Quantum Limits on Sensing and Imaging

Mankei Tsang
eletmk@nus.edu.sg

http://mankei.tsang.googlepages.com/

ECE, Physics, National University of Singapore

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Center for Quantum Information and Control, UNM
Keck Foundation Center for Extreme Quantum Information Theory, MIT
Quantum Waveform Sensing

- Estimation/detection of *classical waveforms* $x(r, t)$ using quantum systems
- Examples: optical interferometry, optical imaging, optomechanical force sensing (gravitational-wave detection), atomic magnetometry, gyroscope, etc.
Optical Phase and Frequency Estimation

\[ \phi(t) = \int_{-\infty}^{\infty} dt' h(t - t') x(t') \]
\[ \langle \delta x^2 \rangle \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{4|h(\omega)|^2 S_{\Delta \hat{I}}(\omega) + 1/S_x(\omega)}, \]

\[ S_{\Delta \hat{I}}^{\text{coh}}(\omega) = \frac{\bar{P}}{\hbar \omega_0}, \quad S_x^{\text{OU}}(\omega) = \frac{\kappa}{\omega^2 + \epsilon^2}. \]  

Achieved by homodyne phase-locked loop + Smoothing [Personick IEEE TIT 17, 240 (1971); Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008); 79, 053843 (2009)]

Wheatley et al., PRL 104, 093601 (2010).

Interferometry, ranging, velocimetry, clock synchronization, coherent comm., etc.
Optomechanical Force Sensing

QCRB [Tsang, Wiseman, and Caves, PRL 106, 090401 (2011)]:

\[
\langle \delta x^2 \rangle \geq \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{1}{(4/\hbar^2)S_{\Delta q}(\omega) + 1/S_x(\omega)}.
\]  

Optical Phase Waveform Detection

- Binary hypothesis testing \( \mathcal{H}_0 : \phi(t) = 0 \), \( \mathcal{H}_1 : \phi(t) = x(t) \)
- Tsang, unpublished:

\[ \mathcal{H}_1 : \phi = x(t) \]
\[ \mathcal{H}_0 : \phi = 0 \]

Photon Counter

nonzero count: choose \( \mathcal{H}_1 \)
zero count: choose \( \mathcal{H}_0 \)

\[ D(-\alpha) \]

\[ P_e \geq \frac{1}{2} \left( 1 - \sqrt{1 - 4P_0 P_1 F} \right), \quad (3) \]

\[ F = \int DxP[x] \left| \langle \psi | \exp \left[ i \int_{t_0}^{T} dt \hat{I}(t) x(t) \right] |\psi \rangle \right|^2. \quad (4) \]

- Kennedy receiver has optimal error exponent for stochastic waveform detection with coherent state.
- Homodyne performance depends on prior waveform statistics.
Optomechanical Force Detection


\[
P_e \geq \frac{1}{2} \left( 1 - \sqrt{1 - 4P_0 P_1 F} \right), \quad F = \int Dx P[x] \left| \langle \psi | T \exp \left[ i \int_{t_0}^{T} dt \hat{q}_0(t) x(t) \right] |\psi \rangle \right|^2
\]

Because the bound is achievable for deterministic \( x(t) \) and not limited by backaction noise, SQL can definitely be overcome.

QNC + Kennedy receiver achieves optimal error exponent
Decoherence

- Significant decoherence/loss rules out any significant quantum enhancement using squeezed state/nonclassical state of light (Durkin et al./Escher et al.)
- No-go for nonclassical light in space optics applications
- No result yet about decoherence in waveform estimation/detection, surprise unlikely
- Quantum illumination (Lloyd/Erkmen/Guha/Shapiro/Giovannetti et al.): up to 6 dB improvement in error exponent!
  - Producing squeezed state requires strong pump with way more photons
  - can be achieved by coherent state with 6 dB more photons
  - Known receivers can’t get to 6 dB
  - Low-photon-number regime only
  - Gaussian noise strengths are assumed to be different under hypotheses for QI to be useful, passive target detection may be better in practice
- Quantum Metrology with POVMs
Quantum Imaging

- **Ghost imaging**: Shih/Shapiro/Erkmen

- **Sub-Rayleigh quantum lithography/imaging**: Boto *et al.*, PRL *85*, 2733 (2000); Tsang, PRL *102*, 253601 (2009); Giovannetti *et al.*, PRA *79*, 013827 (2009)


- **Classical computational sub-Rayleigh imaging**: STORM/PALM [Zhuang, Nature Photon. *3*, 365 (2009)]; STED (Hell), etc.

- Computational imaging for astronomy [Fienup]

- Not much rigorous work in quantum imaging that uses estimation/detection theory
Quantum Camera Design

- start with multi-spatial-mode $\rho_x(r)$ (e.g., multimode thermal or coherent state)

- Record with imaging system/CCD/interferometer/digital holography (model by POVM $E[y(r)]$)

$$P[y(r)|x(r')] = \text{tr} \left\{ E[y(r)] \Phi_{\text{aperture}} \Phi_{\text{differact}} \rho_x(r') \right\}$$  \hspace{1cm} (6)

- Quantum bounds: multiparameter QCRB, etc.

- Do conventional imaging systems saturate these bounds?

- How to implement optimal POVM?
Stellar Interferometry

Estimation of coherence:

\[ \Gamma_{ab} = \langle b^\dagger a \rangle, \]
\[ g^{(1)} = \frac{\langle b^\dagger a \rangle}{\sqrt{\langle b^\dagger b \rangle \langle a^\dagger a \rangle}} \text{ (normalized)}. \]
Old-School Quantum Optics

\[ \rho = \int d^2 \alpha d^2 \beta \Phi(\alpha, \beta) |\alpha, \beta\rangle \langle \alpha, \beta|. \quad (8) \]

\( \Phi(\alpha, \beta) \) is a two-mode zero-mean Gaussian for thermal light, i.e. no entanglement.

Weak thermal light \( \epsilon \equiv \langle a^\dagger a \rangle = \langle b^\dagger b \rangle \ll 1 \) in photon-number basis:

\[ \rho = (1 - \epsilon)|0, 0\rangle \langle 0, 0| + \frac{\epsilon}{2} \left[ |0, 1\rangle \langle 1, 0| + |1, 0\rangle \langle 1, 0| + g^*|0, 1\rangle \langle 1, 0| + g|1, 0\rangle \langle 0, 1| \right] \]

\[ + O(\epsilon^2), \]

\[ P(y|g) = \text{tr} [E(y)\rho]. \quad (9) \]

Classical Fisher information for \( g = g_1 + ig_2 \):

\[ F_{jk} = \left\langle \frac{\partial}{\partial g_j} \ln P \frac{\partial}{\partial g_k} \ln P \right\rangle, \quad \Sigma \geq \frac{1}{M} F^{-1} \quad (11) \]
Bound for Local Measurements

- Nonlocal measurements (direct detection, shared-entanglement): $||F|| \sim \epsilon$.

- A necessary condition for local (LOCC) measurement is the PPT condition applied to the POVM [Terhal et al., PRL 86, 5807 (2001)]. Then $||F|| \leq \epsilon^2 + O(\epsilon^3)$.

- Generalizable to repeated LOCC measurements

- Quantum nonlocality in measurement of nature, even if the state has no entanglement.

- **Quantum Ziv-Zakai bounds** [Tsang, PRL 108, 230401 (2012)]

- **Continuous quantum hypothesis testing** (for tests of physics using continuous quantum measurements) [Tsang, PRL 108, 170502 (2012)]

- **Cavity quantum microwave photonics** [Tsang, PRA 81, 063837 (2010); 84, 043845 (2011)]