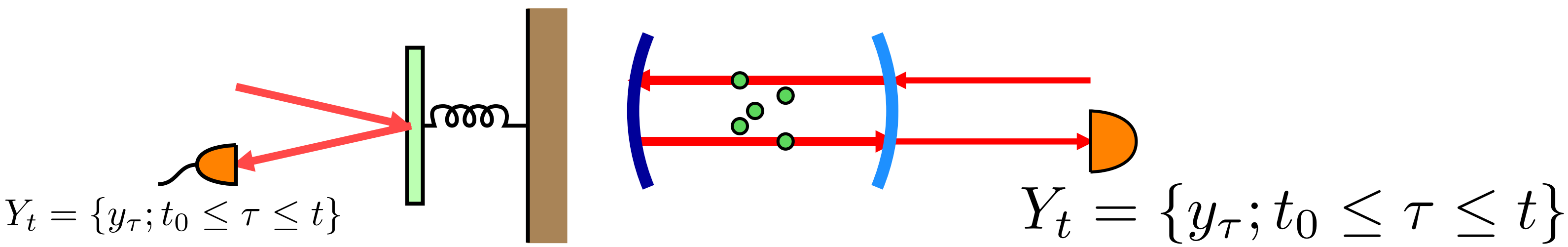
- **Average error probability**:

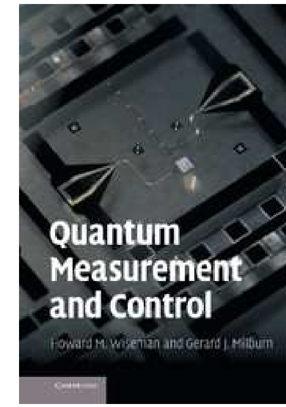
$$\boxed{P_e = P_{10}P_0 + P_{01}P_1} \tag{3}$$

- Alternative: Neyman-Pearson
- Optimal procedure: **likelihood-ratio test**:

$$\text{Decide on } \mathcal{H}_1 \text{ if } \ln\frac{P(Y|\mathcal{H}_1)}{P(Y|\mathcal{H}_0)} \geq \lambda, \tag{4}$$

$$\text{Decide on } \mathcal{H}_0 \text{ if } \ln\frac{P(Y|\mathcal{H}_1)}{P(Y|\mathcal{H}_0)} < \lambda. \tag{5}$$

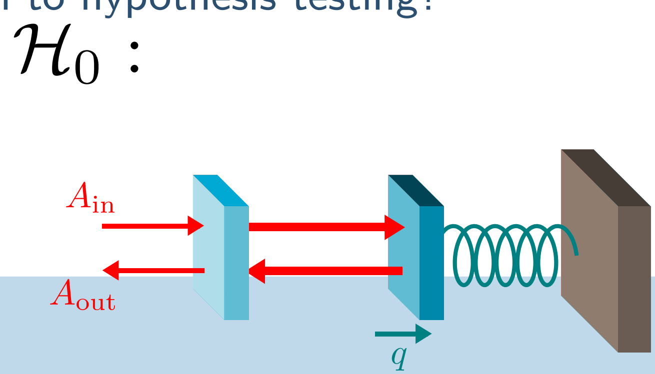


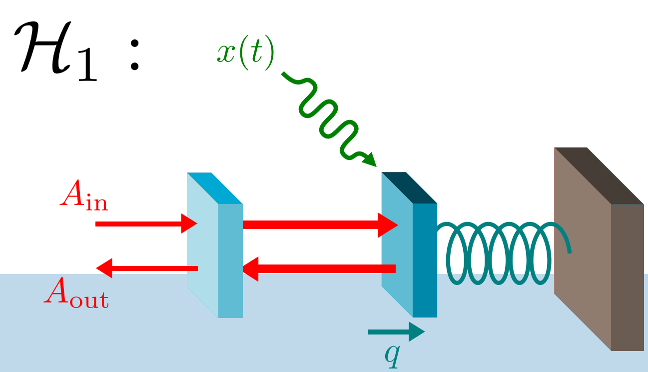


- Quantum control: squeezing, quantum error correction, cooling, sensing, etc.
- Optics, atoms, ions, superconducting microwave circuits, quantum dots, etc.
- linear **Belavkin** equation [Belavkin, Prob. Theor. Appl. **38**, 742 (1993); **39**, 640 (1994); arXiv:quant-ph/0510028]:

$$df_t = \mathcal{L}_t f_t dt + \frac{1}{2}\left(a_t f_t + f_t a_t^\dagger\right) dy_t, \quad \mathbb{E}\left(O_t | Y^t\right) = \frac{\mathrm{tr}\, O_t f_t}{\mathrm{tr}\, f_t}, \quad Y^t \equiv \{y(\tau); 0 \le \tau \le t\}. \tag{6}$$

- Filtering estimation of **quantum observables**.
- Application to hypothesis testing?





Estimator-Correlator Formula

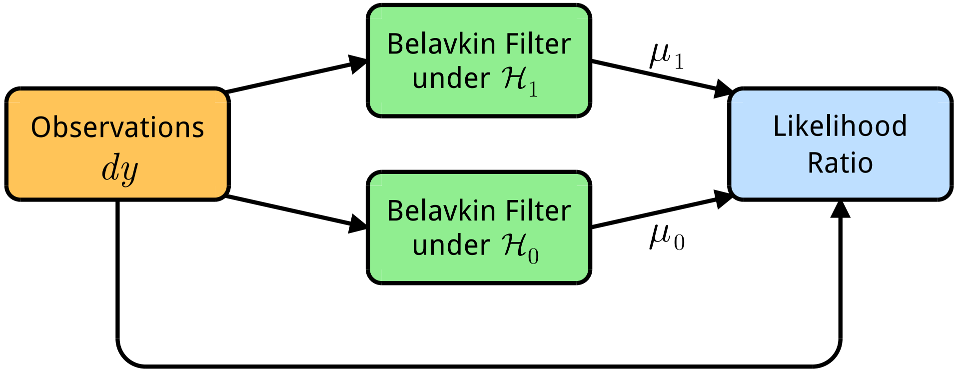
- Tsang, PRL **108**, 230401 (2012):

1. Calculate **assumptive estimates** of **directly measured observable** $q_t \equiv (a_t + a_t^\dagger)/2$:

$$\mu_0(t) \equiv \mathbb{E}(q_t|Y^t, \mathcal{H}_0), \qquad \mu_1(t) \equiv \mathbb{E}(q_t|Y^t, \mathcal{H}_1). \tag{7}$$

2. Use estimator-correlator formula:

$$\boxed{\ln \frac{dP(Y^T|\mathcal{H}_1)}{dP(Y^T|\mathcal{H}_0)} = \int_0^T dy_t\,[\mu_1(t) - \mu_0(t)] - \frac{1}{2}\int_0^T dt\,\left[\mu_1^2(t) - \mu_0^2(t)\right].} \tag{8}$$



- Classical: Duncan, Inf. Control 13, 62 (1968); T. Kailath, IEEE Trans. Inf. Theory 15, 350 (1969); T. Kailath and H.V. Poor, ibid. 44, 2230 (1998).
- Useful when $\mu_j(t)$ is easy to calculate (e.g., Kalman-Bucy filters)
- Similar results for Poissonian measurements

Proof

- Linear Belavkin equation (assuming one hypothesis):

$$df_t = \mathcal{L}_t f_t dt + \frac{1}{2}\left(a_t f_t + f_t a_t^\dagger\right) dy_t, \qquad \mathbb{E}\left(O_t|Y^t\right) = \frac{\operatorname{tr} O_t f_t}{\operatorname{tr} f_t}. \tag{9}$$

- $\mathcal{L}_t$ is Linblad generator, includes Hamiltonian, etc.
- Notice that

$$\operatorname{tr} f_t = \frac{dP(Y^t)}{dP_0(Y^t)}, \tag{10}$$

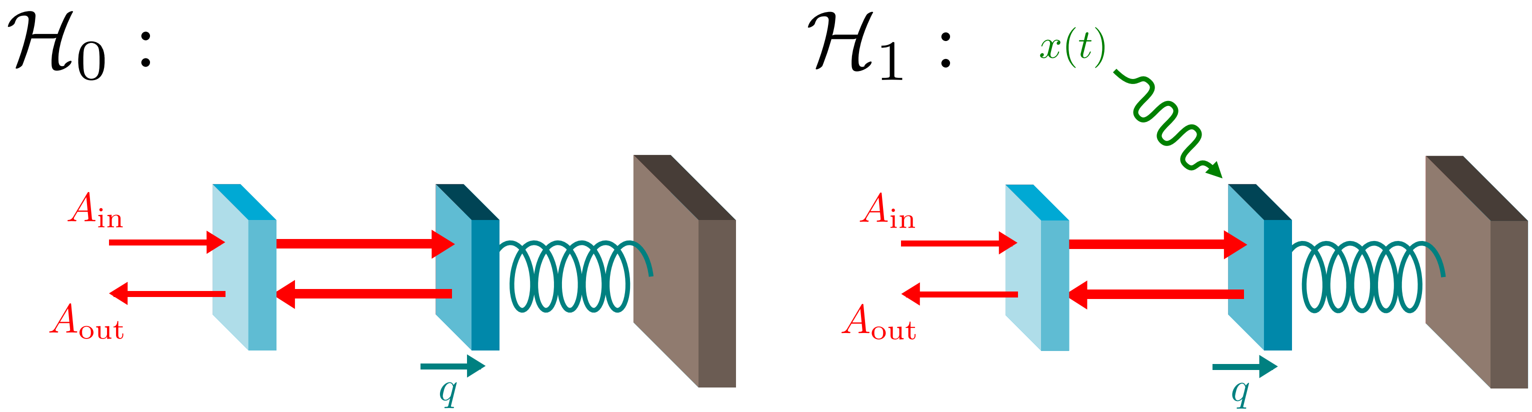
  $dP_0(Y^t)$ is Wiener measure.
- In terms of directly measured observable q_t,

$$d \operatorname{tr} f_t = \operatorname{tr} df_t = dy_t \operatorname{tr}\left(q_t f_t\right) = dy_t\, \mathbb{E}(q_t|Y^t) \operatorname{tr} f_t, \quad q_t \equiv \frac{1}{2}(a_t + a_t^\dagger). \tag{11}$$

- Use Itō calculus:

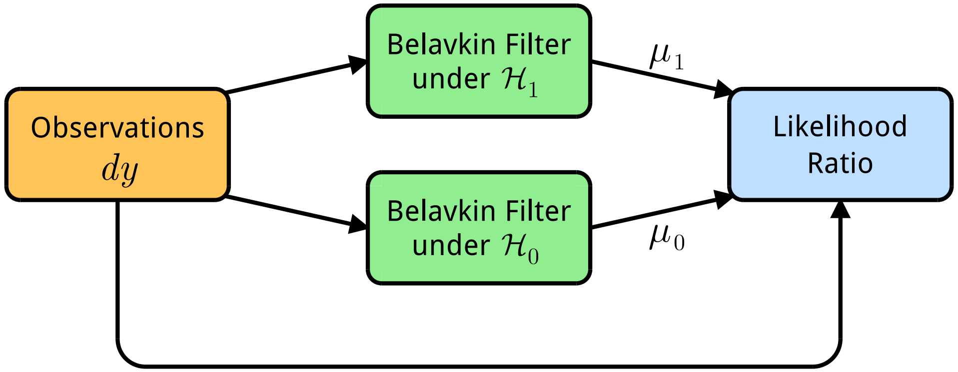
$$\ln \operatorname{tr} f_T = \ln \frac{dP(Y^T)}{dP_0(Y^T)} = \int_0^T dy_t\, \mathbb{E}(q_t|Y^t) - \frac{1}{2}\int_0^T dt\, \mathbb{E}^2(q_t|Y^t). \tag{12}$$

- Barchielli and Lupieri, in *Quantum Stochastics and Information*, eds. Belavkin and Guta, pp. 325–345 (2008).
- For hypothesis testing: Tsang, PRL **108**, 230401 (2012).



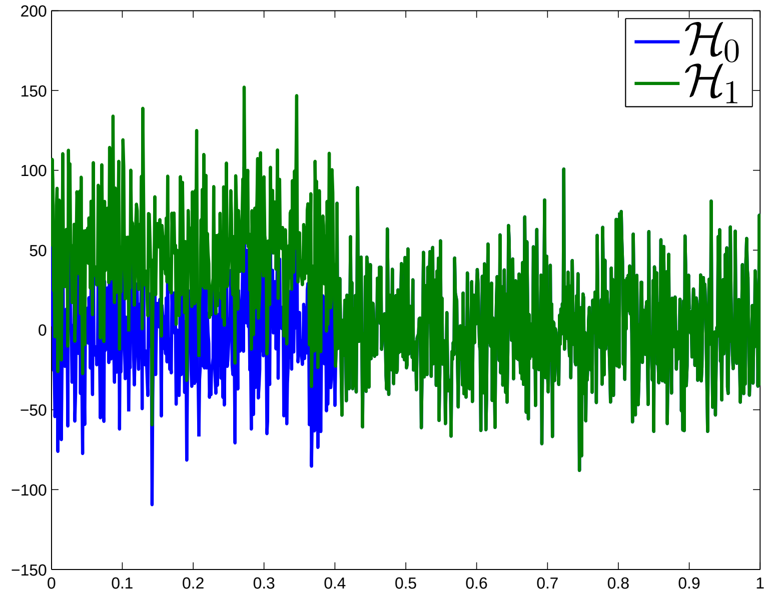
Hypotheses about Hamiltonian:

$$\mathcal{H}_0 : H_0, \qquad \mathcal{H}_1 : H_0 - qx(t). \tag{13}$$



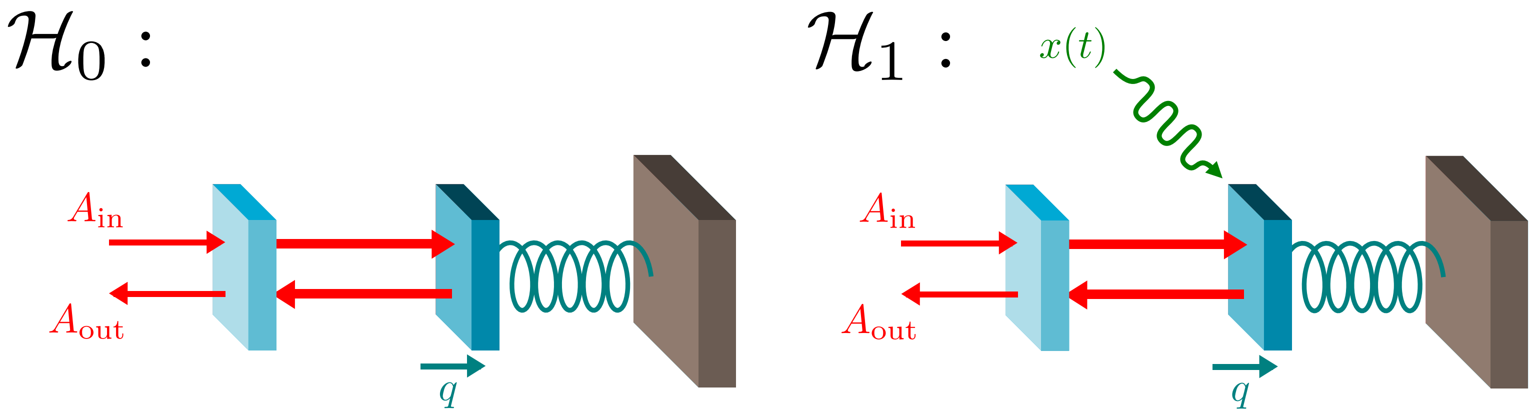
Continuous Qubit Readout

- **Task**: determine if qubit is spin up or down (binary hypothesis testing).
- **Practice**: couple qubit with optical/microwave field then perform continuous measurement.
- **Problem 1**: Noise: $dy(t) = x(t)dt + dW(t)$.
- **Problem 2**: spontaneous decay: $x(t)$ is a binary hidden stochastic process.



- Gambetta *et al.*, PRA **76**, 012325 (2007); D'Anjou and Coish, PRA **89**, 012313 (2014): QND measurement $\to$ classical model, didn't consider stochastic calculus (Stratonovich?)
- Ng and Tsang, PRA **90**, 022325 (2014):
  - Itō calculus
  - Estimator-correlator
  - Analytic solution for $\mathbb{E}(\sigma_z|Y^t, \mathcal{H}_1)$: Zakai equation has the form of geometric Brownian motion.

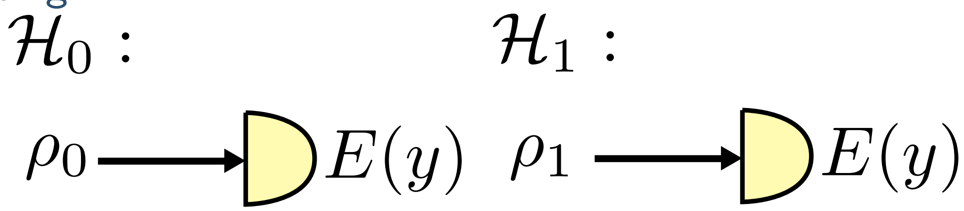
Fundamental Quantum Limits?



- Caves *et al.*, “On the measurement of a weak classical force coupled to a quantum-mechanical oscillator. I. Issues of principle,” Rev. Mod. Phys. **52**, 341 (1980).
- Yuen, “Contractive States and the Standard Quantum Limit for Monitoring Free-Mass Positions,” Phys. Rev. Lett., **51**, 719-722 (1983).
- Caves, “Defense of the standard quantum limit for free-mass position,” Phys. Rev. Lett., **54**, 2465-2468 (1985).
- Ozawa, “Measurement breaking the standard quantum limit for free-mass position,” Phys. Rev. Lett., **60**, 385-388 (1988).
- Braginsky and Khalili, *Quantum Measurement* (Cambridge University Press, Cambridge, 1992).
- **Quantum optimal P_e independent of measurement technique?**

Quantum Detection Bounds

- C. W. Helstrom, Quantum Detection and Estimation Theory, (Academic Press, New York, 1976).
- **Quantum** hypothesis testing:



$$\Pr(Y|\mathcal{H}_0) = \mathrm{tr}[E(Y)\rho_0], \qquad \Pr(Y|\mathcal{H}_1) = \mathrm{tr}[E(Y)\rho_1]. \tag{14}$$

- **Quantum bounds** (Helstrom, Fuchs/van de Graaf):

$$\boxed{\sqrt{P_0 P_1 F} \geq \min_{E(Y)} P_e = \frac{1}{2}\left(1 - ||P_1\rho_1 - P_0\rho_0||_1\right) \geq \frac{1}{2}\left(1 - \sqrt{1 - 4P_0P_1F}\right) \geq P_0 P_1 F.} \tag{15}$$

$||A||_1 \equiv \mathrm{tr}\sqrt{A^\dagger A}$, $F \equiv \left(\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_0\sqrt{\rho_1}}\right)^2$.

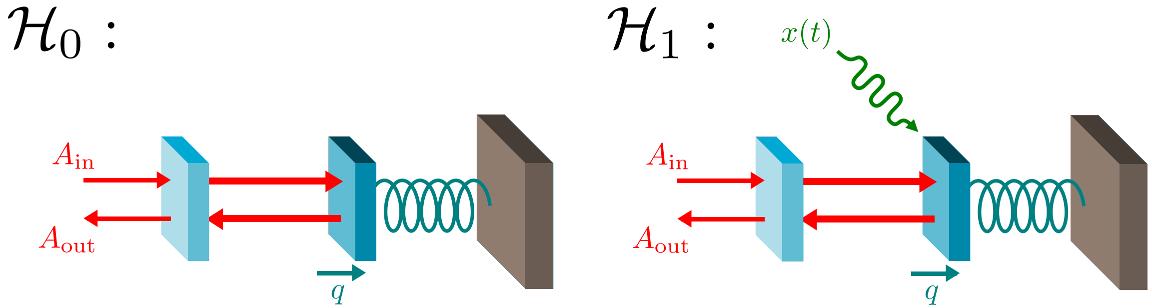
- For pure states,

$$\boxed{||P_1\rho_1 - P_0\rho_0||_1 = \sqrt{1 - 4P_0P_1F}, \quad F = |\langle\psi_0|\psi_1\rangle|^2\,,} \tag{16}$$

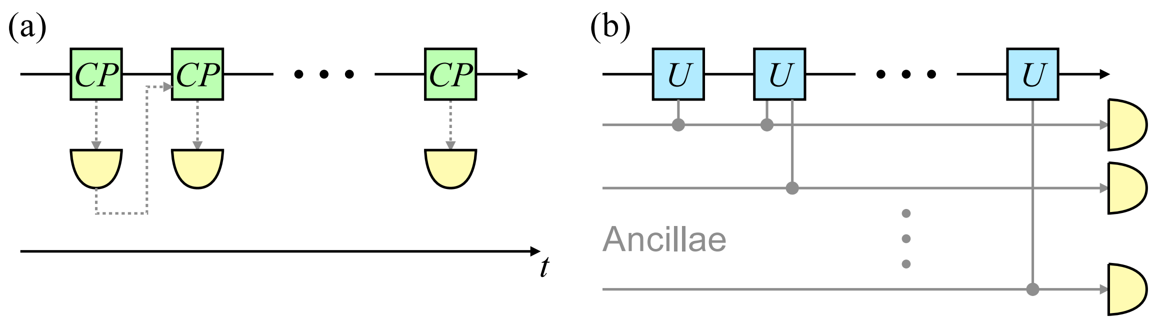
and **there exists a POVM such that the fidelity bound is attained.**

Helstrom Bound for Waveform Detection

- How to apply Helstrom's bound to waveform detection?



- **Larger Hilbert space**: Purification, deferred measurements:



- **Unitary discrimination**:

$$\rho_0 = U_0|\psi\rangle\langle\psi|U_0^\dagger, \qquad \rho_1 = U_1|\psi\rangle\langle\psi|U_1^\dagger, \tag{17}$$

$$U_0 = \mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt H_0(t)\right], \qquad U_1 = \mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt H_1(x(t),t)\right], \tag{18}$$

$$F = \left|\langle\psi|U_0^\dagger U_1|\psi\rangle\right|^2. \tag{19}$$

- **Interaction picture**:

$$U_0^\dagger U_1 = \mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt H_I(t)\right], \qquad H_I(t) \equiv U_0^\dagger(t,t_0)\left[H_1(t) - H_0(t)\right]U_0(t,t_0). \tag{20}$$

- Suppose $H_1 - H_0 = -qx(t)$,

$$F = \left|\left\langle \mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt x(t) q_I(t)\right]\right\rangle\right|^2. \tag{21}$$

- Assume $q_I(t)$ has **linear** dynamics (H_0 quadratic with respect to Z, a vector of canonical coordinate operators),

$$q_I(t) = g(t,t_0)Z + \int dt' g(t,t')J(t'), \tag{22}$$

- Cascading **displacement operators**:

$$\mathcal{T}\exp\left[\frac{1}{i\hbar}\int dt x(t) q_I(t)\right] \approx e^{i\phi} e^{-\frac{i}{\hbar} dt x(t_N) g(t_N,t_0) Z} \dots e^{-\frac{i}{\hbar} dt x(t_1) g(t_1,t_0) Z} \tag{23}$$

$$= e^{i\phi'}\exp\left\{-\frac{i}{\hbar}\left[\int dt x(t) g(t,t_0)\right] Z\right\} \tag{24}$$

Analytic Expression

- **Characteristic function** is Fourier transform of **Wigner**:

$$\left\langle e^{-i\kappa Z} \right\rangle = \int dz W(z) \exp(-i\kappa z). \tag{25}$$

- Assume initial state (light, mechanical) is **Gaussian**, Fourier transform of $W(z)$ is Gaussian:

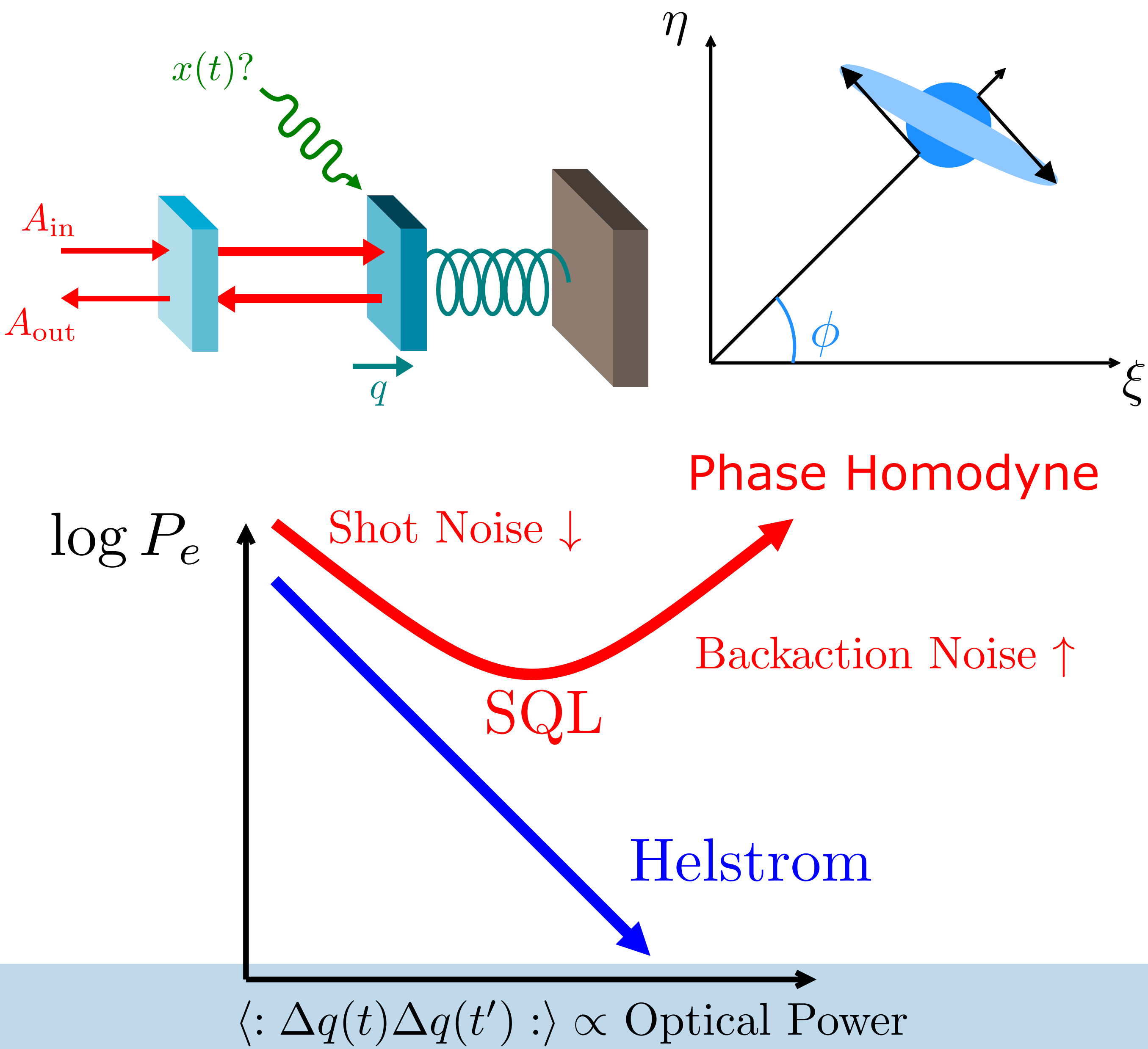
$$\boxed{F = \exp\left[-\frac{1}{\hbar^2} \int dt \int dt' x(t) \left\langle : \Delta q_I(t) \Delta q_I(t') : \right\rangle x(t')\right].} \tag{26}$$

- Tsang and Nair, PRA **86**, 042115 (2012).
- Helstrom bound is independent of measurement technique, and it is **guaranteed to be attainable**.
- In optomechanics,

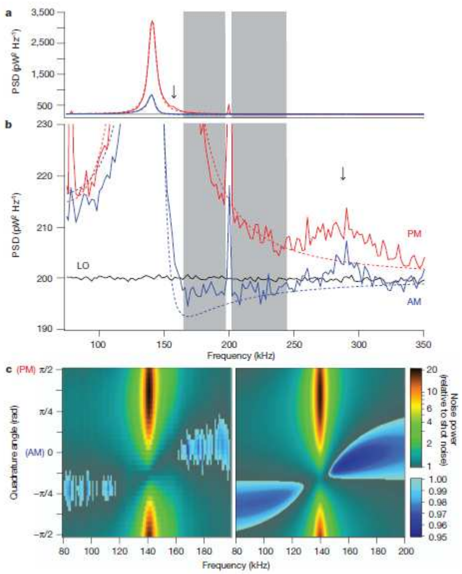
$$\frac{dq}{dt} = \frac{p}{m}, \qquad \frac{dp}{dt} = -m\omega_m^2 q + \kappa a^\dagger a, \qquad \frac{da}{dt} = -\frac{\gamma}{2} a - i\omega_0 a + \sqrt{\gamma} A_{\text{in}}, \tag{27}$$

$\langle : \Delta q_I(t) \Delta q_I(t') : \rangle$ is a function of **input optical state** and **dynamics**.

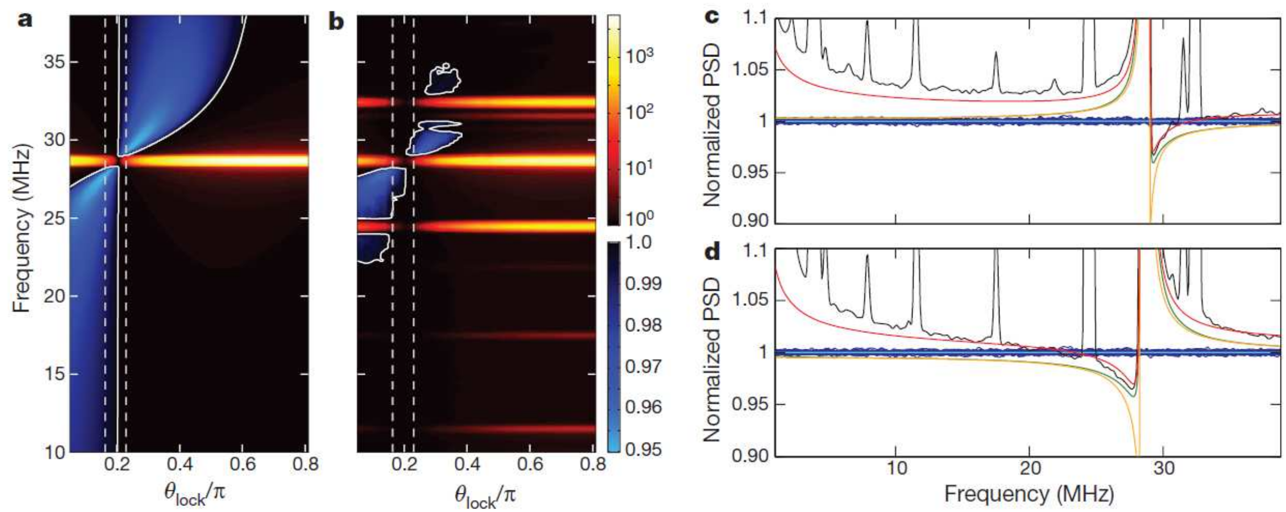
Standard Quantum Limit



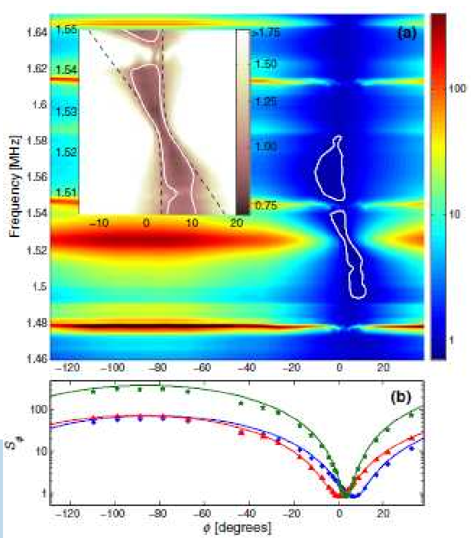
Ponderomotive Squeezing



Brooks *et al.*, Nature **488**, 476 (2012)

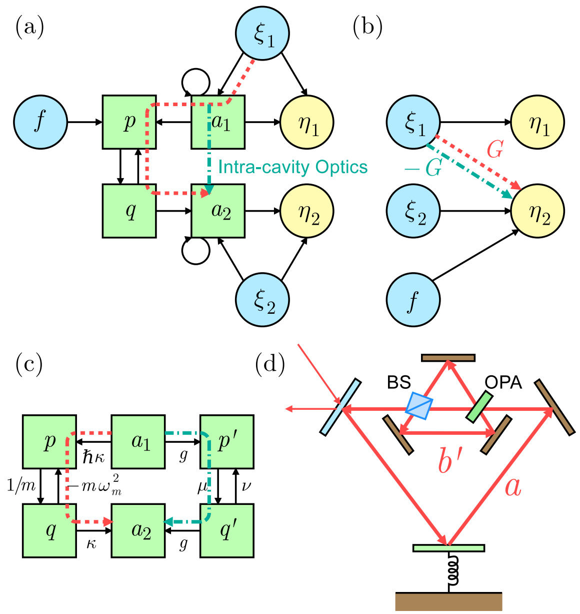


Safavi-Naeini *et al.*, Nature **500**, 185 (2013)

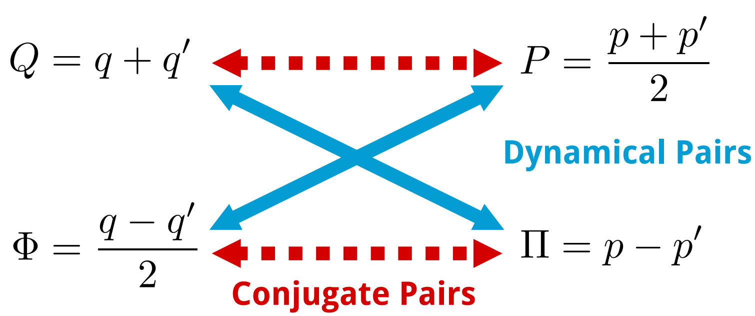


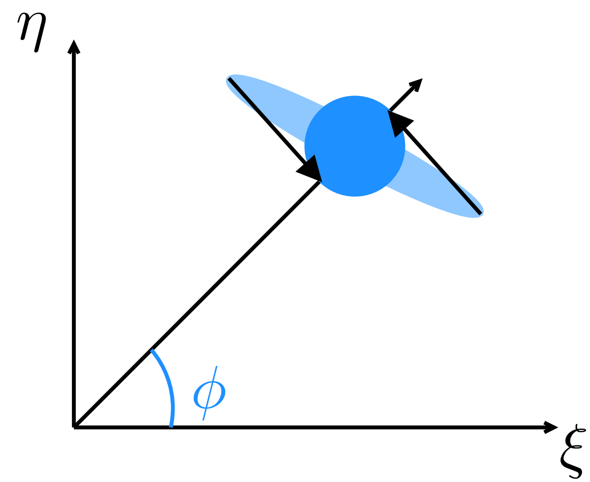
Purdy *et al.*, PRX **3**, 031012 (2013)

Overcoming SQL



Positive-mass oscillator $\{q, p\}$
and negative-mass oscillator $\{q', p'\}$

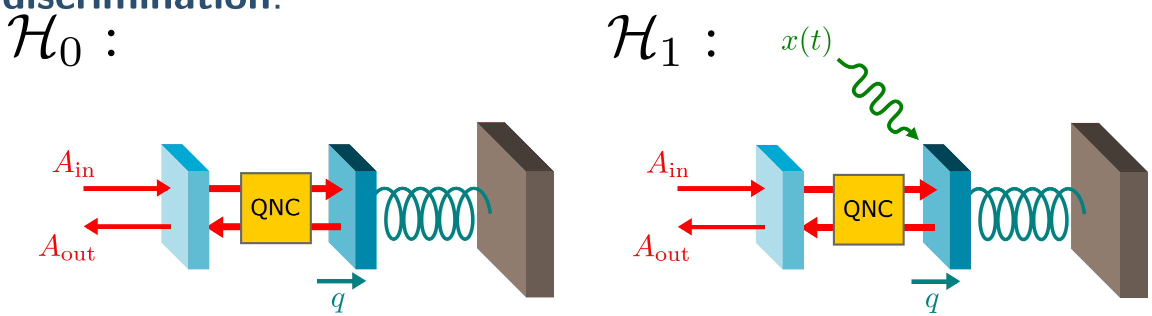




- Kimble *et al.*, PRD **65**, 022002 (2001)
- **Quantum Noise Cancellation**: Tsang and Caves, PRL **105**, 123601 (2010)
- **Quantum-Mechanics-Free Subsystem**: Tsang and Caves, PRX **2**, 031016 (2012); see also Appendix D, Gough and James, IEEE TAC **54**, 2530 (2009).
- **Undo ponderomotive squeezing**

Homodyne is Suboptimal

- Suppose input field is in coherent state, output is also coherent state after QNC.
- **Coherent-state discrimination**:

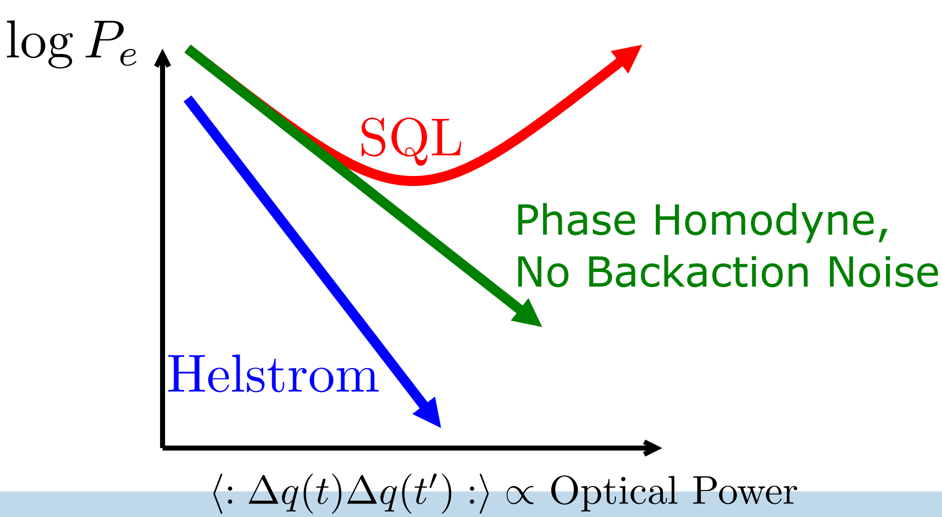


$$\mathcal{H}_0 : |\mathcal{A}_{\rm out}[x(t) = 0]\rangle, \qquad \mathcal{H}_1 : |\mathcal{A}_{\rm out}[x(t)]\rangle. \tag{28}$$

- Figure of merit: **error exponent** $-\ln P_e$

- $$\boxed{-\ln P_e^{\text{Homodyne}} \approx \frac{1}{2}\left(-\ln P_e^{\text{Helstrom}}\right)} \tag{29}$$

  Optimal measurement can save **half of optical power**



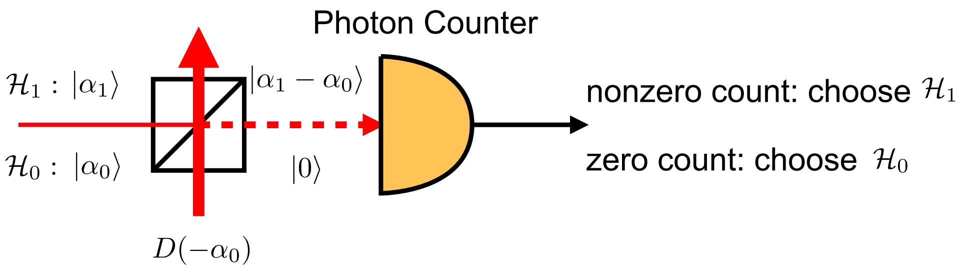
Quantum Optimal Measurements

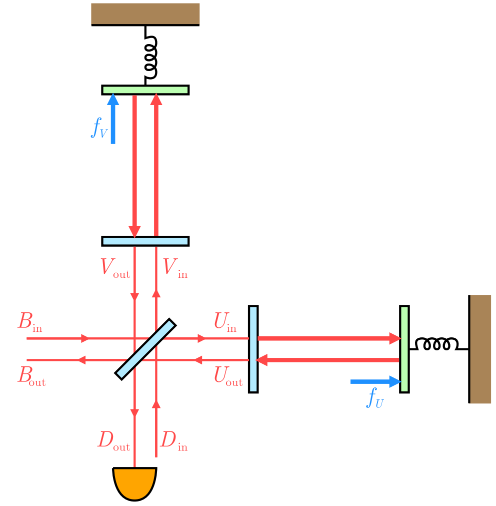
- Kennedy: null the field for $\mathcal{H}_0$:

$$\mathcal{H}_0 : |0\rangle, \qquad \mathcal{H}_1 : |\mathcal{A}_{\text{out}}[x(t)] - \mathcal{A}_{\text{out}}[0]\rangle, \tag{30}$$

- Error probabilities:

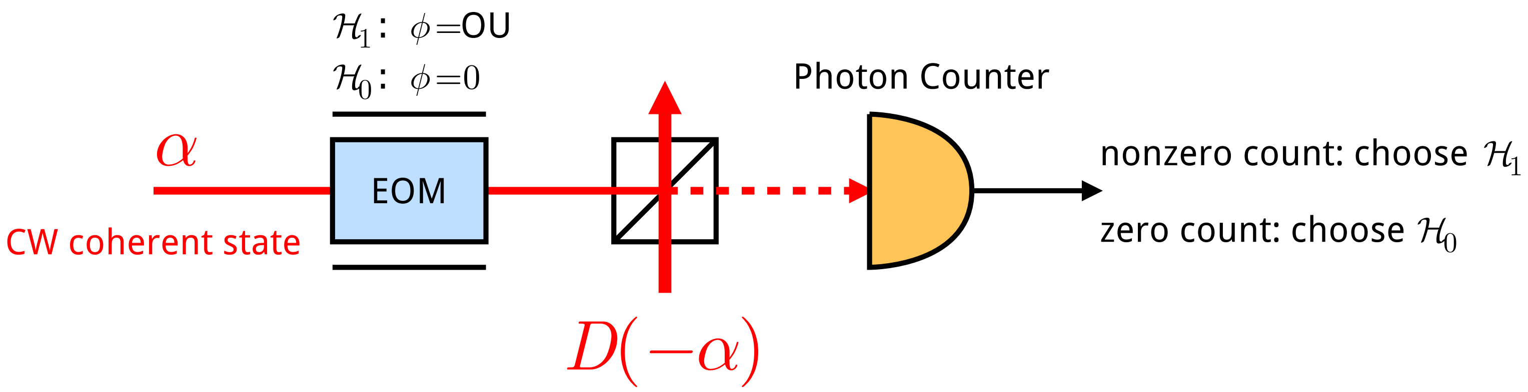
$$P_{10} = |\langle N \neq 0 | N = 0\rangle|^2 = 0, \qquad \boxed{P_{01} = |\langle 0 | \mathcal{A}_{\text{out}}[x(t)] - \mathcal{A}_{\text{out}}[0]\rangle|^2 = F.} \tag{31}$$

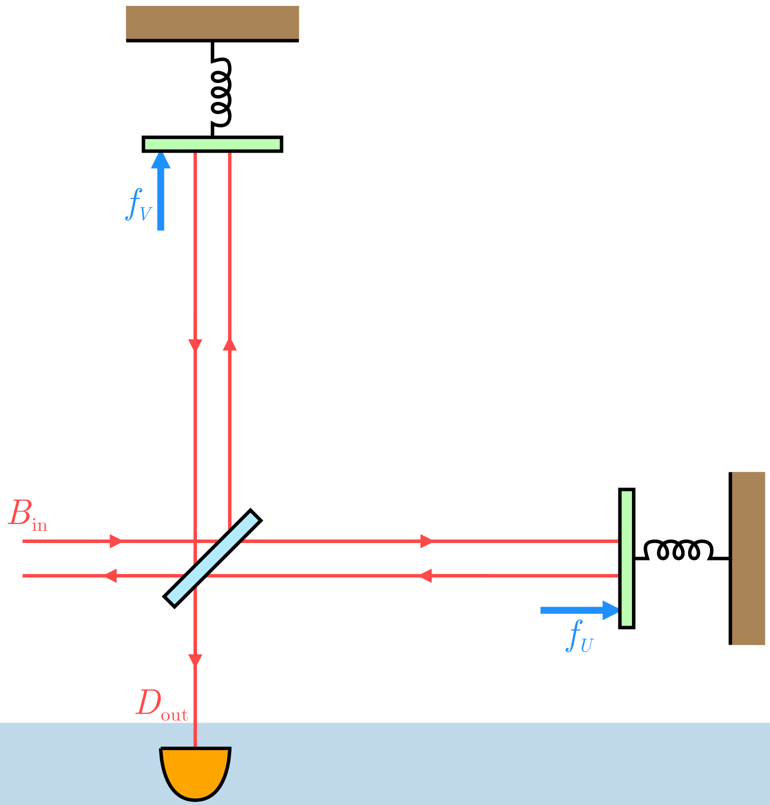




$$\boxed{-\ln P_e^{\text{Kennedy}} \approx -\ln P_e^{\text{Helstrom}} - (-\ln P_0)} \tag{32}$$

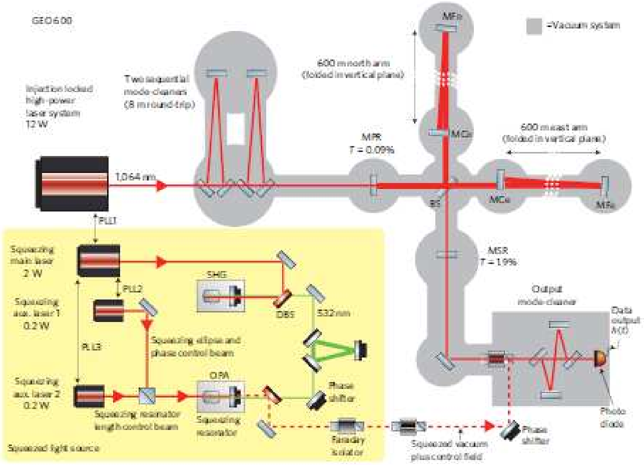
- M. Tsang and R. Nair, PRA **86**, 042115 (2012).
- Kennedy has optimal error exponent even for **stochastic** $x(t)$ detection.
- Dolinar?

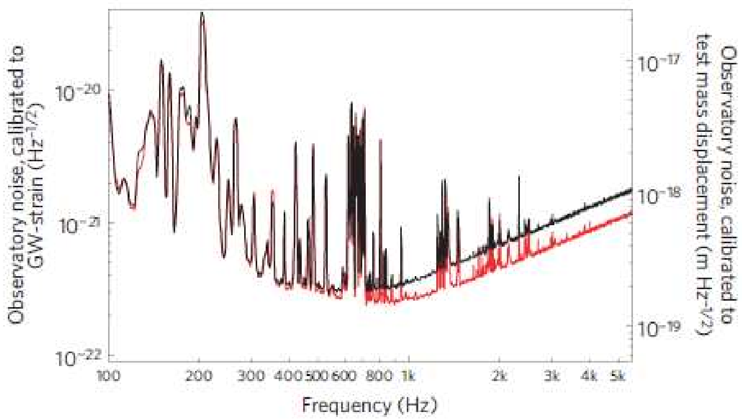

$\mathcal{H}_1$: ϕ=OU
$\mathcal{H}_0$: $\phi=0$
Photon Counter
α
EOM
CW coherent state
nonzero count: choose $\mathcal{H}_1$
zero count: choose $\mathcal{H}_0$
$D(-\alpha)$


f_V
B_{in}
f_U
D_{out}

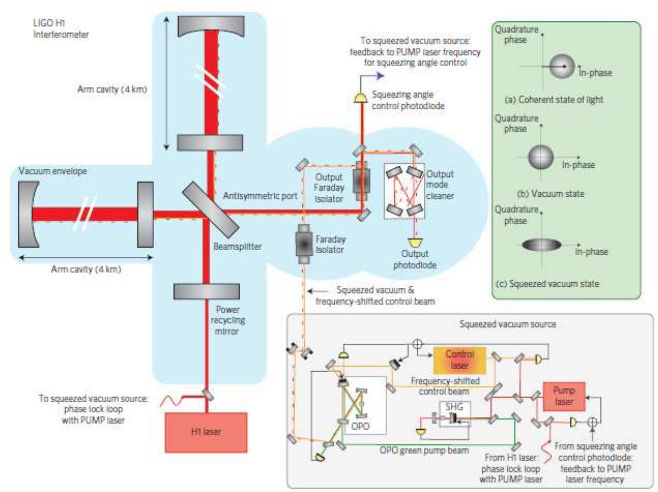
Squeezed Input State

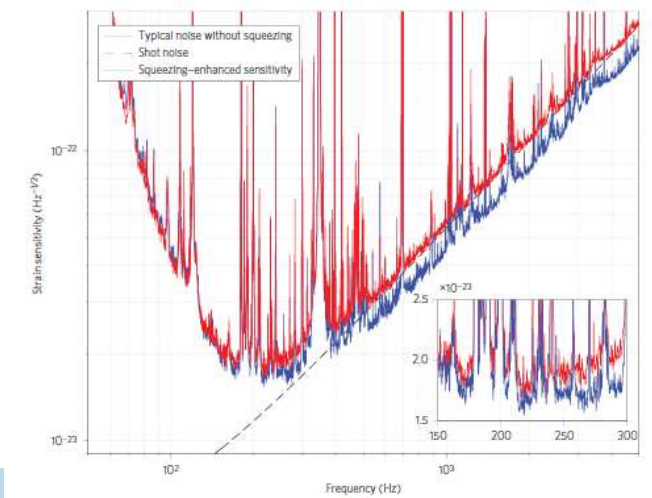
- Caves, PRD **23**, 1693 (1981).
- **GEO 600**: Nature Phys. **7**, 962 (2011)





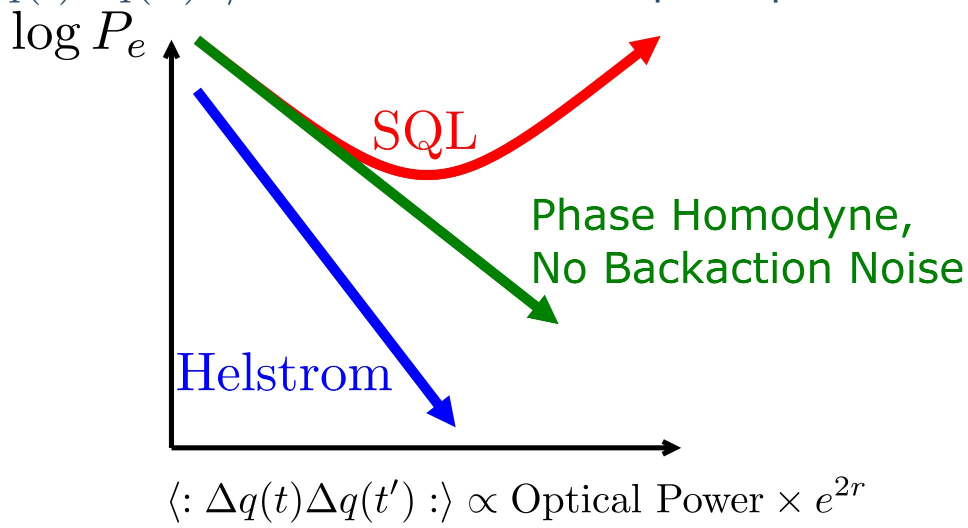
- **LIGO Hanford**: Nature Photon. **7**, 613 (2013)





Squeezed Input State

- Squeezing increases $\langle : \Delta q(t)\Delta q(t') : \rangle$ without increase in optical power



- LIGO nowhere near SQL yet
- Optimal detection for squeezed state?
- **Fundamental quantum limit** with respect to optical power?
  - Optical loss: Tsang, NJP **15**, 073005 (2013).
  - Heisenberg limit to detection and estimation?

Quantum Ziv-Zakai Bounds

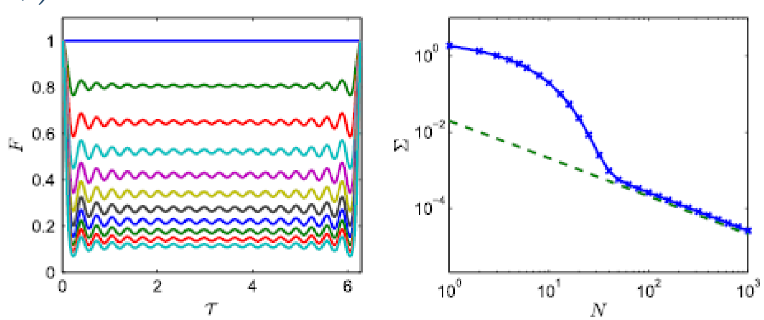
- Parameter estimation: prior $P(x)$, likelihood $P(y|x)$
- Ziv-Zakai bound on MSE:

$$\boxed{\mathbb{E}\left[x-\tilde{x}(y)\right]^2 \geq \frac{1}{2}\int_0^\infty d\tau\tau\int_{-\infty}^{\infty} dx\left[P(x)+P(x+\tau)\right]P_e(x,x+\tau)} \tag{33}$$

$P_e(x,x+\tau) =$ average error probability for binary hypothesis testing:

$$P(y|\mathcal{H}_0)=P(y|x), \quad P(y|\mathcal{H}_1)=P(y|x+\tau), \quad P_0=\frac{P(x)}{P(x)+P(x+\tau)}, \quad P_1=1-P_0. \tag{34}$$

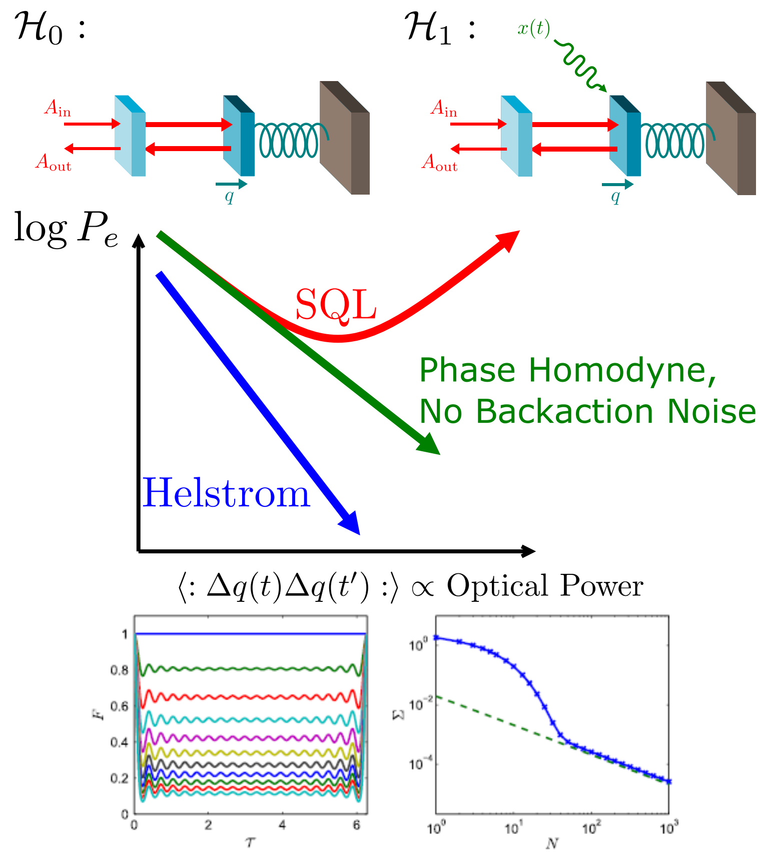
- Quantum: $P_e(x,x+\tau) \geq \frac{1}{2}\left[1-\sqrt{1-4P_0P_1F(x,x+\tau)}\right]$
- prove **Heisenberg limit** MSE $\geq C/\langle N\rangle^2$.
- Rivas-Luis: $\left(\sqrt{1-\epsilon}|0\rangle+\sqrt{\epsilon}|\psi\rangle\right)^{\otimes\nu}$, **super-Heisenberg** QCRB



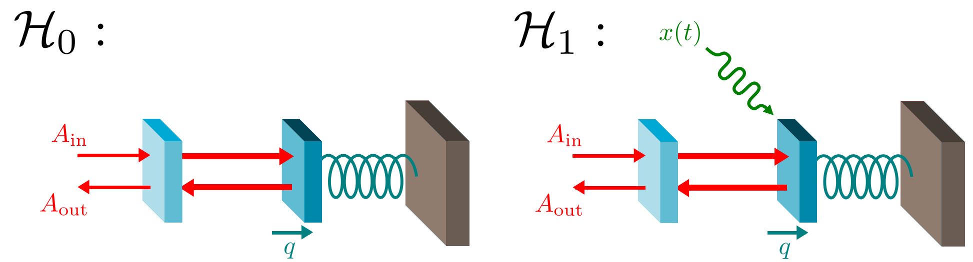
- Tsang, PRL **108**, 230401 (2012).
- **Multiparameter extensions possible [Berry, Tsang, Hall, Wiseman, arXiv:1409.7877].**

References

- **Estimator-Correlator Formula**
  - Tsang, PRL **108**, 170502 (2012).
  - Ng and Tsang, PRA **90**, 022325 (2014).
- **Quantum Detection Bounds**
  - **Detection**: Tsang and Nair, PRA **86**, 042115 (2012); Tsang, NJP **15**, 073005 (2013).
  - **QZZB**: Tsang PRL **108**, 230401 (2012); Berry, Tsang, Hall, Wiseman, arXiv:1409.7877.
- http://mankei.tsang.googlepages.com/
- eletmk@nus.edu.sg



Stochastic Waveform Detection



- What if we don't know $x(t)$ exactly?
- Fidelity upper/lower bounds still hold:

$$F = \int dP[x(t)] \left| \left\langle \mathcal{T} \exp\left[\frac{1}{i\hbar}\int dt x(t) q_I(t)\right]\right\rangle\right|^2 \tag{35}$$

- Coherent state under $\mathcal{H}_0$, mixed state under $\mathcal{H}_1$:

$$\mathcal{H}_0 : |\mathcal{A}_{\rm out}[0]\rangle, \qquad \mathcal{H}_1 : \int dP[x(t)] |\mathcal{A}_{\rm out}[x(t)]\rangle\langle\mathcal{A}_{\rm out}[x(t)]|. \tag{36}$$

- Kennedy receiver still has near-optimal error exponent:

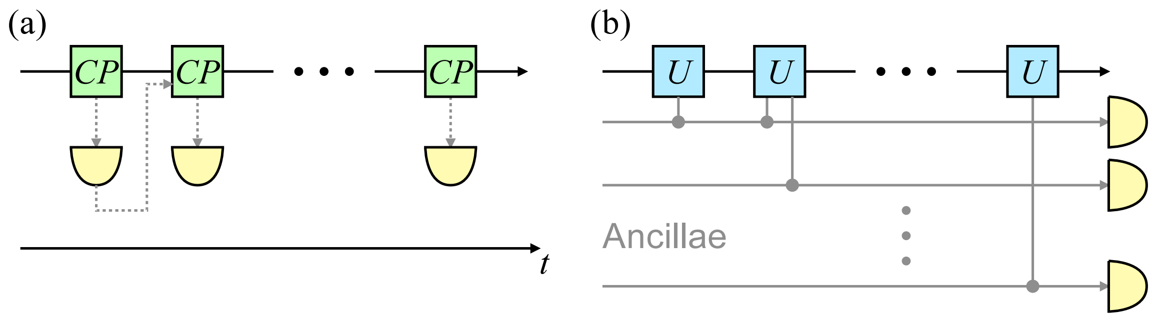
$$\mathcal{H}_0 : |0\rangle, \qquad \mathcal{H}_1 : \int dP[x(t)] |\mathcal{A}_{\rm out}[x(t)] - \mathcal{A}_{\rm out}[0]\rangle\langle\mathcal{A}_{\rm out}[x(t)] - \mathcal{A}_{\rm out}[0]|,$$

$$P_{10} = 0, \qquad \boxed{P_{01} = \int dP[x(t)]\, |\langle 0|\mathcal{A}_{\rm out}[x(t)] - \mathcal{A}_{\rm out}[0]\rangle|^2 = F.} \tag{37}$$

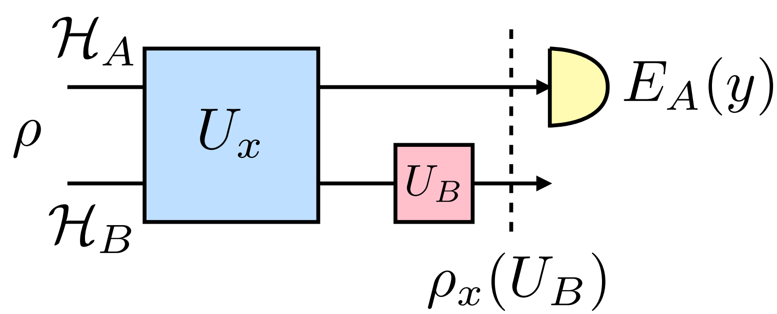
- M. Tsang and R. Nair, PRA **86**, 042115 (2012).

Open Systems

- Larger Hilbert space: bounds assume **everything** can be measured.



- For **open systems**, bounds are still **valid** but **not tight**
- Mixed states: $||P_1\rho_1 - P_0\rho_0||_1$, $F = \left(\mathrm{tr}\sqrt{\sqrt{\rho_1}\rho_0\sqrt{\rho_1}}\right)^2$, **hopeless** to compute
- **Modified purification**:



$$F = \max_{U_B} \left| \langle\psi|U_0^\dagger U_B U_1|\psi\rangle \right|^2 \quad (38)$$

- Choose U_B to tighten lower bound.
- Hard to find optimal U_B, lower bound may not be achievable, useful as no-go only.
- Escher *et al.*, Nature Phys. **7**, 406 (2011) (quantum Fisher information)