Mismatched Quantum Filtering and Entropic Information *

[arXiv:1310.0291]

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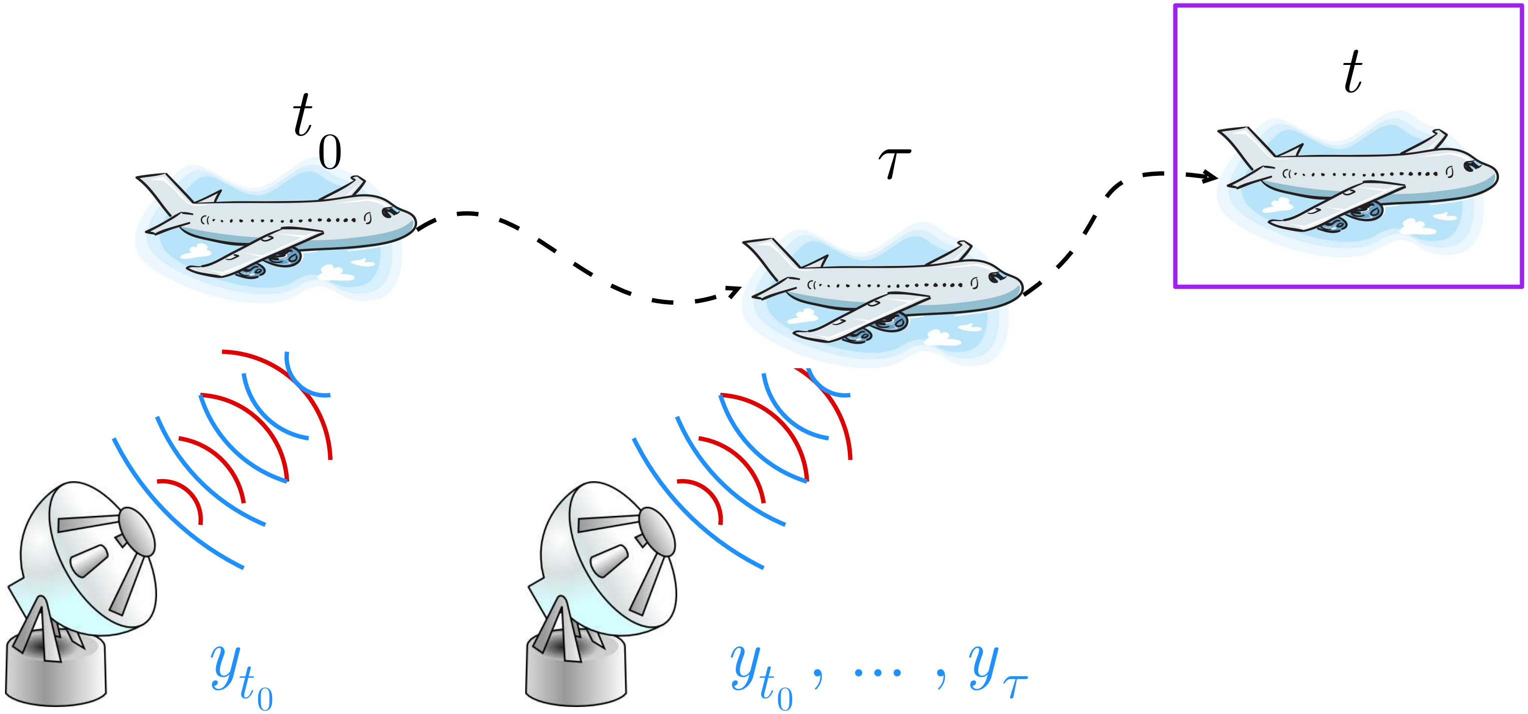
Regret

Extra cost of making a bad decision

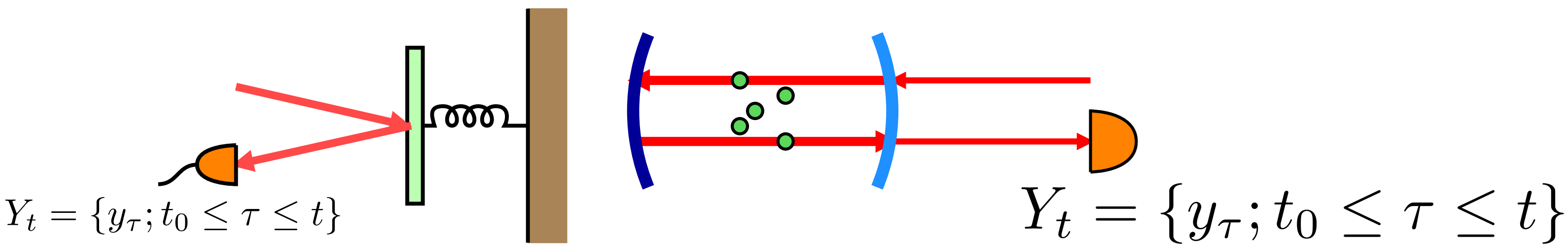


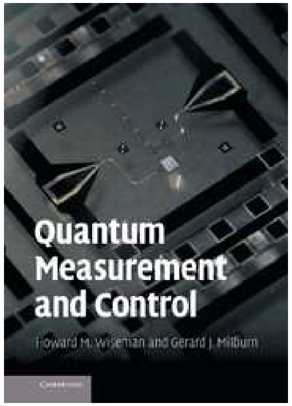
mean-square error (**mse**) $= \mathbb{E}\left[X - \check{X}(Y)\right]^2$

minimum mean-square error (**mmse**) $= \mathbb{E}\left[X - \mathbb{E}\left(X|Y\right)\right]^2$

Regret = mse−mmse

Quantum Filtering





- Quantum control: squeezing, quantum error correction, cooling, sensing, etc.
- Optics, atoms, ions, superconducting microwave circuits, quantum dots, etc.
- linear **Belavkin** equation:

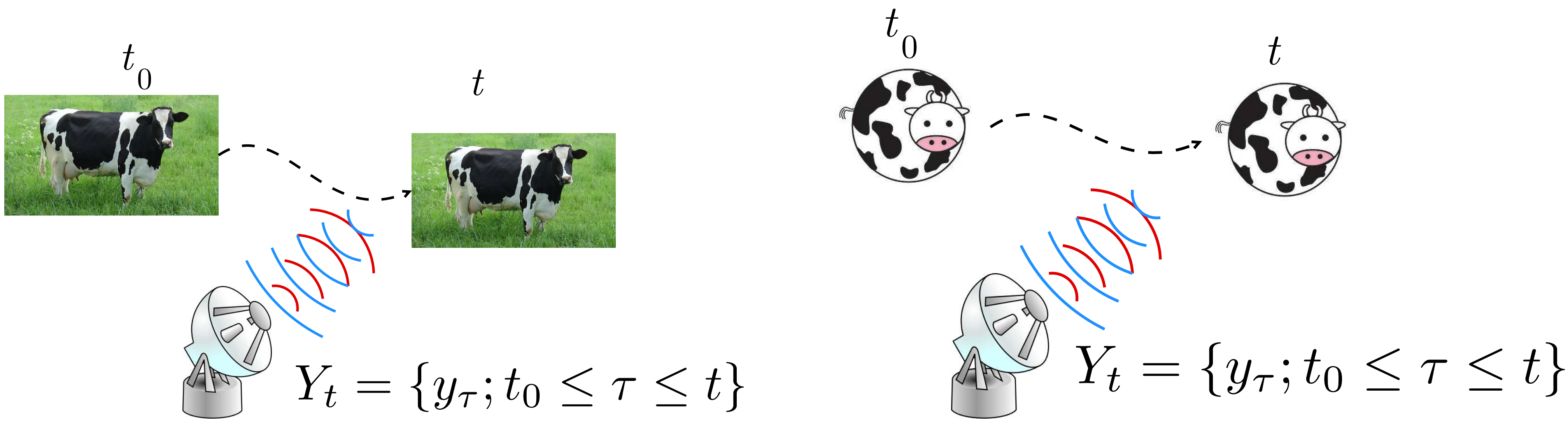
$$df_t = \mathcal{L}_t f_t dt + \frac{1}{2}\left(a_t f_t + f_t a_t^\dagger\right) dy_t, \qquad \mathbb{E}\left(q_t|Y_t\right) = \frac{\operatorname{tr} q_t f_t}{\operatorname{tr} f_t}. \tag{1}$$

- Minimum mean-square error:

$$\boxed{\mathsf{mmse}_t = \mathbb{E}\left[q_t - \mathbb{E}\left(q_t|Y_t\right)\right]^2.} \tag{2}$$

- Consider $q_t = (a_t + a_t^\dagger)/2$, the **directly measured** observable.
- $y_t - \int_0^t d\tau\, \mathbb{E}(q_\tau|Y_\tau)$ is a Wiener process (*innovation process*).
- Belavkin, Prob. Theor. App. **38**, 742 (1993); **39**, 640 (1994) [arXiv:quant-ph/0510028].
- Wiseman and Milburn; Bouten, van Handel, James, SIAM J. Contr. Optim. **46**, 2199 (2007); etc.

Filter Mismatch Regret



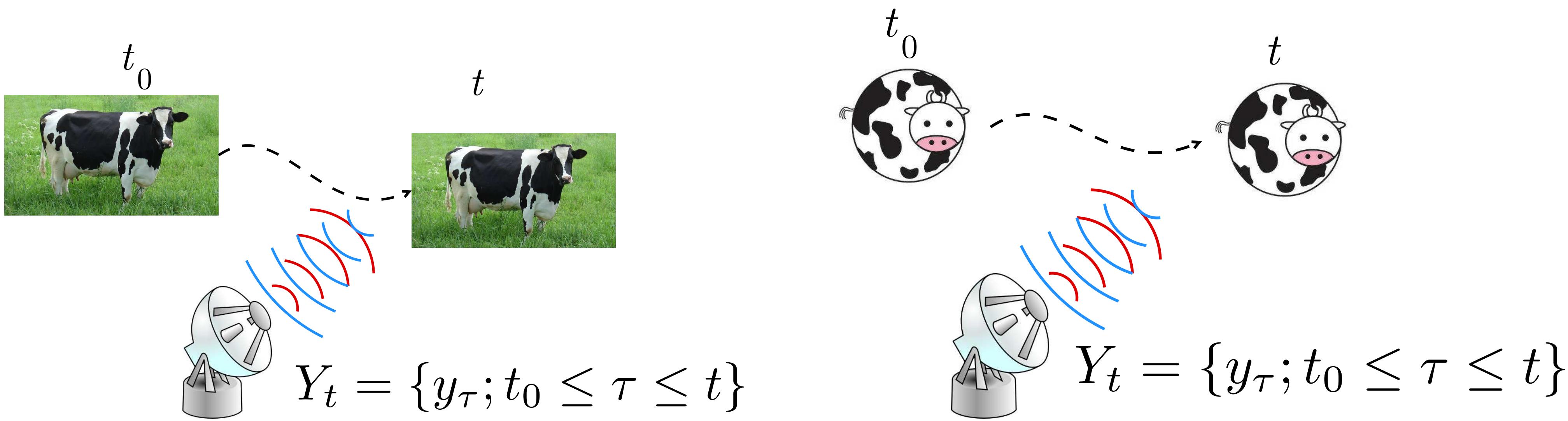
- Assume a wrong model $(\rho'_0, a'_t, \mathcal{L}'_t)$:

$$df'_t = \mathcal{L}'_t f'_t dt + \frac{1}{2}\left(a'_t f'_t + f'_t a'^{\dagger}_t\right) dy_t, \qquad \mathbb{E}'\left(q'_t | Y_t\right) = \frac{\operatorname{tr} q'_t f'_t}{\operatorname{tr} f'_t}. \tag{3}$$

- Regret due to filter mismatch:

$$\boxed{R \equiv \frac{1}{2}\int_0^T dt \left\{\mathbb{E}\left[q_t - \mathbb{E}'\left(q'_t | Y_t\right)\right]^2 - \mathsf{mmse}_t\right\}.} \tag{4}$$

Regret = Relative Entropy



$$\boxed{R = D(dP||dP') \equiv \int dP(Y_T) \ln \frac{dP(Y_T)}{dP'(Y_T)}} \tag{5}$$

Proof [arXiv:1310.0291]

- tr f_t is likelihood ratio:

$$\operatorname{tr} f_t = \frac{dP(Y_t)}{dP_0(Y_t)}, \qquad \operatorname{tr} f'_t = \frac{dP'(Y_t)}{dP_0(Y_t)}, \tag{6}$$

  $dP_0 =$ Wiener measure.
- From Belavkin equation,

$$d \operatorname{tr} f_t = \operatorname{tr} df_t = \operatorname{tr} \mathcal{L}_t f_t dt + \frac{1}{2} \operatorname{tr} \left(a_t f_t + f_t a_t^\dagger \right) dy_t = \mathbb{E}\left(q_t|Y_t\right)\left(\operatorname{tr} f_t\right) dy_t. \tag{7}$$

- Itō calculus:

$$\ln \operatorname{tr} f_T = \int_0^T dy_t \, \mathbb{E}\left(q_t|Y_t\right) - \frac{1}{2}\int_0^T dt \, \mathbb{E}^2\left(q_t|Y_t\right), \tag{8}$$

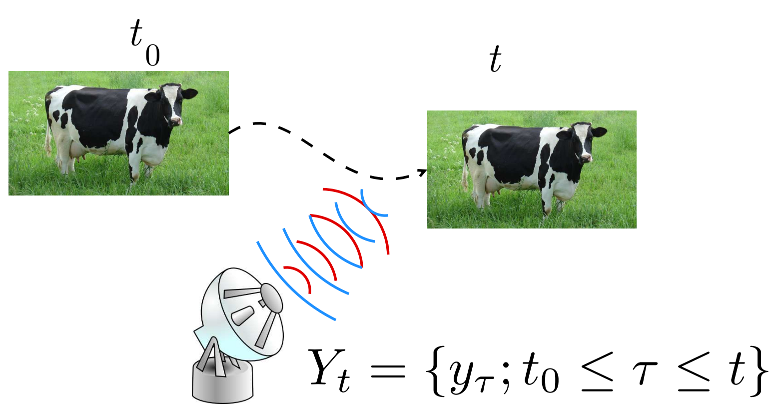
$$\ln \frac{dP(Y_T)}{dP'(Y_T)} = \int_0^T dy_t \left[\mathbb{E}\left(q_t|Y_t\right) - \mathbb{E}'\left(q'_t|Y_t\right)\right] - \frac{1}{2}\int_0^T dt \left[\mathbb{E}^2\left(q_t|Y_t\right) - \mathbb{E}'^2\left(q'_t|Y_t\right)\right] \tag{9}$$

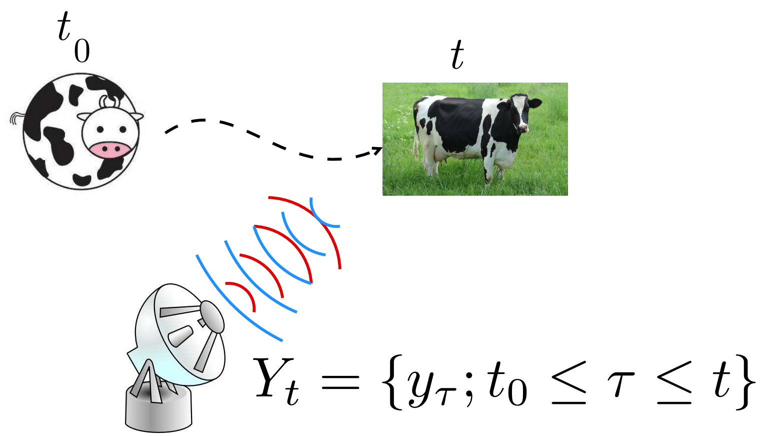
  [Tsang, Physical Review Letters **108**, 170502 (2012)].
- Relative entropy is expected log-likelihood ratio:

$$D(dP||dP') \equiv \mathbb{E} \ln \frac{dP(Y_T)}{dP'(Y_T)}. \tag{10}$$

- Use basic properties of **quantum conditional expectation** and **innovation process** $\mathbb{E}[dy_t g(Y_t)] = \mathbb{E}[\mathbb{E}(dy_t|Y_t)g(Y_t)] = \mathbb{E}[dt\,\mathbb{E}(q_t|Y_t)g(Y_t)] = dt\,\mathbb{E}[q_t g(Y_t)]$

Example: Mismatched Initial Condition





- Suppose dynamics and measurement models are accurate ($\mathcal{L}_t = \mathcal{L}_t'$, $a_t = a_t'$), only $\rho_0' \neq \rho_0$,

$$dP(Y_T) = \operatorname{tr} d\mu(Y_T)\rho_0, \qquad dP'(Y_T) = \operatorname{tr} d\mu(Y_T)\rho_0'. \tag{11}$$

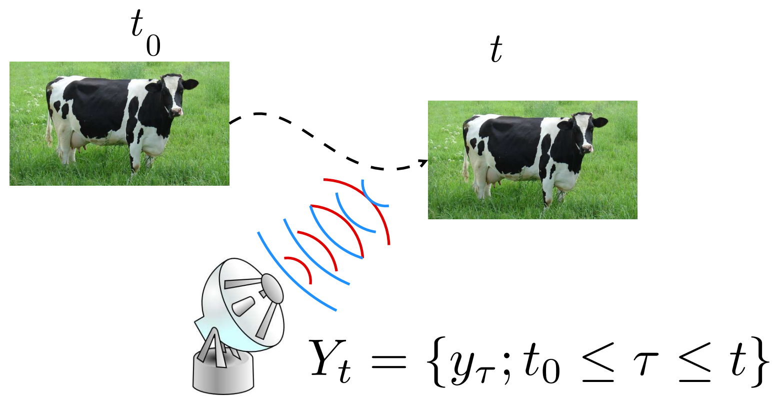
- Monotonicity:

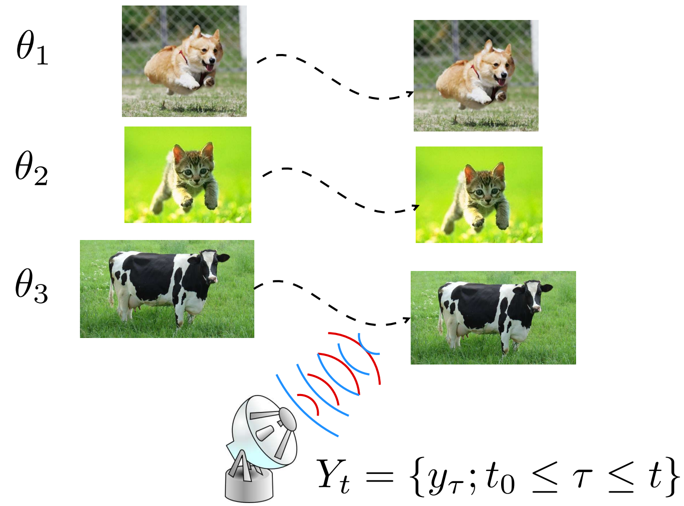
$$D(dP||dP') \le D(\rho_0||\rho_0') = \operatorname{tr} \rho_0 \left(\ln \rho_0 - \ln \rho_0'\right). \tag{12}$$

- Regret is **upper-bounded**:

$$\boxed{R \le \operatorname{tr} \rho_0 \left(\ln \rho_0 - \ln \rho_0'\right).} \tag{13}$$

Ignorance Regret = Mutual Information





- True model is one of $\{\rho_0^\theta, a_t^\theta, \mathcal{L}_t^\theta\}$, with prior $d\pi(\theta)$.
- If I know θ, $\mathbb{E}(q_t|Y_t) = \mathbb{E}(q_t|Y_t, \theta)$.
- If I don't know θ, mse is minimized by Bayesian $\mathbb{E}'(q_t'|Y_t) = \mathbb{E}_\theta \left[\mathbb{E}(q_t|Y_t, \theta)|Y_t\right]$.
- Observation probability measures: $dP(Y_T) = dP_\theta(Y_T)$, $dP'(Y_T) = \mathbb{E}_\theta\, dP_\theta(Y_T)$.
- Regret due to **ignorance** = **mutual information**:

$$\boxed{\min_{\{\rho_0', a_t', \mathcal{L}_t'\}} \mathbb{E}_\theta\, R = \mathbb{E}_\theta\, D(dP_\theta || \mathbb{E}_\theta\, dP_\theta) \equiv I(\theta; Y).} \tag{14}$$

- **Measure of parameter importance**.
- Holevo bound for mismatched initial condition:

$$\min_{\{\rho_0'\}} \mathbb{E}_\theta\, R = I(\theta; Y) \le \mathbb{E}_\theta\, D(\rho_0^\theta || \mathbb{E}_\theta\, \rho_0^\theta). \tag{15}$$

Worst-Case Ignorance Regret = Channel Capacity

- Worst-case Bayesian regret (maximin regret):

$$\max_{d\pi} \min_{\{\rho_0', a_t', \mathcal{L}_t'\}} \mathbb{E}_\theta R = \max_{d\pi} I(\theta; Y) \equiv C, \tag{16}$$

  which is **channel capacity**.
- maximin is equal to minimax (von Neumann minimax theorem):

$$\boxed{\max_{d\pi} \min_{\{\rho_0', a_t', \mathcal{L}_t'\}} \mathbb{E}_\theta R = \min_{\{\rho_0', a_t', \mathcal{L}_t'\}} \max_{d\pi} \mathbb{E}_\theta R = C.} \tag{17}$$

- Redundancy-capacity theorem:

$$C = \max_{d\pi} \min_{dP'} \mathbb{E}_\theta D(dP_\theta || dP') = \min_{dP'} \max_{d\pi} \mathbb{E}_\theta D(dP_\theta || dP'). \tag{18}$$

Poissonian Measurements

- Filtering equation (reference probability):

$$df_t = \mathcal{L}_t f_t dt + \left(a_t f_t a_t^\dagger - f_t\right)(dy_t - dt)\,, \qquad \mathbb{E}\left(q_t|Y_t\right) = \frac{\mathrm{tr}\, q_t f_t}{\mathrm{tr}\, f_t}, \qquad \mathrm{tr}\, f_t = \frac{dP(Y_t)}{dP_0(Y_t)}. \tag{19}$$

dP_0 =standard Poisson measure. Directly measured observable is $q_t = a_t^\dagger a_t$.

- Define loss function

$$\boxed{L(q, \check{q}) \equiv q \ln \frac{q}{\check{q}} - q + \check{q}.} \tag{20}$$

- Define regret:

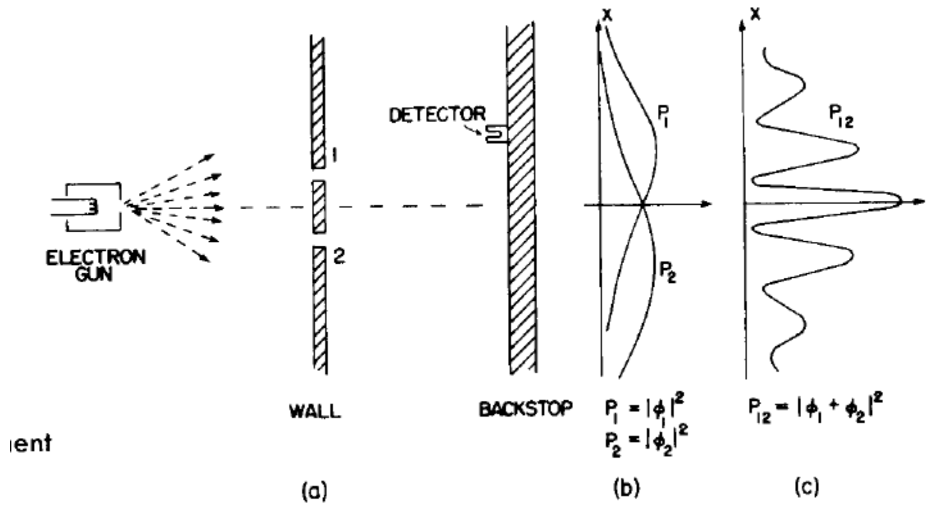
$$R_L \equiv \int_0^T dt \left\{L\left[q_t, \mathbb{E}'(q_t'|Y_t)\right] - L\left[q_t, \mathbb{E}(q_t|Y_t)\right]\right\}. \tag{21}$$

- **Same identity**:

$$\boxed{R_L = D(dP||dP').} \tag{22}$$

- Filtering:
  - T. E. Duncan, "On the calculation of mutual information," SIAM Journal on Applied Mathematics 19, 215–220 (1970).
  - T. Weissman, "The relationship between causal and noncausal mismatched estimation in continuous-time AWGN channels," IEEE Transactions on Information Theory, vol. 56, no. 9, pp. 4256–4273, 2010.
  - R. Atar and T. Weissman, "Mutual information, relative entropy, and estimation in the Poisson channel," IEEE Transactions on Information Theory, vol. 58, no. 3, pp. 1302–1318, 2012.
  - A. No and T. Weissman, "Minimax Filtering via Relations between Information and Estimation," arXiv:1301.5096, ArXiv e-prints, Jan. 2013.
- Smoothing (non-causal estimation):
  - D. Guo, S. Shamai, and S. Verdu, "Mutual information and minimum mean-square error in Gaussian channels," IEEE Transactions on Information Theory, vol. 51, no. 4, pp. 1261–1282, april 2005.
  - —, "Mutual information and conditional mean estimation in Poisson channels," IEEE Transactions on Information Theory, vol. 54, no. 5, pp. 1837–1849, 2008.
  - S. Verdu, "Mismatched estimation and relative entropy," IEEE Transactions on Information Theory, vol. 56, no. 8, pp. 3712–3720, 2010.

Quantum Smoothing?

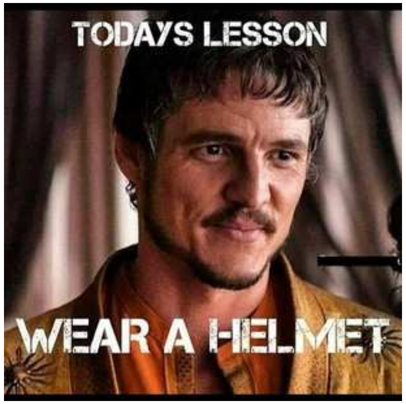


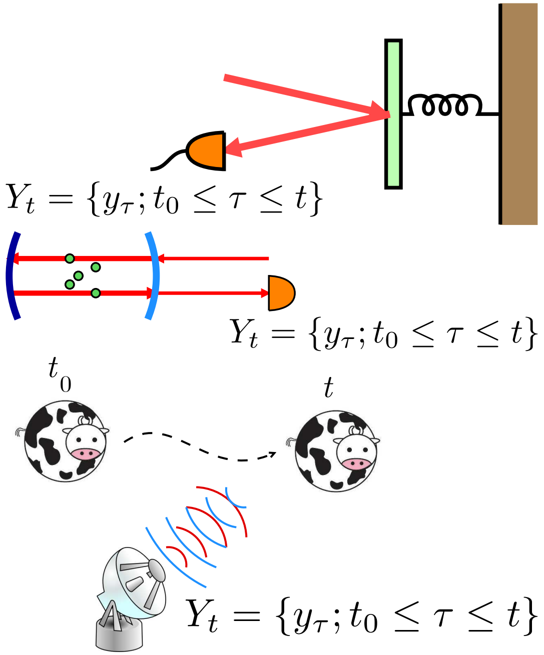
> . . . **one may not say that an electron goes either through hole 1 or hole 2.** If one does say that, and starts to make any deductions from the statement, he will make errors in the analysis. This is the logical tightrope on which we must walk if we wish to describe nature successfully.
>
> — Chapter 37 "Quantum Behavior," *Feynman Lectures on Physics Vol. I.*

- Tsang, "Time-Symmetric Quantum Theory of Smoothing," Physical Review Letters **102**, 250403 (2009); Physical Review A **80**, 033840 (2009); **81**, 013824 (2010).
- —, "A Bayesian quasi-probability approach to inferring the past of quantum observables," arXiv:1403.3353.

Conclusion

- **Regret = Relative Entropy**
- Quantum upper bound on regret
- Mutual information, channel capacity
- **Tsang, arXiv:1310.0291.**





- **Information Theory for Quantum Dynamical Systems**
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