Quantum Metrology Kills Rayleigh’s Criterion

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Quantum Volterra Filters

- Belavkin
  - Optimal
  - **Curse of Dimensionality**
- Quantum Volterra filters [Tsang, PRA 92, 062119 (2015)]
  - Polynomial (e.g., linear, **Wiener-Kolmogorov**)
  - Heisenberg picture, QND principle
  - Works for any nonlinear/non-Markovian systems
  - Error guarantees
  - Depends on finite-order correlation functions
- (Zhang *et al.*, arXiv:1407.8108: Volterra series for quantum input-output relations)


Seth Lloyd of the Massachusetts Institute of Technology in the US is impressed. ‘This is awesome work and I am amazed that it has’t been done before,’ he says. ‘Perhaps everyone thought it was too good to be true.’
Imaging of One Point Source

\[ X \]

\[ X_1 \]

\[ \phi \]

\[ \frac{M \lambda}{\sin \phi} \]

\[ -M X_1 \]

\[ x \]
Point-Spread Function of Hubble Space Telescope
Inferring Position of One Point Source

- Classical source
- Given $N$ detected photons, **mean-square error**:

\[
\Delta X_1^2 = \frac{\sigma^2}{N},
\]

(1)

\[
\sigma \sim \frac{\lambda}{\sin \phi}.
\]

(2)

- Frieden (1966), Helstrom (1970), Lindegren (1978), Bobroff (1986), ...
- PALM, STED, STORM, etc.: isolate emitters. Locate centroids.
- https://www.youtube.com/watch?v=2R21l9SFrU (25:45)
- Special fluorophores
- slow
- doesn’t work for stars
- For a review, see Moerner, PNAS 104, 12596 (2007).
Two Point Sources

\[ \Lambda(x) = \frac{1}{2} \left[ |\psi_1(x)|^2 + |\psi_2(x)|^2 \right] \]

Rayleigh's criterion (1879): requires \( \theta_2 \gtrsim \sigma \) (heuristic)
Centroid and Separation Estimation

- Incoherent sources, Poisson statistics
- \( X_1 = \theta_1 - \theta_2 / 2, \quad X_2 = \theta_1 + \theta_2 / 2. \)
- Cramér-Rao bound for centroid:
  \[
  \Delta \theta_1^2 \geq \frac{\sigma^2}{N}.
  \]
- CRB for separation estimation: two regimes
  - \( \theta_2 \gg \sigma: \)
    \[
    \Delta \theta_2^2 \geq \frac{4 \sigma^2}{N},
    \]
  - \( \theta_2 \ll \sigma: \)
    \[
    \Delta \theta_2^2 \to \frac{4 \sigma^2}{N} \times \infty
    \]
- Rayleigh’s curse
- PALM/STED/STORM: avoid Rayleigh

Beyond Rayleigh’s criterion: A resolution measure with application to single-molecule microscopy

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Rayleigh’s criterion is extensively used in optical microscopy for various experimental factors that affect the acquired data such as

(a)\[\theta_2\]

(b)\[\psi_1(x), \psi_2(x), \sigma, \Delta \theta_2^2 \geq \frac{4 \sigma^2}{N}, \Delta \theta_2^2 \to \frac{4 \sigma^2}{N} \times \infty
\]
Cramér-Rao bounds:

\[ \Delta \theta_1^2 \geq \frac{1}{J_{11}^{(\text{direct})}} \]

\[ \Delta \theta_2^2 \geq \frac{1}{J_{22}^{(\text{direct})}} \]  

\[ J^{(\text{direct})} \] is Fisher information for CCD

Gaussian PSF, similar behavior for other PSF
CCD is just one measurement method. Quantum mechanics allows infinite possibilities.

Quantum state of light: density matrix (operator) $\rho(\theta)$ (positive-semidefinite), $M$ temporal modes: $\rho^{\otimes M}$

Born’s rule (generalized) $P(Y|\theta) = \text{tr}[E(Y)\rho^{\otimes M}(\theta)]$, $E(Y)$ is called a positive operator-valued measure (POVM).

Helstrom (1967): For any POVM

$$\Sigma \geq J^{-1} \geq K^{-1},$$

$$J_{\mu\nu} = \int dY P(Y|\theta) \left[ \frac{\partial}{\partial \theta_\mu} \ln P(Y|\theta) \right] \left[ \frac{\partial}{\partial \theta_\nu} \ln P(Y|\theta) \right],$$

$$K_{\mu\nu} = M \text{Re} \left( \text{tr} \mathcal{L}_\mu \mathcal{L}_\nu \rho \right),$$

$$\frac{\partial \rho}{\partial \theta_\mu} = \frac{1}{2} \left( \mathcal{L}_\mu \rho + \rho \mathcal{L}_\mu \right).$$

Ultimate amount of information in the photons


Mixed states:

$$\rho = \sum_n D_n |e_n\rangle \langle e_n|,$$

$$\mathcal{L}_\mu = 2 \sum_{n,m; D_n + D_m \neq 0} \frac{\langle e_n| \frac{\partial \rho}{\partial \theta_\mu} |e_m\rangle}{D_n + D_m} |e_n\rangle \langle e_m|.$$
- Mandel and Wolf, *Optical Coherence and Quantum Optics*; Goodman, *Statistical Optics*

- Thermal sources, e.g., stars, fluorescent particles.

- Coherence time $\sim 10$ fs. Within each coherence time interval, *average photon number* $\epsilon \ll 1$ at optical frequencies (visible, UV, X-ray, etc.).

  \[
  \rho = (1 - \epsilon) |\text{vac}\rangle \langle \text{vac}| + \frac{\epsilon}{2} (|\psi_1\rangle \langle \psi_1| + |\psi_2\rangle \langle \psi_2|) + O(\epsilon^2)
  \]

  \[
  \langle \psi_1 | \psi_2 \rangle \neq 0, \quad (13)
  \]

  \[
  |\psi_1\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_1) |x\rangle, \quad |\psi_2\rangle = \int_{-\infty}^{\infty} dx \psi(x - X_2) |x\rangle. \quad (14)
  \]

- Quantum state at image plane:

- *derive from zero-mean Gaussian P function*

- Multiphoton coincidence: *rare*, little information as $\epsilon \ll 1$ (homeopathy)

- Similar model for stellar interferometry in Gottesman, Jennewein, Croke, PRL 109, 070503 (2012); Tsang, PRL 107, 270402 (2011).
Quantum and classical Fisher information

\[ \frac{\mathcal{K}_{11}}{N/4\sigma^2} \]

\[ \frac{\mathcal{J}^{(\text{direct})}_{11}}{N/4\sigma^2} \]

\[ \frac{\mathcal{K}_{22}}{N/4\sigma^2} \]

\[ \frac{\mathcal{J}^{(\text{direct})}_{22}}{N/4\sigma^2} \]

Cramér-Rao bounds on separation error

\[ \text{Quantum} \left( \frac{1}{\mathcal{K}_{22}} \right) \]

\[ \text{Direct imaging} \left( \frac{1}{\mathcal{J}^{(\text{direct})}_{22}} \right) \]

Tsang, Nair, and Lu, Physical Review X 6, 031033 (2016)

\[ \Delta \theta^2_2 \geq \frac{1}{\mathcal{K}_{22}} = \frac{1}{N\Delta k^2}. \] (15)

Nair and Tsang, e-print arXiv:1604.00937 (accepted by PRL): thermal sources with arbitrary \( \epsilon \)

Hayashi ed., Asymptotic Theory of Quantum Statistical Inference; Fujiwara JPA 39, 12489 (2006): there exists a POVM such that \( \Delta \theta^2_\mu \rightarrow 1/\mathcal{K}_{\mu\mu}, M \rightarrow \infty \).
project the photon in **Hermite-Gaussian** basis:

\[ E_1(q) = |\phi_q \rangle \langle \phi_q | , \]  

\[ |\phi_q \rangle = \int_{-\infty}^{\infty} dx \phi_q(x) |x\rangle , \]  

\[ \phi_q(x) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} H_q \left( \frac{x}{\sqrt{2}\sigma} \right) \exp \left( -\frac{x^2}{4\sigma^2} \right) . \] 

Assume PSF \( \psi(x) \) is Gaussian (common).

\[ \frac{1}{J_{22}^{(HG)}} = \frac{1}{K_{22}} = \frac{4\sigma^2}{N} . \] 

**Maximum-likelihood estimator** can saturate the classical bound asymptotically for large \( N \).
Spatial-Mode Demultiplexing (SPADE)

![Diagram of spatial-mode demultiplexing](image)

- $\psi(x + \frac{\theta_2}{2})$
- $\psi(x - \frac{\theta_2}{2})$

- $q_{\text{max}}$
- $q > q_{\text{max}}$

- $\beta_0$
- $\beta_1$
- $\beta_2$

- $x$
- $z$

- Image plane
Binary SPADE

Classical Fisher information

Fisher information for sinc PSF

\[
\mathcal{J}_{22}^{(b)} = \frac{N}{4\sigma^2}
\]

\[
\mathcal{J}_{22}^{(direct)} = \frac{\pi^2 N}{3W^2}
\]

\[
\mathcal{J}_{22}^{(HG)} = \mathcal{K}_{22}
\]
$\theta^2 / \sigma^2$

Mean-square error / $(4\sigma^2 / L)$

Simulated errors for SPADE

$1 / \mathcal{J}^{(HG)}_{22} = 1 / \mathcal{K}'_{22}$

$L = 10$

$L = 20$

$L = 100$

Simulated errors for binary SPADE

$L = 10$

$L = 20$

$L = 100$

- $L =$ number of detected photons
- biased, $< 2 \times$ CRB.
Van Trees inequality for any biased/unbiased estimator (e-print arXiv:1605.03799)

Quantum/SPADE: $\sup_{\theta} \Sigma_{22}(\theta) \geq \frac{4\sigma^2}{N}$,  

Direct imaging: $\sup_{\theta} \Sigma_{22}^{(\text{direct})}(\theta) \geq \frac{\sigma^2}{\sqrt{N}}$.  \hspace{1cm} (20)
- **SuperLocalization via Image-in**VER**sion interferometry**
- Laser Focus World, Feb 2016 issue.
Ang, Nair, Tsang, e-print arXiv:1606.00603
Misalignment

- $\xi \equiv |\hat{\theta}_1 - \theta|/\sigma \ll 1$
- Overhead photons $N_1 \sim 1/\xi^2$
- $\xi = 0.1, \ N_1 \sim 100$. 
- CRB for $X_s = \theta_1 \pm \theta_2/2$
### Theoretical Follow-up

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<td>Quantum, Bayesian, Chernoff</td>
<td>SPADE, SLIVER</td>
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</tbody>
</table>

**Other groups:**

- Rehacek et al. (Europe), arXiv:1606.08332.
Experiments

  - SLIVER
  - Laser, classical noise

  - Mode heterodyne
  - Laser

  - SPADE
  - single-photon sources, close to quantum limit

  - SPADE
  - laser, close to quantum limit
Design quantum computer to
- Maximize information extraction
- Reduce classical computational complexity
Quantum Metrology Kills Rayleigh’s Criterion

Cramér-Rao bounds on separation error

\[ \text{Mean-square error} / \left( 4\sigma^2 / N \right) \]

Quantum \( (1/K_{22}) \)

Direct imaging \( (1/J_{22}^{(direct)}) \)

FAQ: https://sites.google.com/site/mankeitsang/news/rayleigh/faq

email: mankei@nus.edu.sg
\[ \epsilon \ll 1 \] Approximation

- Chap. 9, Goodman, *Statistical Optics*:

“If the count degeneracy parameter is much less than 1, it is highly probable that there will be either zero or one counts in each separate coherence interval of the incident classical wave. In such a case the classical intensity fluctuations have a negligible ”bunching” effect on the photo-events, for (with high probability) the light is simply too weak to generate multiple events in a single coherence cell.

- Zmuidzinas (https://pma.caltech.edu/content/jonas-zmuidzinas), JOSA A 20, 218 (2003):

“It is well established that the photon counts registered by the detectors in an optical instrument follow statistically independent **Poisson** distributions, so that the fluctuations of the counts in different detectors are uncorrelated. To be more precise, this situation holds for the case of thermal emission (from the source, the atmosphere, the telescope, etc.) in which the mean photon occupation numbers of the modes incident on the detectors are low, \( n \ll 1 \). In the high occupancy limit, \( n \gg 1 \), photon bunching becomes important in that it changes the counting statistics and can introduce correlations among the detectors. We will discuss only the first case, \( n \ll 1 \), which applies to most astronomical observations at optical and infrared wavelengths."

- See also Labeyrie et al., *An Introduction to Optical Stellar Interferometry*, etc.
- Fluorescent particles: Pawley ed., *Handbook of Biological Confocal Microscopy*, Ram, Ober, Ward (2006), etc., may have **antibunching**, but Poisson model is fine because of \( \epsilon \ll 1 \).