Quantum Optical Temporal Phase Estimation by Homodyne Phase-Locked Loops

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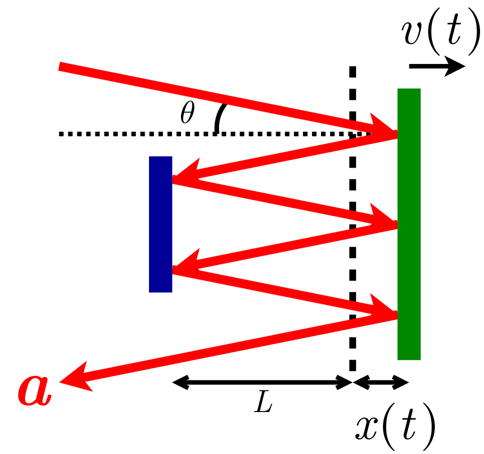
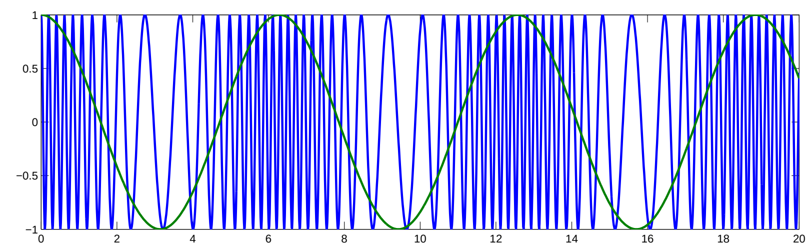
Motivation

- Sensing
  - $x(t) \propto \phi(t)$
  - $v(t) \propto \dot{\phi}(t)$
- Metrology
  - Clock stability determined by $\dot{\phi}(t)$ fluctuations.
- Coherent Communications
  - PM: $m(t) \propto \phi(t)$
  - FM: $m(t) \propto \dot{\phi}(t)$





Quantization of 1D Optical Fields

- Frequency modes:

$$[\hat{a}(\omega), \hat{a}^\dagger(\omega')] = \delta(\omega - \omega'). \tag{1}$$

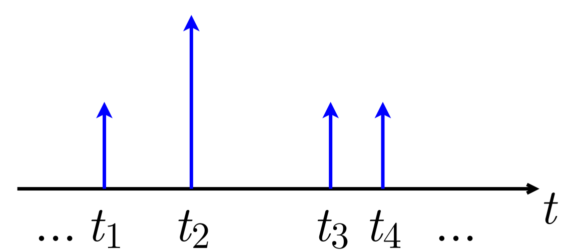
- Rotating-wave approximation:

$$\hat{A}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \hat{a}(\omega) \exp[-i(\omega - \omega_0)t], \qquad [\hat{A}(t), \hat{A}^\dagger(t)] = \delta(t - t'). \tag{2}$$

- Continuous-time Fock states:

$$|dn(t)\rangle \equiv \prod_j \frac{1}{\sqrt{dn(t_j)!}} \left[\hat{A}^\dagger(t_j)\sqrt{dt}\right]^{dn(t_j)} |0\rangle,$$

$$\hat{A}^\dagger(t)\hat{A}(t)|dn(t)\rangle = \left[\sum_j dn(t_j)\delta(t - t_j)\right] |dn(t)\rangle. \tag{3}$$



- J. H. Shapiro, Quantum Semiclass. Opt. **10**, 567 (1998).

Temporal-Phase POVM

- Generalizing Susskind-Glogower phase states:

$$|\phi(t)\rangle = \sum_{dn(t)} \exp\left[i\int_{-\infty}^{\infty} dt \frac{dn(t)}{dt}\phi(t)\right]|dn(t)\rangle \tag{4}$$

$$= \sum_{dn(t)} \exp\left[i\sum_j dn(t_j)\phi(t_j)\right]|dn(t)\rangle \tag{5}$$

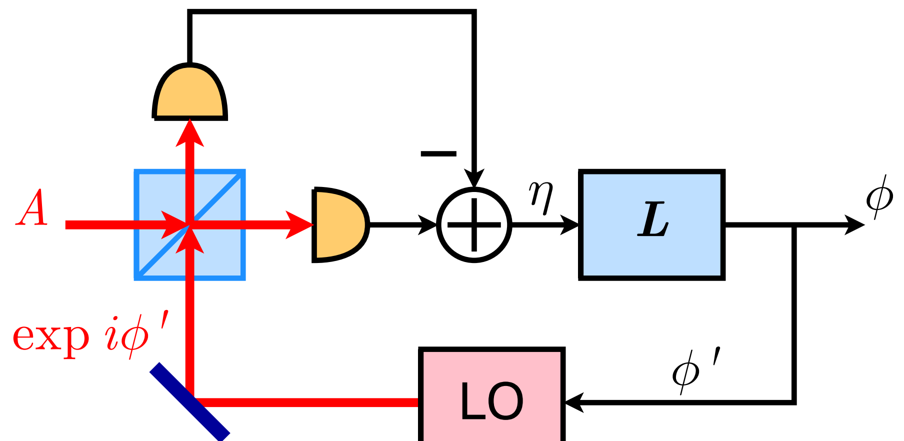
- Temporal-Phase POVM:

$$\hat{\Pi}[\phi(t)] \equiv |\phi(t)\rangle\langle\phi(t)|, \qquad P[\phi(t)] = \mathrm{Tr}\left\{\hat{\rho}\hat{\Pi}[\phi(t)]\right\}, \tag{6}$$

$$\int D\phi(t)\ \hat{\Pi}[\phi(t)] = \hat{1}, \qquad D\phi(t) = \lim_{\delta t\to 0}\prod_k \frac{d\phi(t+k\delta t)}{2\pi}. \tag{7}$$

- Difficult to realize experimentally.

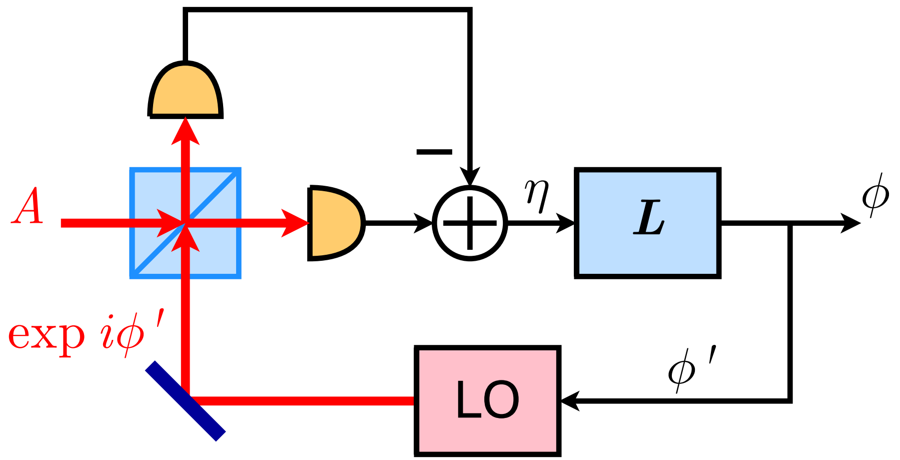
Adaptive Homodyne Detection



- Single-mode phase measurement:
  - Wiseman, Phys. Rev. Lett. **75**, 4587, (1995).
  - Armen *et al.*, Phys. Rev. Lett. **89**, 133602 (2002).
- Phase-Locked Loop (PLL)
  - Viterbi, *Principles of Coherent Communications* (McGraw-Hill, New York, 1966).
  - H. L. Van Trees, *Detection, Estimation, and Modulation Theory, Part I* (Wiley, New York, 2001); *Part II: Nonlinear Modulation Theory* (Wiley, New York, 2002).
  - A. B. Baggeroer, *State Variables and Communication Theory* (MIT Press, Cambridge, 1970).

Phase-Locked Loop Design for Coherent States

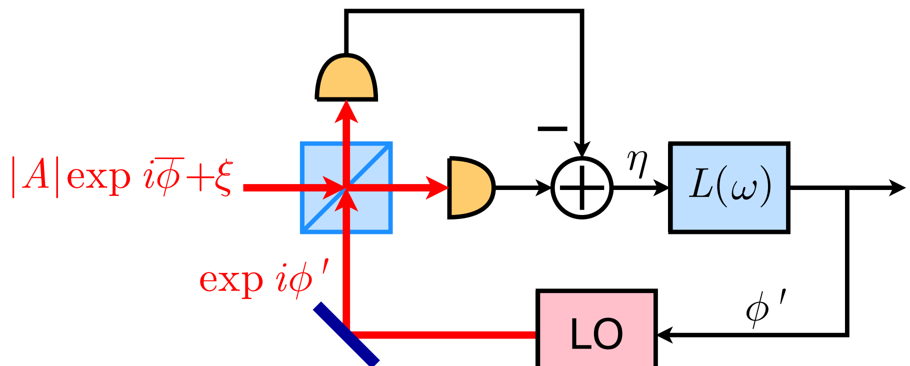
- Wigner distribution for coherent states is Gaussian.



- Upon homodyne detection, a coherent state can be regarded as a classical signal with additive white Gaussian noise,

$$\eta(t) = \sin\left[\bar{\phi}(t) - \phi'(t)\right] + z(t), \quad \langle z(t)z(\tau)\rangle = \frac{1}{4|\alpha|^2}\delta(t-\tau), \quad |\alpha|^2 = \frac{\mathcal{P}}{\hbar\omega_0}. \qquad (8)$$

Wiener Filtering



- Let the mean phase be a classical stationary Gaussian random process:

$$\langle \bar{\phi}(t)\bar{\phi}(\tau)\rangle = K_{\bar{\phi}}(t-\tau), \qquad S_{\bar{\phi}}(\omega) = \int_{-\infty}^{\infty} dt\, K_{\bar{\phi}}(t)\exp(i\omega t), \tag{9}$$

- Assume that the PLL is phase locked,

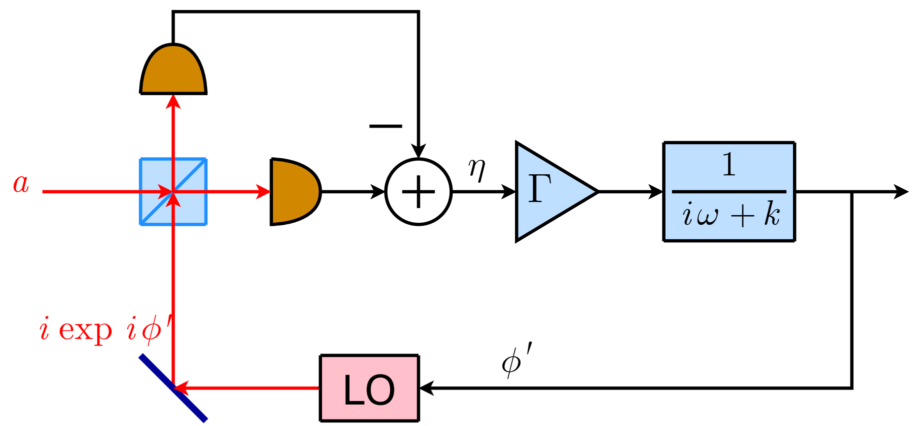
$$\langle [\bar{\phi}(t)-\phi'(t)]^2\rangle \ll 1, \qquad \eta(t) \approx 2|\alpha| \left[\bar{\phi}(t)-\phi'(t)\right] + z(t). \tag{10}$$

- $L(\omega)$ can be designed using the Wiener filtering technique.

Example: Ornstein-Uhlenbeck Process

- Power spectral density of mean phase $\bar{\phi}(t)$:

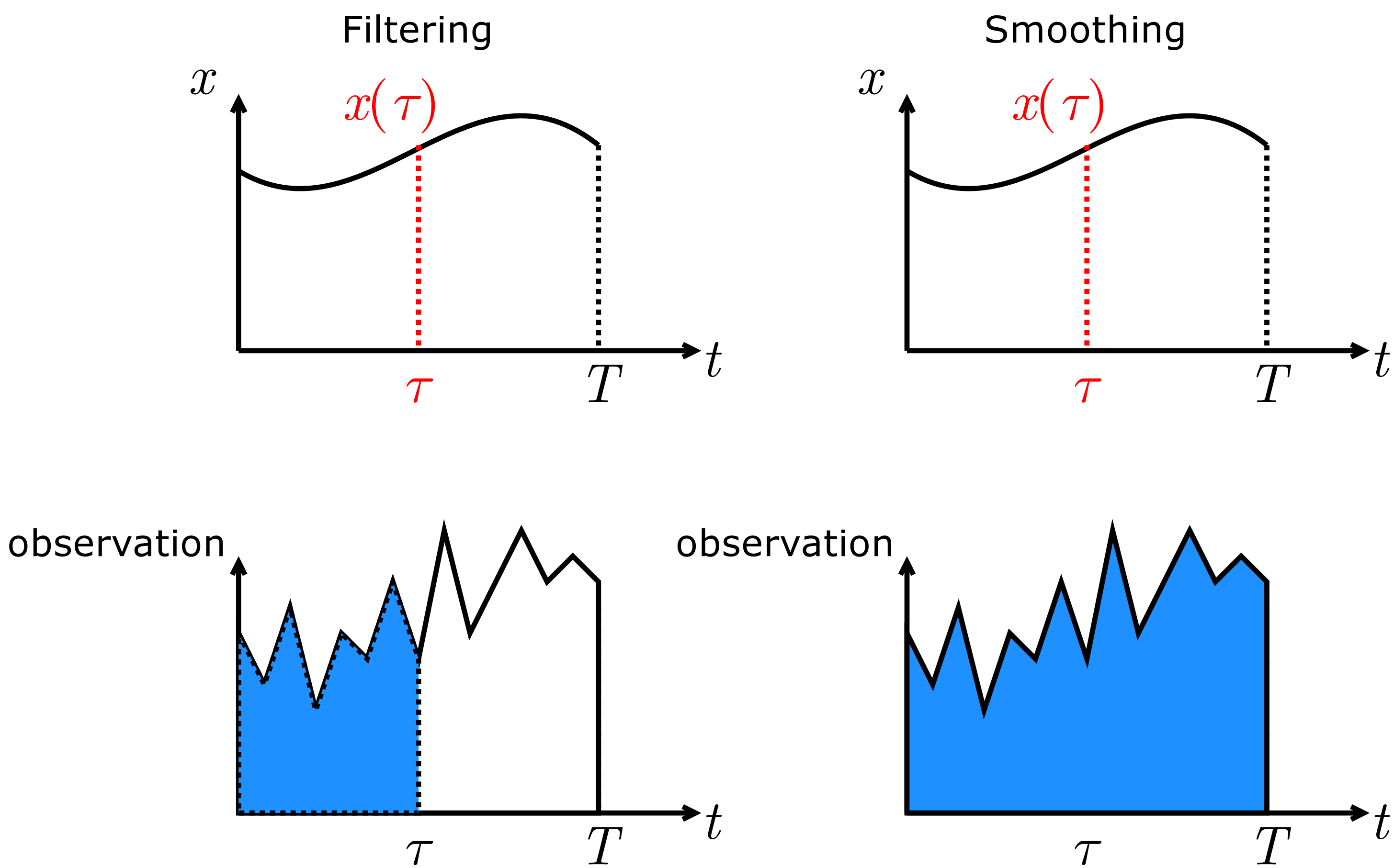
$$S_{\bar{\phi}}(\omega) = \frac{\kappa}{\omega^2 + k^2}. \tag{11}$$



$$\Gamma = \sqrt{\frac{4\kappa\mathcal{P}}{\hbar\omega_0}} - k, \qquad \langle (\bar{\phi} - \phi')^2 \rangle = \frac{\hbar\omega_0 k}{4\mathcal{P}} \left(\sqrt{\frac{4\kappa\mathcal{P}}{\hbar\omega_0 k^2}} - 1 \right) \approx \frac{1}{2}\sqrt{\frac{\hbar\omega_0\kappa}{\mathcal{P}}}. \tag{12}$$

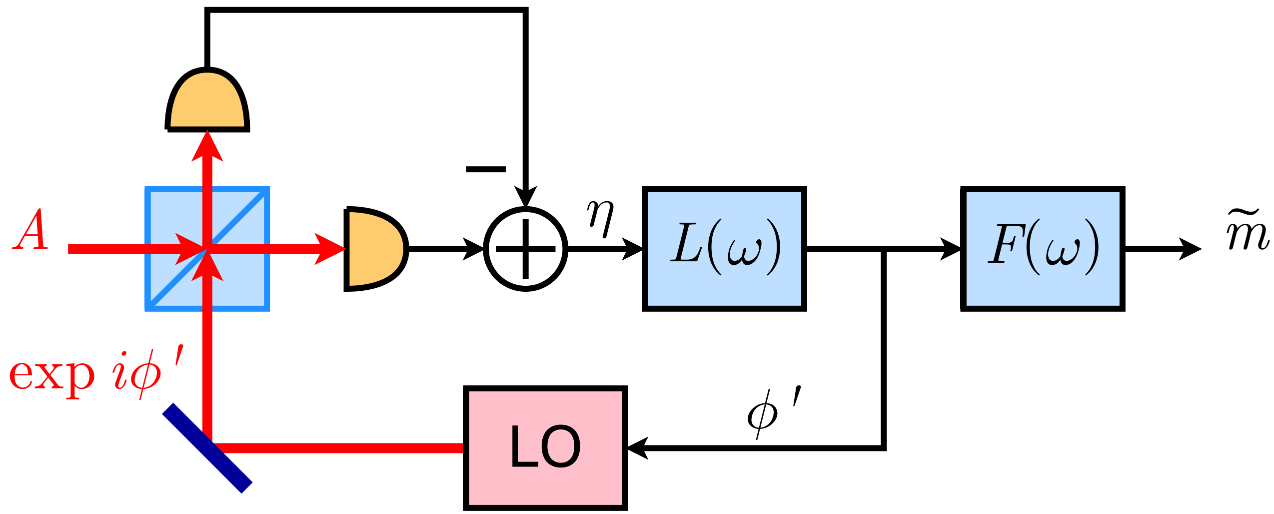
- The special case of $k \to 0$ (Wiener process) has been studied by Berry and Wiseman.
- Use Kalman-Bucy filtering for Gaussian non-stationary random processes.

Smoothing



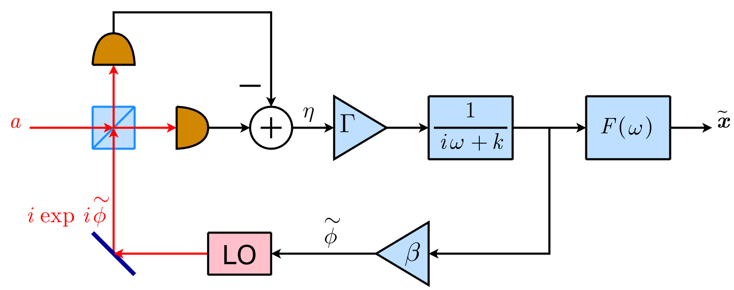
Frequency-Domain Smoothing

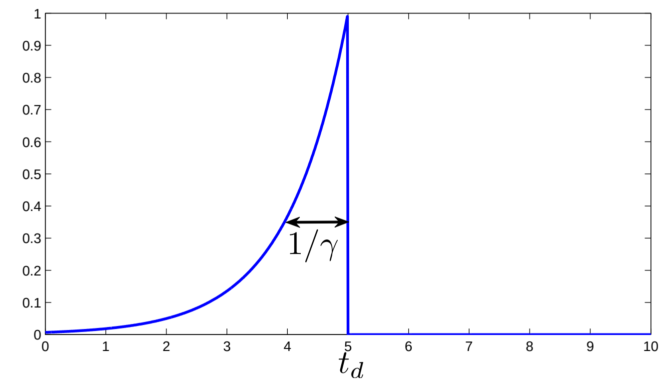
- Smoothing can be achieved by PLL + post-loop filter:



Example: Ornstein-Uhlenbeck Process

- For phase modulation, let $\bar{\phi}(t) = \beta m(t)$.





$$F(\omega) = \frac{k + \gamma}{-i\omega + \gamma} \exp(-i\omega t_d), \qquad \gamma \equiv \left(\frac{4\kappa\beta^2\mathcal{P}}{\hbar\omega_0} + k^2\right)^{1/2}. \tag{13}$$

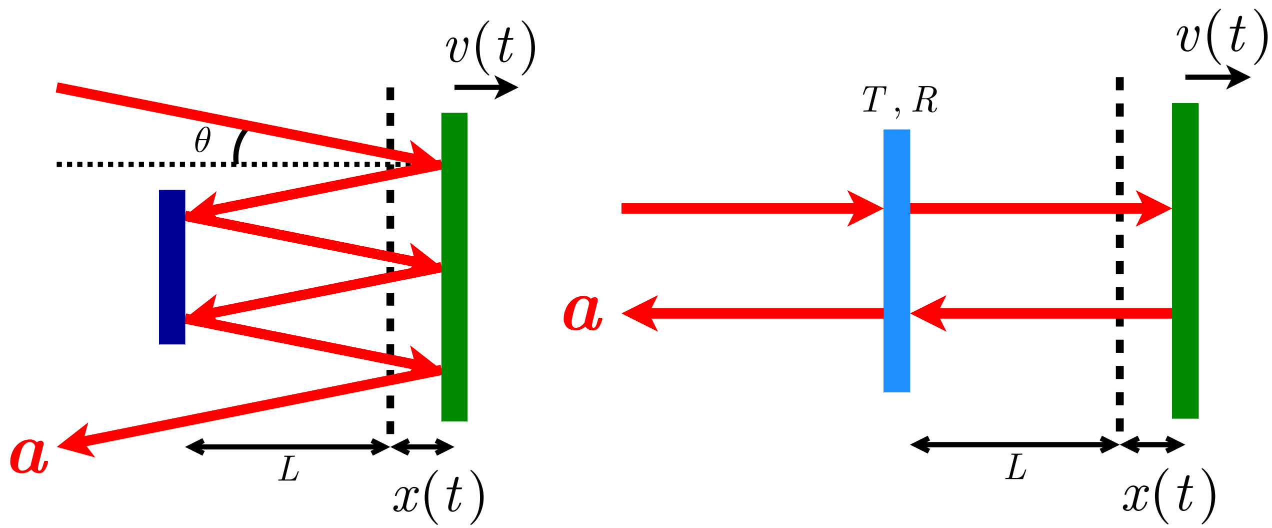
- “Irreducible Error”:

$$\left\langle (m - \widetilde{m})^2 \right\rangle = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{S_m(\omega)}{4|\alpha|^2\beta^2 S_m(\omega) + 1} = \frac{\kappa}{2\gamma} \approx \frac{1}{4\beta}\sqrt{\frac{\hbar\omega_0\kappa}{\mathcal{P}}} \tag{14}$$

~ 3 dB better than Wiener or Kalman-Bucy filtering.

- State-variable approach: Bryson-Frazier or Mayne-Fraser-Potter smoothing

Multipass Position and Velocity Sensing



- Multipass constant phase measurements:
  - Giovannetti, Lloyd, and Maccone, Phys. Rev. Lett. **96**, 010401 (2006).
  - Higgins *et al.*, Nature **450**, 393 (2007).
- With a homodyne PLL, we can continuously monitor the mirror position and velocity simultaneously at the quantum limit using a high-power coherent state.
- If the optical beam hits the target multiple times before $x(t)$ and $v(t)$ change significantly, $\beta \propto M$, and the SNR can be increased.

Conclusion

- Temporal-Phase POVM
- Phase-Locked Loop Design Using Estimation Theory
- General Quantum Theory of Smoothing
- References:
  - Tsang, Shapiro, and Lloyd, Phys. Rev. A **78**, 053820 (2008),
  - Tsang, Shapiro, and Lloyd, Phys. Rev. A **79**, 053843 (2009),
  - Tsang, submitted to Phys. Rev. Lett. (e-print arXiv:0904.1969).
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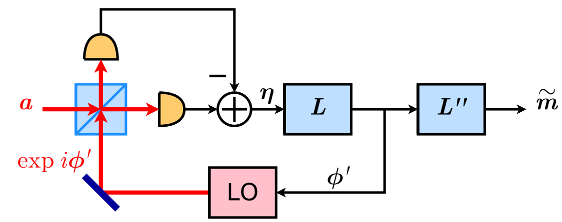
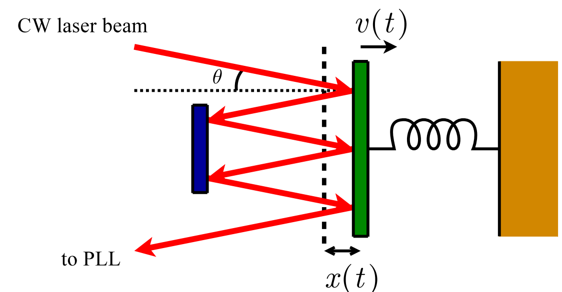




Maximum A Posteriori (MAP) Estimation

- To be more general, let

$$\bar{\phi}(t) = \int_{-\infty}^{\infty} d\tau h(t-\tau)m(\tau), \qquad \langle m(t)m(\tau)\rangle = K_m(t,\tau). \tag{15}$$

$m(t)$ is the message we wish to estimate. For example,

$$h(t-\tau) = \beta\delta(t-\tau), \qquad \bar{\phi}(t) = \beta m(t), \qquad (\mathrm{PM}) \tag{16}$$

$$h(t-\tau) = -2\pi\mathcal{F}\int_{t_0}^{t} du\delta(u-\tau), \qquad \bar{\phi}(t) = -2\pi\mathcal{F}\int_{t_0}^{t} d\tau\, m(\tau). \qquad (\mathrm{FM}) \tag{17}$$

- MAP estimation: solve for the "most likely" message given our full measurement record:

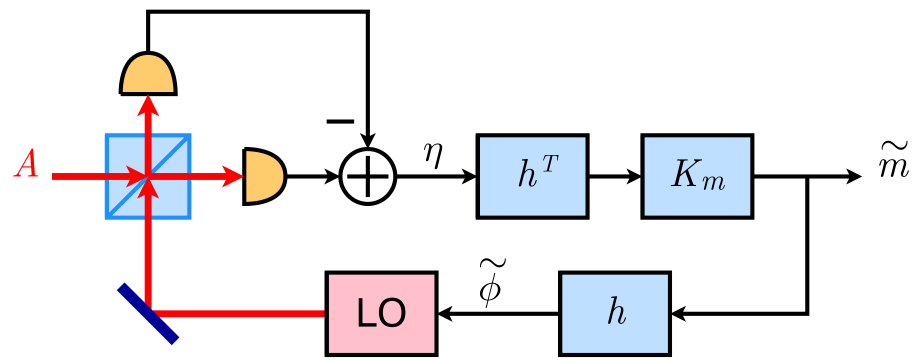
$$\frac{\delta}{\delta m(t)}\left\{\ln P[m(t)|A(\tau)]\right\}_{m(t)=\tilde{m}(t)} = 0, \tag{18}$$

$$\frac{\delta}{\delta m(t)}\left\{\ln W[A(\tau)|m(t)] + \ln P[m(t)]\right\}_{m(t)=\tilde{m}(t)} = 0. \tag{19}$$

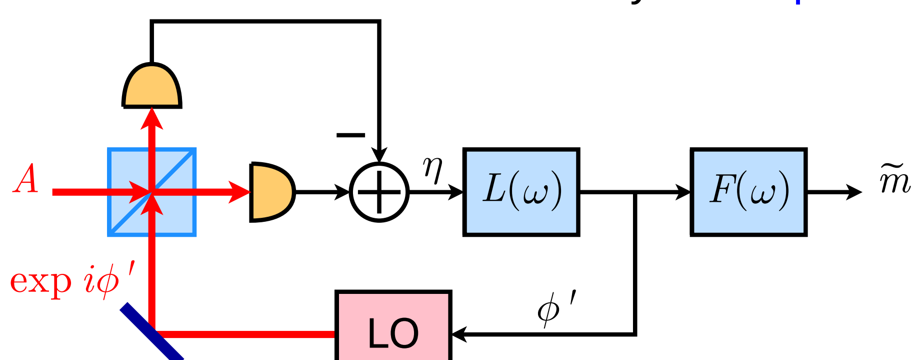
Phase-Locked Loop Design via MAP Estimation

- For coherent states, the MAP equation becomes

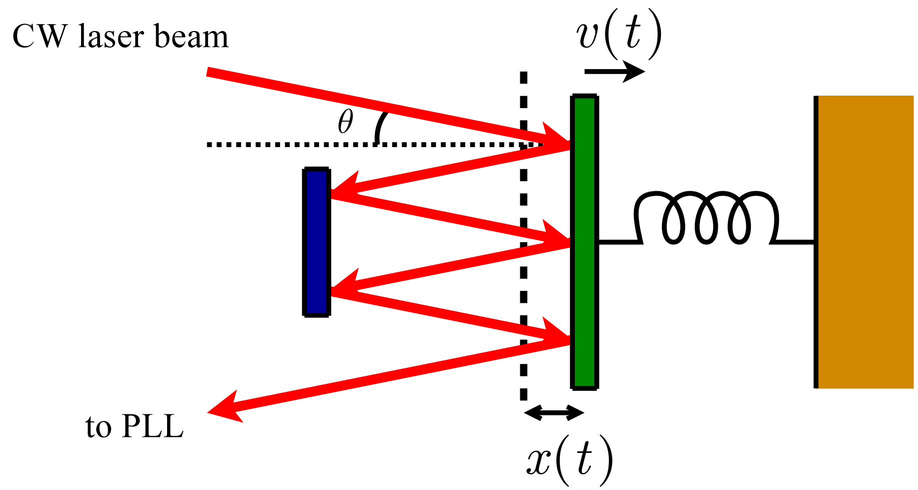
$$\tilde{m}(t) = 2|\alpha| \int d\tau du\, K_m(t,\tau) h(u-\tau) \eta[A(u), \tilde{\phi}(u)], \tag{20}$$



- The feedback filter $h^T * K_m * h$ is non-causal, this PLL is unrealizable.
- Linearizing η again, MAP estimation can be achieved by PLL + post-loop filter:



Quantum-Limited Position and Velocity Estimation



- System model:

$$\frac{dx}{dt} = v, \quad \frac{dv}{dt} = -\omega_m^2 x + \frac{2M\hbar\omega_0 \cos\theta}{mc} I(t), \quad \langle I(t)I(\tau)\rangle_{\mathrm{coh}} = \frac{\mathcal{P}}{\hbar\omega_0}\delta(t-\tau). \tag{21}$$

- Equivalent observation process by homodyne PLL:

$$y = (2Mk_0\cos\theta)x(t) + w(t), \qquad \langle w(t)w(\tau)\rangle \approx \frac{\hbar\omega_0}{4\mathcal{P}}\delta(t-\tau). \tag{22}$$

- The mirror quantum state remains a Gaussian state under these approximations, and we can use Kalman-Bucy filtering.

Kalman-Bucy Filtering Errors

- The Kalman-Bucy covariances at steady state $t \to \infty$ are

$$\left\langle \Delta x^2 \right\rangle = \frac{\hbar}{2m\omega_m} \frac{\sqrt{2}}{Q} \left[(1+Q^2)^{1/2} - 1 \right]^{1/2}, \tag{23}$$

$$\frac{1}{2} \left\langle \Delta x \Delta v + \Delta v \Delta x \right\rangle = \frac{\hbar}{2m} \frac{1}{Q} \left[(1+Q^2)^{1/2} - 1 \right], \tag{24}$$

$$\left\langle \Delta v^2 \right\rangle = \frac{\hbar\omega_m}{2m} \frac{\sqrt{2}}{Q} \left[(1+Q^2)^{1/2} - 1 \right]^{1/2} (1+Q^2)^{1/2}, \tag{25}$$

$$Q \equiv \frac{8M^2\omega_0\mathcal{P}\cos^2\theta}{m\omega_m^2 c^2}. \tag{26}$$

- Previously derived using a general QND measurement model in
  - Belavkin and Staszewski, Phys. Lett. A **140**, 359 (1989).
  - Doherty *et al.*, Phys. Rev. A **60**, 2380 (1999).
- At steady state, the conditioned mirror quantum state is a pure Gaussian state:

$$\left\langle \Delta x^2 \right\rangle \left\langle \Delta v^2 \right\rangle - \left(\frac{1}{2} \left\langle \Delta x \Delta v + \Delta v \Delta x \right\rangle \right)^2 = \frac{\hbar^2}{4m^2}. \tag{27}$$

Quantum-Limited Smoothing

- With post-processing, classical estimation theory predicts improved performance.
- Smoothing errors:

$$\langle \Delta x^2 \rangle = \frac{\hbar}{8m\omega_m}\left[\frac{1}{(1+iQ)^{1/2}} + \frac{1}{(1-iQ)^{1/2}}\right], \tag{28}$$

$$\langle \Delta v^2 \rangle = \frac{\hbar\omega_m}{8m}\left[(1+iQ)^{1/2} + (1-iQ)^{1/2}\right], \tag{29}$$

- Uncertainty product:

$$\langle \Delta x^2 \rangle \langle \Delta v^2 \rangle = \frac{\hbar^2}{32m^2}\left[1 + \frac{1}{(1+Q^2)^{1/2}}\right] < \frac{\hbar^2}{4m^2}. \tag{30}$$

- Resolution of paradox: We estimate the position and velocity of the mirror some time in the past, but the past quantum state of the mirror has been irreversibly destroyed.
  - We can't measure the mirror more accurately in the past without further disturbing it.
  - We can't clone the past quantum state of the mirror and store it for future comparisons.
  - We can't reverse the quantum dynamics of the mirror, because we have measured the phase and the radiation pressure force becomes unknown to us.

Delayed Estimation of Classical Information

- We can still estimate a classical force $F_{\text{ext}}(t)$ with delay:

$$m\frac{dv}{dt} = -m\omega_m^2 x + F_{\text{rad}}(t) + F_{\text{ext}}(t). \quad (31)$$

- For delayed estimation, current quantum trajectory theory needs to be modified.
  - Belavkin, Carmichael, Wiseman and Milburn, ...

$$\hat{\rho}_c(t), |\widetilde{\psi}(t)\rangle \text{ given } \eta(\tau), \tau < t. \quad (32)$$

  - For smoothing, we need

$$\text{conditioned ``quantum state'' at time } t, \text{ given } \eta(\tau),\ t_0 \le \tau \le T. \quad (33)$$

- Idea: Use two quantum states, one traveling forward in time from t_0, and one traveling backward in time from T, ala Aharonov *et al.*

Fundamental Quantum Limits

- A band-limited random process:

$$S_m(\omega) = \begin{cases} 1/b, & |\omega| \leq \pi b, \\ 0, & |\omega| > \pi b, \end{cases} \qquad \mathcal{N} \equiv \frac{\mathcal{P}}{\hbar\omega_0 b}, \qquad \Lambda(r) \equiv \left(\mathcal{N} - \sinh^2 r\right)\exp(2r). \tag{34}$$

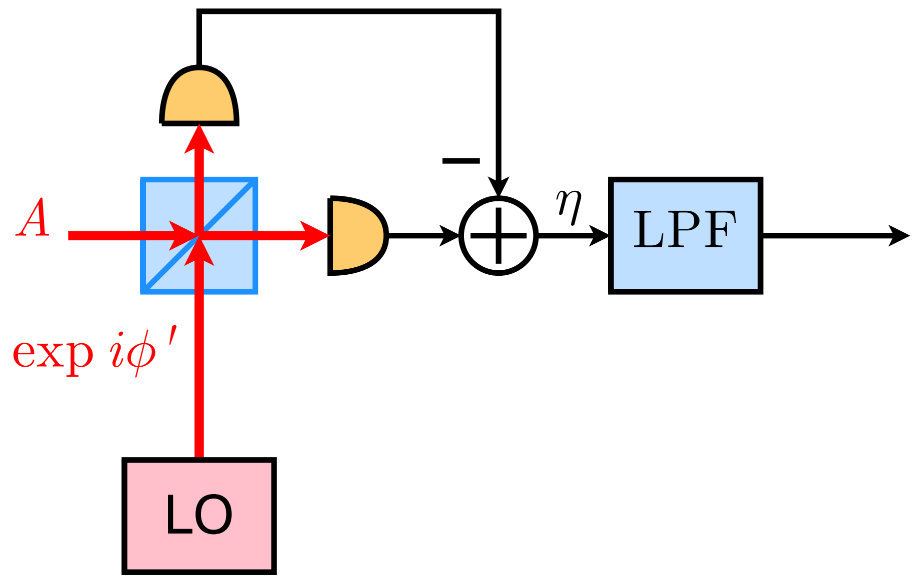
	SQL SNR	Squeezed	Threshold	Max. SNR
Homodyne PLL, PM	$4\beta^2\mathcal{N}$	$4\beta^2\Lambda$	$\frac{\exp(4r)}{\Lambda}\ln(1+\beta^2\Lambda) \ll 1$	$\ll 8\beta^2\mathcal{N}^2/\ln\mathcal{N}$
Homodyne PLL, FM	$12\beta^2\mathcal{N}$	$12\beta^2\Lambda$	$\frac{\exp(4r)}{\Lambda}\ln(1+\beta^2\Lambda) \ll 1$	$\ll 24\beta^2\mathcal{N}^2/\ln\mathcal{N}$
POVM + PLL, PM	$4\beta^2\mathcal{N}$	$4\beta^2\Lambda$	$\frac{1}{\Lambda}\ln(1+\beta^2\Lambda) \ll 1$	$4\beta^2\mathcal{N}(\mathcal{N}+1)$
POVM + PLL, FM	$12\beta^2\mathcal{N}$	$12\beta^2\Lambda$	$\frac{1}{\Lambda}\ln(1+\beta^2\Lambda) \ll 1$	$12\beta^2\mathcal{N}(\mathcal{N}+1)$

- Increasing modulation index β can enhance the SNR, but the optical bandwidth is also increased,

$$\mathrm{PM}: \bar{\phi}(t) = \beta m(t), \qquad \mathrm{FM}: \bar{\phi}(t) = -\pi\beta b\int_{-\infty}^{t} d\tau\, m(\tau), \tag{35}$$

$$A(t) = |\alpha|\exp[i\bar{\phi}(t)], \qquad \text{Optical } B \sim (\beta+1)b. \tag{36}$$

Homodyne Detection

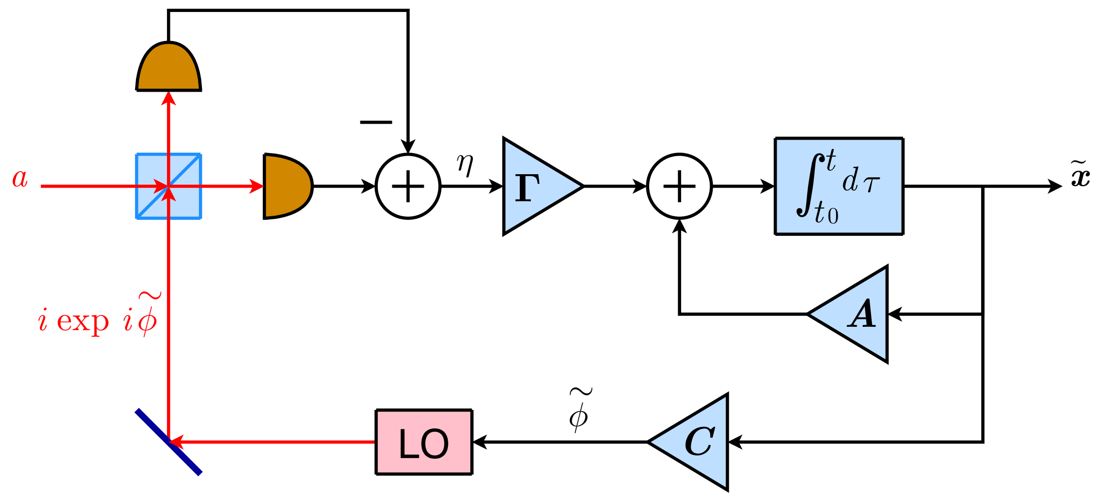


- Output of homodyne detection:

$$\langle \hat{\eta}(t) \rangle \propto -i \left\langle \hat{a} \exp(-i\phi') - \hat{a}^\dagger \exp(i\phi') \right\rangle = 2|\alpha| \sin[\bar{\phi}(t) - \phi'(t)]. \quad (37)$$

- Statistics of $\hat{\eta}(t)$ obey Wigner distribution.
- Does not work if $\bar{\phi}(t)$ has large fluctuations.

Kalman-Bucy Filtering



- Model $\bar{\phi}(t)$ as solution of stochastic differential equations:

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}, \qquad \langle \boldsymbol{u}(t) \otimes \boldsymbol{u}(\tau) \rangle = \boldsymbol{U}\delta(t-\tau), \qquad \bar{\phi}(t) = \boldsymbol{C}(t) \cdot \boldsymbol{x}(t), \tag{38}$$

- Again linearizing $\eta(t) \approx \bar{\phi} - \widetilde{\phi} + z$, use $\eta(t)$ as the Kalman-Bucy "innovation", and obtain the Kalman-Bucy variance equation for $\boldsymbol{\Sigma}(t) \equiv \langle [\boldsymbol{x}(t) - \widetilde{\boldsymbol{x}}(t)] \otimes [\boldsymbol{x}(t) - \widetilde{\boldsymbol{x}}(t)] \rangle$ and "gain,"

$$\frac{d\boldsymbol{\Sigma}}{dt} = \boldsymbol{A}\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\boldsymbol{A}^T - \frac{4\mathcal{P}}{\hbar\omega_0}\boldsymbol{\Sigma}\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{\Sigma} + \boldsymbol{B}\boldsymbol{U}\boldsymbol{B}^T, \qquad \boldsymbol{\Gamma} = \frac{4\mathcal{P}}{\hbar\omega_0}\boldsymbol{\Sigma}\boldsymbol{C}^T. \tag{39}$$