Quantum Metrology Kills Rayleigh’s Criterion *

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Imaging of One Point Source
Point-Spread Function of Hubble Space Telescope
Inferring Position of One Point Source

- Classical source
- Given $N$ detected photons, mean-square error:

\[ \Delta X_1^2 = \frac{\sigma^2}{N}, \quad (1) \]

\[ \sigma \sim \frac{\lambda}{\sin \phi}. \quad (2) \]

- Helstrom, Lindegren (astrometry), Bobroff, ...
- PALM, STED, STORM, etc.: isolate emitters. Locate centroids.
- https://www.youtube.com/watch?v=2R2ll9SRCeo (25:45)
- Special fluorophores
- slow
- doesn't work for stars
- For a review, see Moerner, PNAS 104, 12596 (2007).
Two Point Sources

Rayleigh's criterion (1879): requires $\theta_2 \gtrsim \sigma$ (heuristic)

- Incoherent sources, Poisson statistics
- \( X_1 = \theta_1 - \theta_2 / 2, \ X_2 = \theta_1 + \theta_2 / 2 \).
- Cramér-Rao bound for centroid:
  \[
  \Delta \theta_1^2 \geq \frac{\sigma^2}{N}.
  \] (3)

- CRB for separation estimation: two regimes
  - \( \theta_2 \gg \sigma \):
    \[
    \Delta \theta_2^2 \geq \frac{4\sigma^2}{N},
    \] (4)
  - \( \theta_2 \ll \sigma \):
    \[
    \Delta \theta_2^2 \rightarrow \frac{4\sigma^2}{N} \times \infty
    \] (5)

- Rayleigh’s curse
- PALM/STED/STORM: avoid Rayleigh
Cramér-Rao bounds:

\[ \Delta \theta_1^2 \geq \frac{1}{J_{11}^{(\text{direct})}} \]

\[ \Delta \theta_2^2 \geq \frac{1}{J_{22}^{(\text{direct})}} \]  \hspace{1cm} (6)

\( J^{(\text{direct})} \) is Fisher information for CCD

Gaussian PSF, similar behavior for other PSF
Quantum Information

- CCD is just one measurement method. **Quantum mechanics allows infinite possibilities.**
- Helstrom: For any measurement (POVM) of an optical state $\rho^{\otimes M}$ ($M$ here is number of copies)

\[
\Sigma \geq J^{-1} \geq K^{-1},
\]

\[
K_{\mu\nu} = M \operatorname{Re} (\operatorname{tr} L_\mu L_\nu \rho),
\]

\[
\frac{\partial \rho}{\partial \theta_\mu} = \frac{1}{2} (L_\mu \rho + \rho L_\mu).
\]

- **Ultimate amount of information in the photons**
- Mixed states:

\[
\rho = \sum_n D_n |e_n\rangle \langle e_n|,
\]

\[
L_\mu = 2 \sum_{n,m; D_n + D_m \neq 0} \frac{\langle e_n | \frac{\partial \rho}{\partial \theta_\mu} | e_m \rangle}{D_n + D_m} |e_n\rangle \langle e_m|.
\]
Mandel and Wolf, *Optical Coherence and Quantum Optics*; Goodman, *Statistical Optics*

- Thermal sources, e.g., stars, fluorescent particles.
- Coherence time $\sim 10$ fs. Within each coherence time interval, **average photon number $\epsilon \ll 1$ at optical frequencies** (visible, UV, X-ray, etc.).

\[ \rho = (1 - \epsilon) |\text{vac}\rangle \langle \text{vac}| + \frac{\epsilon}{2} (|\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|) + O(\epsilon^2) \]

\[ \langle \psi_1 | \psi_2 \rangle \neq 0, \quad (12) \]

|\psi_1\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_1) |x\rangle, \quad |\psi_2\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_2) |x\rangle. \quad (13) \]

- Quantum state at image plane:

- derive from zero-mean Gaussian P function
- Multiphoton coincidence: **rare**, little information as $\epsilon \ll 1$ (homeopathy)
- Similar model for stellar interferometry in Gottesman, Jennewein, Croke, PRL 109, 070503 (2012); Tsang, PRL 107, 270402 (2011).
Quantum and classical Fisher information

Cramér-Rao bounds on separation error

- Tsang, Nair, and Lu, e-print arXiv:1511.00552

- Nair and Tsang, e-print arXiv:1604.00937: thermal sources with arbitrary $\epsilon$

- Hayashi ed., *Asymptotic Theory of Quantum Statistical Inference*; Fujiwara JPA 39, 12489 (2006): there exists a POVM such that $\Delta \theta^2_\mu \to 1/\mathcal{K}_{\mu\mu}$, $M \to \infty$. 

\[
\Delta \theta^2_2 \ge \frac{1}{\mathcal{K}_{22}} = \frac{1}{N \Delta k^2}.
\]
project the photon in **Hermite-Gaussian** basis:

\[ E_1(q) = |\phi_q\rangle \langle \phi_q| , \quad (15) \]

\[ |\phi_q\rangle = \int_{-\infty}^{\infty} dx \phi_q(x) |x\rangle , \quad (16) \]

\[ \phi_q(x) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} H_q \left( \frac{x}{\sqrt{2}\sigma} \right) \exp \left( -\frac{x^2}{4\sigma^2} \right) . \quad (17) \]

Assume PSF \( \psi(x) \) is Gaussian (common).

\[
\frac{1}{J_{22}^{(HG)}} = \frac{1}{K_{22}} = \frac{4\sigma^2}{N} . \quad (18)
\]

**Maximum-likelihood estimator** can saturate the classical bound asymptotically for large \( N \).
Spatial-Mode Demultiplexing (SPADE)
Binary SPADE

**Classical Fisher information**

\[
\mathcal{J}_{22}^{(HG)} = \mathcal{K}_{22}
\]

\[
\mathcal{J}_{22}^{(direct)}
\]

\[
\mathcal{J}_{22}^{(b)}
\]

**Fisher information for sinc PSF**

\[
\mathcal{K}_{22}
\]

\[
\mathcal{J}_{22}^{(direct)}
\]

\[
\mathcal{J}_{22}^{(b)}
\]
Numerical Performance of Maximum-Likelihood Estimators

Simulated errors for SPADE

Simulated errors for binary SPADE

$L = \text{number of detected photons}$

biased, $< 2 \times \text{CRB.}$
Van Trees inequality for any biased/unbiased estimator (e-print arXiv:1605.03799)

Quantum/SPADE: \[ \sup_\theta \Sigma_{22}^{(\text{SPADE})}(\theta) \geq \frac{4\sigma^2}{N}, \]

Direct imaging: \[ \sup_\theta \Sigma_{22}^{(\text{direct})}(\theta) \geq \frac{\sigma^2}{\sqrt{N}}. \quad (19) \]
- **SuperLocalization via Image-in**VERsion interferometry
- Laser Focus World, Feb 2016 issue.
Ang, Nair, Tsang, e-print arXiv:1606.00603
Misalignment

- $\xi \equiv |\hat{\theta}_1 - \theta|/\sigma \ll 1$
- Overhead photons $N_1 \sim 1/\xi^2$
- $\xi = 0.1$, $N_1 \sim 100$.
- CRB for $X_s = \theta_1 \pm \theta_2/2$

Fisher information for misaligned SPADE

Simulated errors for misaligned binary SPADE

Cramér-Rao bounds on localization error

object plane

image plane

SPADE

image plane

photon-counting array

$\hat{\theta}_1$
## Follow-up

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<td>2D</td>
<td>Thermal (any frequency)</td>
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<td>1D</td>
<td>Weak thermal, lasers</td>
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<td>Quantum</td>
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</table>

- More to come!
Experiments

  - SLIVER
  - Laser, classical noise

  - Mode heterodyne
  - Laser

  - SPADE
  - single-photon sources, quantum
Design quantum computer to
- Maximize information extraction
- Reduce classical computational complexity
Quantum Metrology with Classical Sources

- **Thermal/fluorescent/laser sources, linear optics, photon counting**
- Compare with other QIP applications:
  - QKD (Bennett *et al.*)
  - Nonclassical-state metrology (Yuen, Caves)
  - Shor’s algorithm
  - Quantum simulations (Feynman, Lloyd)
  - Boson sampling (Aaronson)

- Microscopy (physics, chemistry, biology, engineering), telescope (astronomy), radar/lidar (military), etc.
- Classical sources are more robust to losses.
- Current microscopy limited by **photon shot noise** in EMCCD (see, e.g., Pawley *ed.*, *Handbook of Biological Confocal Microscopy*).
- Ground telescopes limited by **atmospheric turbulence**, space telescopes are diffraction/shot-noise-limited.
- **Linear optics/photon-counting technology** is mature

Although any given scheme can be explained by a semiclassical model, quantum metrology remains a powerful tool for exploring the **ultimate performance**.
Quantum Metrology Kills Rayleigh’s Criterion

Cramér-Rao bounds on separation error

- Quantum $(1/\mathcal{K}_{22})$
- Direct imaging $(1/J_{22}^{(direct)})$

FAQ: https://sites.google.com/site/mankeitsang/news/rayleigh/faq
email: mankei@nus.edu.sg
Chap. 9, Goodman, *Statistical Optics*:

“If the count degeneracy parameter is much less than 1, it is highly probable that there will be either zero or one counts in each separate coherence interval of the incident classical wave. In such a case the classical intensity fluctuations have a negligible "bunching" effect on the photo-events, for (with high probability) the light is simply too weak to generate multiple events in a single coherence cell.

Zmuidzinas (https://pma.caltech.edu/content/jonas-zmuidzinas), JOSA A 20, 218 (2003):

“It is well established that the photon counts registered by the detectors in an optical instrument follow statistically independent Poisson distributions, so that the fluctuations of the counts in different detectors are uncorrelated. To be more precise, this situation holds for the case of thermal emission (from the source, the atmosphere, the telescope, etc.) in which the mean photon occupation numbers of the modes incident on the detectors are low, \( n \ll 1 \). In the high occupancy limit, \( n \gg 1 \), photon bunching becomes important in that it changes the counting statistics and can introduce correlations among the detectors. We will discuss only the first case, \( n \ll 1 \), which applies to most astronomical observations at optical and infrared wavelengths.”


See also Labeyrie *et al.*, *An Introduction to Optical Stellar Interferometry*, etc.

Fluorescent particles: Pawley *ed.*, *Handbook of Biological Confocal Microscopy*, Ram, Ober, Ward (2006), etc., may have antibunching, but Poisson model is fine because of \( \epsilon \ll 1 \).