Quantum Sensing and Imaging

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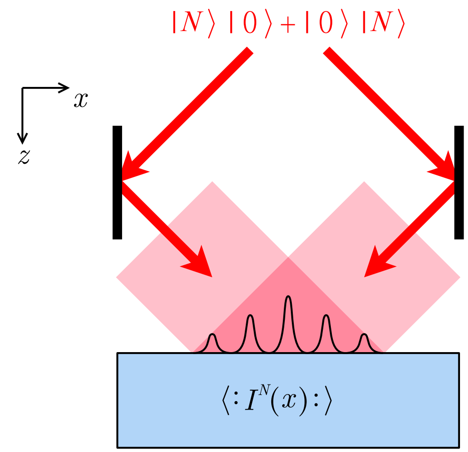
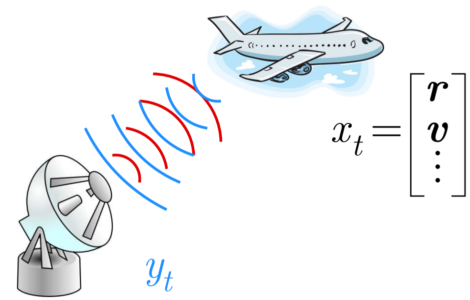
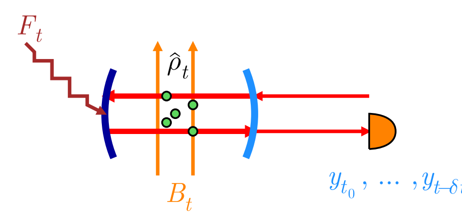
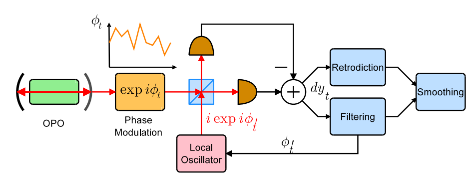
Outline

- Super-resolution Quantum Imaging
  - Tsang, PRA **75**, 043813 (2007).
  - Tsang, PRL **101**, 033602 (2008).
  - Tsang, PRL **102**, 253601 (2009).
- Quantum Smoothing Theory for Optimal Quantum Sensing
  - Tsang, Shapiro, and Lloyd, PRA **78**, 053820 (2008).
  - Tsang, Shapiro, and Lloyd, PRA **79**, 053843 (2009).
  - Tsang, PRL **102**, 250403 (2009).
  - Tsang, PRA **80**, 033840 (2009).
  - Tsang, PRA **81**, 013824 (2010).



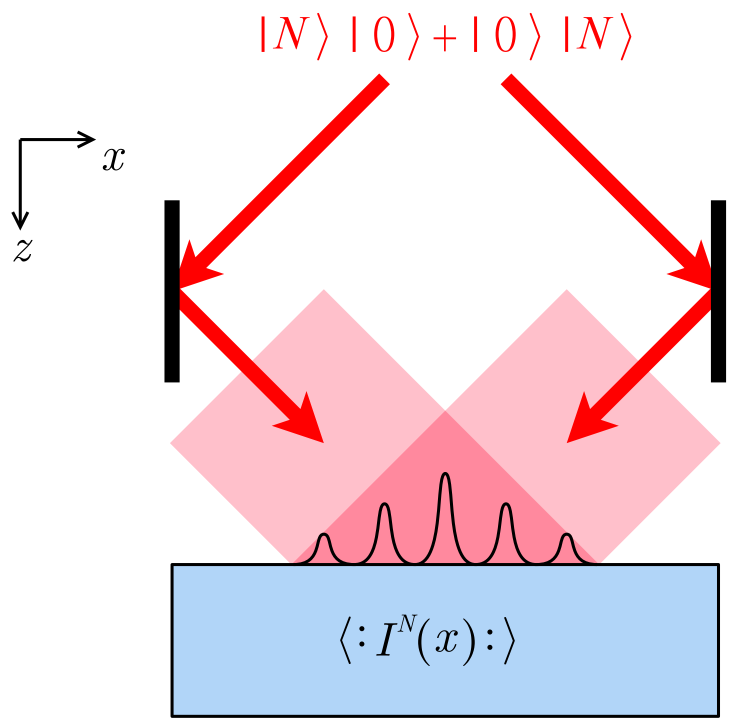






Quantum Lithography

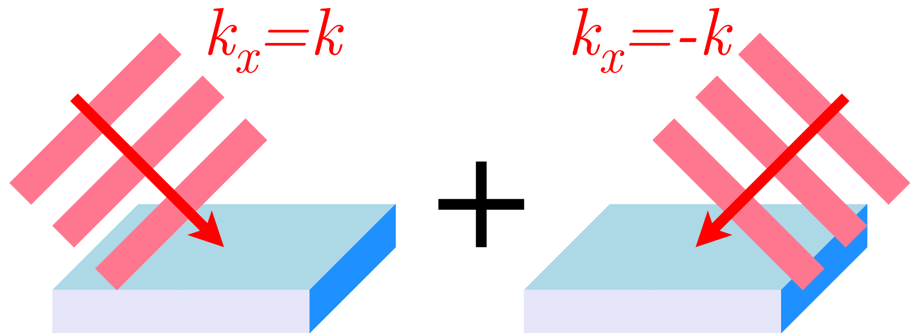
- Boto, Dowling *et al.*, PRL **85**, 2733 (2000).



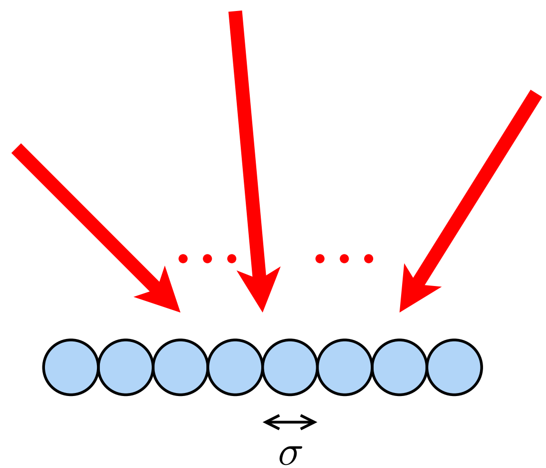
- For path-entangled photons, fringe period $\sim \lambda/N$

Multiphoton-Absorption Rate of NOON State

- $|\Psi\rangle \propto |N\rangle_k |0\rangle_{-k} + |0\rangle_k |N\rangle_{-k}$



- Positively correlated in momentum, anti-correlated in space



- Photons don't arrive at the same absorber as likely as uncorrelated photons.

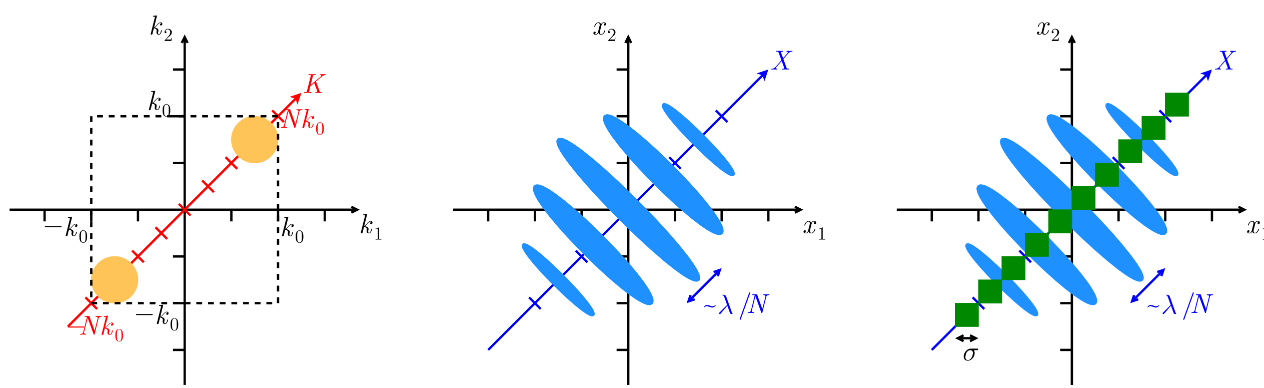
Configuration-Space Picture

- with paraxial approximation:

$$\phi(k_1,\dots,k_N) = \frac{1}{\sqrt{N!}}\langle 0|\hat{a}(k_1)\dots\hat{a}(k_N)|\Psi\rangle, \quad [\hat{a}(k),\hat{a}^\dagger(k')] = \delta(k-k'), \tag{1}$$

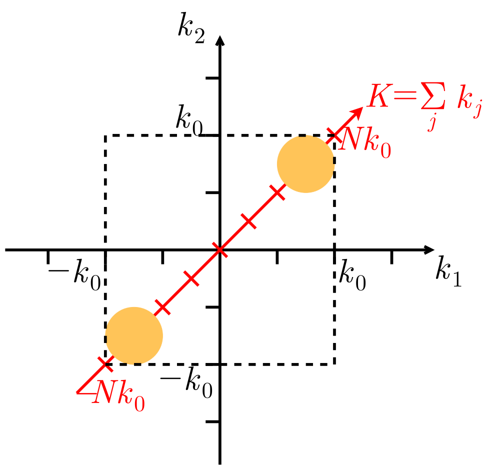
$$\psi(x_1,\dots,x_N) \equiv \int \frac{d^N k}{(2\pi)^{N/2}}\phi(k_1,\dots,k_N)\exp(ik_1x_1+\dots+ik_Nx_N), \tag{2}$$

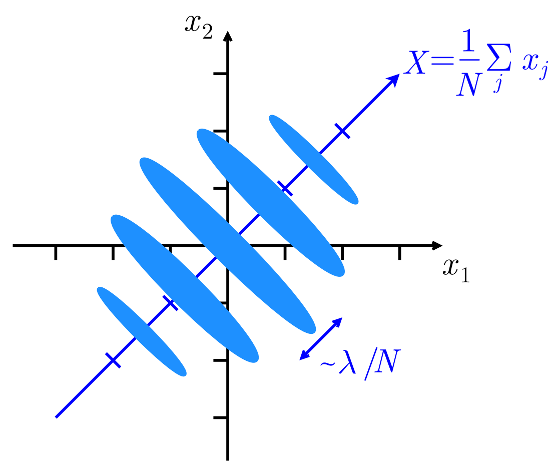
$$\left\langle : \hat{I}^N(x) : \right\rangle = \eta^N N! |\psi(x,x,\dots,x)|^2 \tag{3}$$

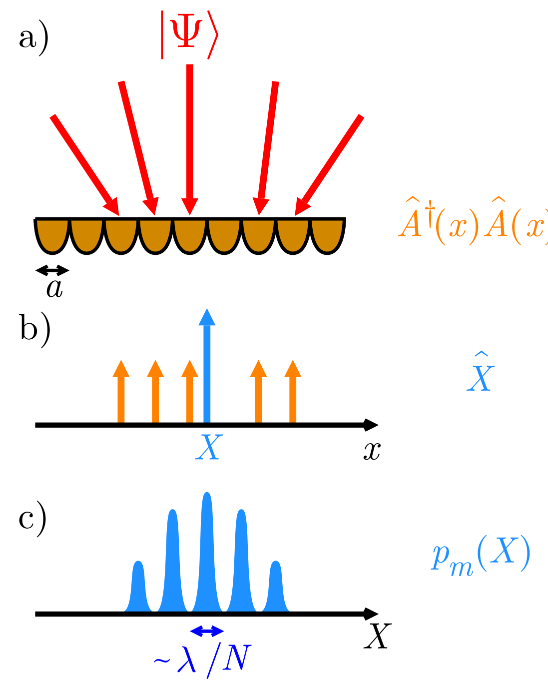


- Peak multi-photon absorption rate of NOON state is a factor of 2^{N-1} lower than that of N independent photons [Tsang, PRA **75**, 043813 (2007)].
- Coherent fields achieve the highest peak multi-photon absorption rate [Tsang, PRL **101**, 033602 (2008)].

Centroid Measurements







- Tsang, PRL **102**, 253601 (2009); Anisimov and Dowling, Physics **2**, 52 (2009).
- Zhuang, “Nano-imaging with STORM,” Nature Phys. **3**, 365 (2009).
- Giovannetti, Lloyd, Maccone, and Shapiro, “Sub-Rayleigh quantum imaging,” PRA **79**, 013827 (2009).

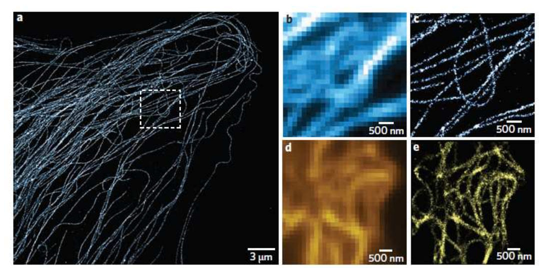
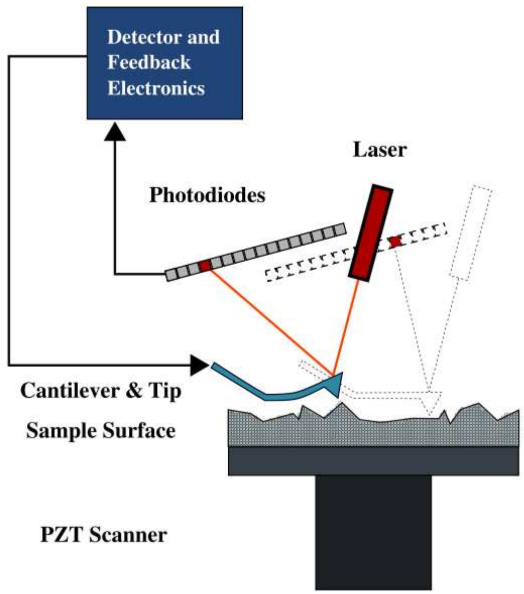


Figure 2 | STORM imaging of cells with photo-switchable dyes and fluorescent proteins. **a**, STORM image of microtubules in a BS-C-1 cell. The microtubules are immunostained with antibodies that are labelled with photo-switchable Alexa Fluor 647. **b**,**c**, Conventional (**b**) and STORM (**c**) images correspond to the boxed region in **a**. **d**,**e**, Conventional (**d**) and STORM (**e**) images of vimentin in a BS-C-1 cell. The vimentin filaments are labelled with a photo-switchable fluorescent protein, mEos2. A moderate aggregation of the vimentin filaments is observed, potentially caused by the fluorescent protein tags.

[Zhuang, Nature Phys. **3**, 365 (2009)]

Atomic Force Microscopy

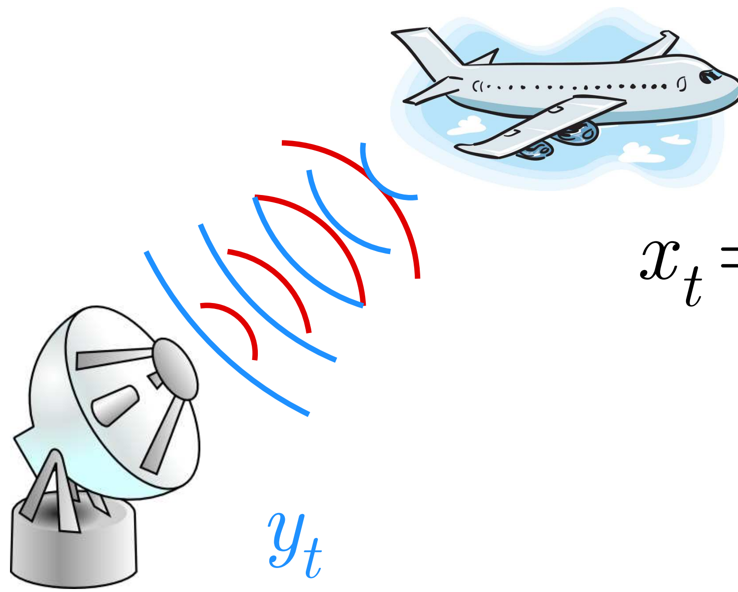


[Wikipedia]

- Squeezing of transverse laser beam position:
  - Treps *et al.*, “Quantum Laser Pointer,” Science **301**, 940 (2003).
- Centroid measurements allow detection of non-Gaussian signatures of light and the cantilever.

Tracking an Aircraft

- or submarine, terrorist, criminal, mosquito, cancer cell, nanoparticle, . . .



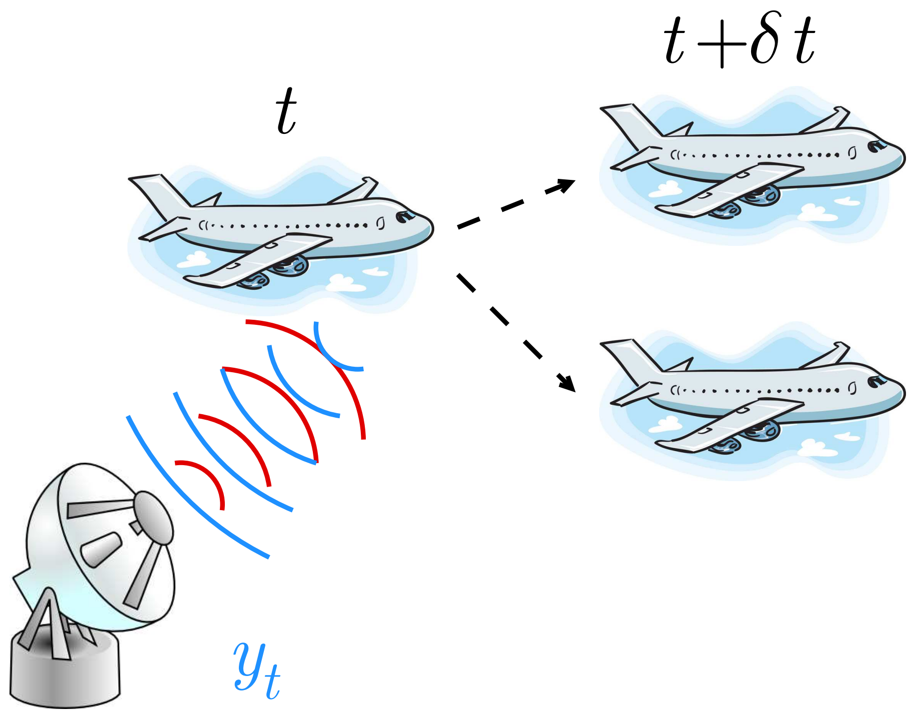
$$x_t = \begin{bmatrix} \boldsymbol{r} \\ \boldsymbol{v} \\ \vdots \end{bmatrix}$$

- Use Bayes theorem:

$$P(x_t|y_t) = \frac{P(y_t|x_t)P(x_t)}{\int dx_t (\text{numerator})}, \tag{4}$$

- $P(y_t|x_t)$ from observation noise, $P(x_t)$ from *a priori* information.

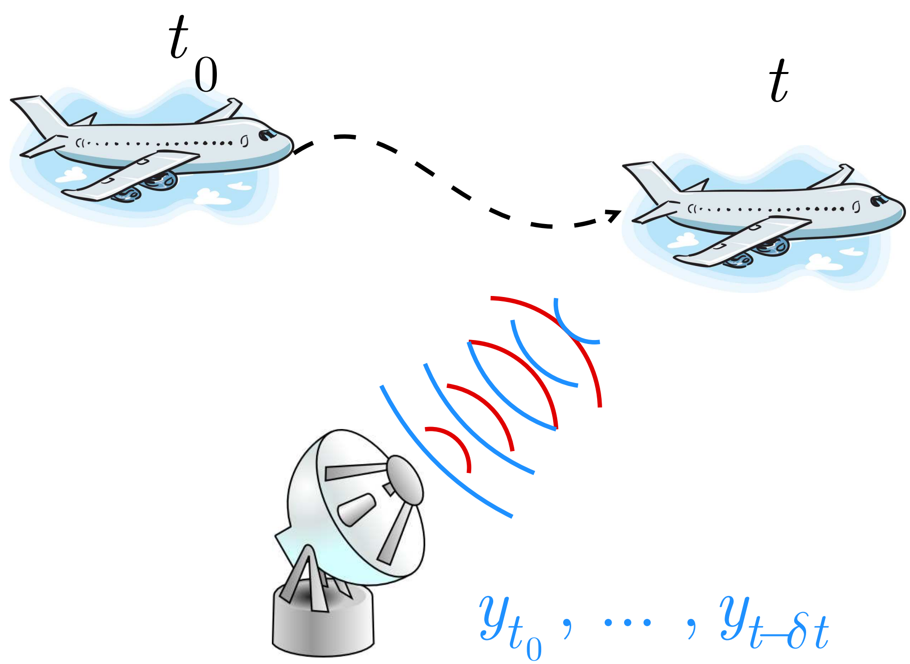
Prediction



- Assume x_t is a Markov process, use Chapman-Kolmogorov equation:

$$P(x_{t+\delta t}|y_t) = \int dx_t P(x_{t+\delta t}|x_t) P(x_t|y_t) \tag{5}$$

Filtering: Real-Time Estimation

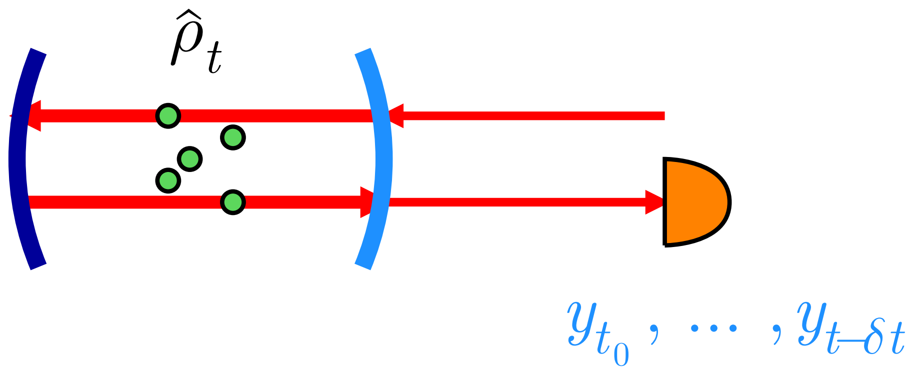


- Applying Bayes theorem and Chapman-Kolmogorov equation repeatedly, we can obtain

$$P(x_t | y_{t-\delta t}, \ldots, y_{t_0+\delta t}, y_{t_0}) \tag{6}$$

- Useful for control, weather and finance forecast, etc.
- Wiener, Stratonovich, Kalman, Kushner, etc.

Quantum Filtering



- Use "quantum Bayes theorem,"

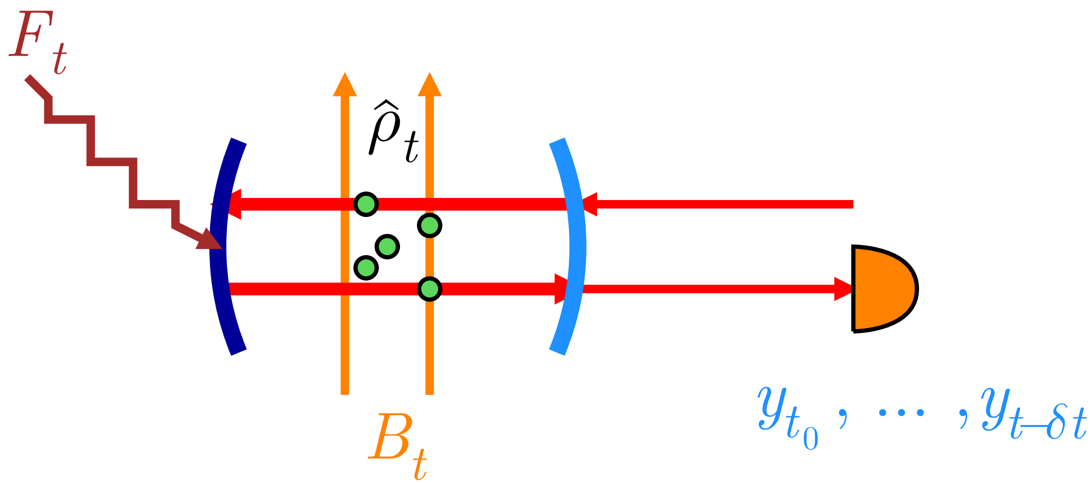
$$\hat{\rho}_t(|y_t) = \frac{\hat{M}(y_t)\hat{\rho}_t\hat{M}^\dagger(y_t)}{\mathrm{tr}(\text{numerator})} \tag{7}$$

- Use a completely positive map to evolve the system state (analogous to Chapman-Kolmogorov),

$$\hat{\rho}_{t+\delta t}(|y_t) = \sum_\mu \hat{K}_\mu \hat{\rho}_t(|y_t)\hat{K}_\mu^\dagger \tag{8}$$

- Belavkin, Barchielli, Carmichael, Caves, Milburn, Wiseman, Mabuchi, etc.
- Useful for cavity QED, quantum optics, etc.

Hybrid Classical-Quantum Filtering



- Define hybrid density operator $\hat{\rho}(x_t)$, $P(x_t) = \mathrm{tr}[\hat{\rho}(x_t)]$, $\hat{\rho}_t = \int dx_t \hat{\rho}(x_t)$
- Use generalized quantum Bayes theorem and positive map + Chapman-Kolmogorov:

$$\hat{\rho}(x_t|y_t) = \frac{\hat{M}(y_t|x_t)\hat{\rho}(x_t)\hat{M}^\dagger(y_t|x_t)}{\int dx_t\, \mathrm{tr}(\text{numerator})}, \tag{9}$$

$$\hat{\rho}(x_{t+\delta t}|y_t) = \int dx_t P(x_{t+\delta t}|x_t) \sum_\mu \hat{K}_\mu(x_t)\hat{\rho}(x_t|y_t)\hat{K}_\mu^\dagger(x_t) \tag{10}$$

Hybrid Filtering Equations

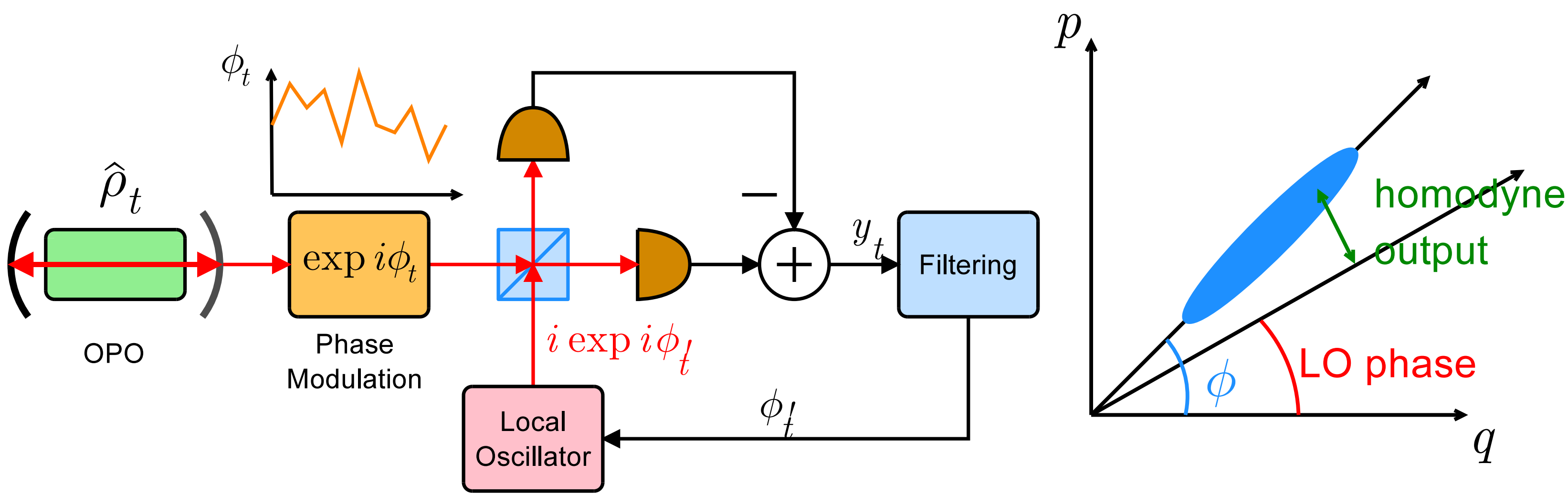
- Hybrid Belavkin equation (analogous to Kushner-Stratonovich):

$$
\begin{aligned}
d\hat{\rho}(x,t) &= dt\left[\mathcal{L}_0 + \mathcal{L}(x) - \frac{\partial}{\partial x_\mu}A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu}B_{\mu\nu}\right]\hat{\rho}(x,t) \\
&\quad + \frac{dt}{8}\left[2\hat{C}^T R^{-1}\hat{\rho}(x,t)\hat{C}^\dagger - \hat{C}^{\dagger T}R^{-1}\hat{C}\hat{\rho}(x,t) - \hat{\rho}(x,t)\hat{C}^{\dagger T}R^{-1}\hat{C}\right] \\
&\quad + \frac{1}{2}\left(dy_t - \frac{dt}{2}\left\langle \hat{C} + \hat{C}^\dagger\right\rangle\right)^T R^{-1}\left[\left(\hat{C} - \langle\hat{C}\rangle\right)\hat{\rho}(x,t) + \text{H.c.}\right] \qquad (11)
\end{aligned}
$$

- Linear version (analogous to Duncan-Mortensen-Zakai):

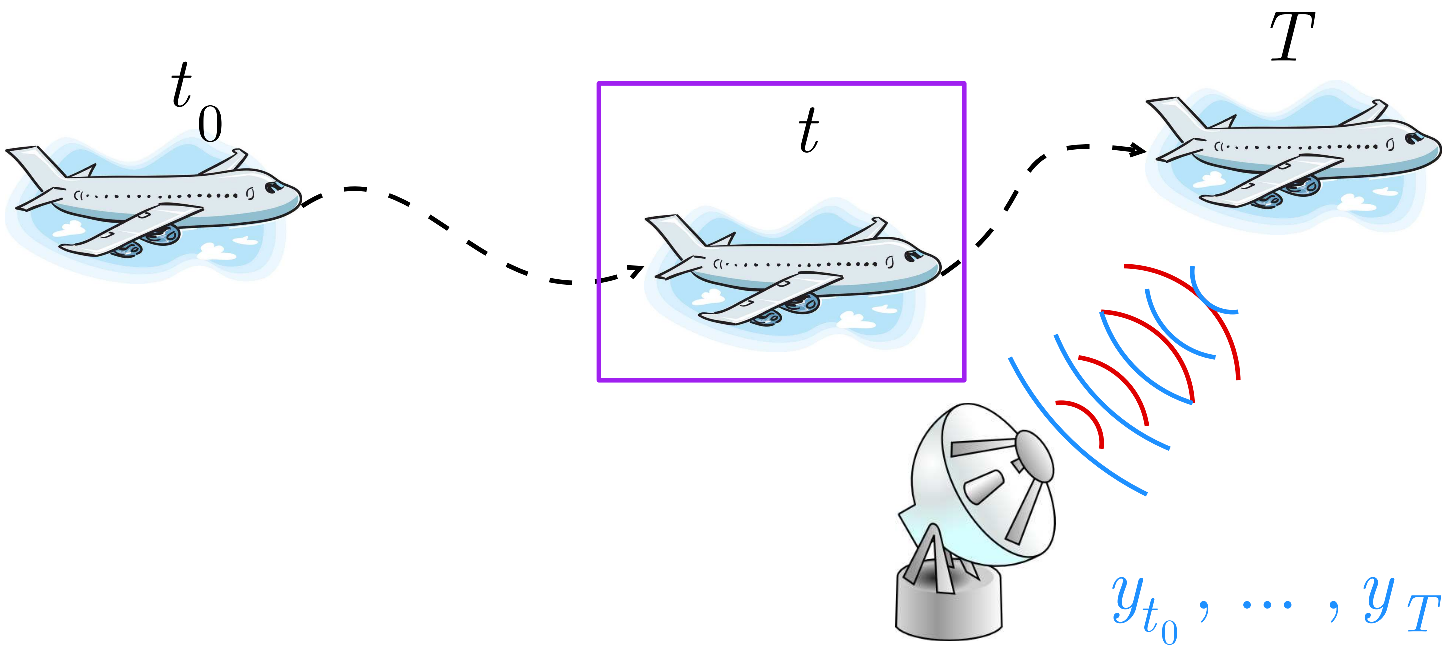
$$
\begin{aligned}
d\hat{f}(x,t) &= dt\left[\mathcal{L}_0 + \mathcal{L}(x) - \frac{\partial}{\partial x_\mu}A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu}B_{\mu\nu}\right]\hat{f}(x,t) \\
&\quad + \frac{dt}{8}\left[2\hat{C}^T R^{-1}\hat{f}(x,t)\hat{C}^\dagger - \hat{C}^{\dagger T}R^{-1}\hat{C}\hat{\rho}(x,t) - \hat{f}(x,t)\hat{C}^{\dagger T}R^{-1}\hat{C}\right] \\
&\quad + \frac{1}{2}dy_t^T R^{-1}\left[\hat{C}\hat{f}(x,t) + \text{H.c.}\right], \quad \hat{\rho}(x,t) = \frac{\hat{f}(x,t)}{\int dx\,\mathrm{tr}[\hat{f}(x,t)]}. \qquad (12)
\end{aligned}
$$

Phase-Locked Loop Design



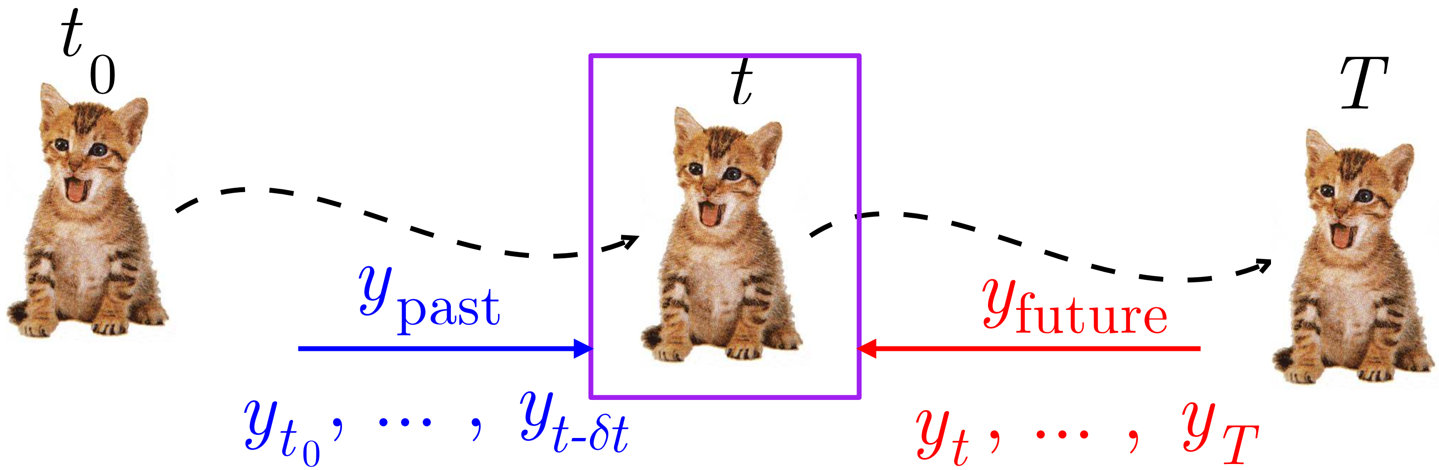
- Personick, IEEE Trans. Inform. Th. **IT-17**, 240 (1971); Wiseman, PRL **75**, 4587 (1995); Armen *et al.*, PRL **89**, 133602 (2002).
- Berry and Wiseman, PRA **65**, 043803 (2002); **73**, 063824 (2006).
- Tsang, Shapiro, and Lloyd, PRA **78**, 053820 (2008); **79**, 053843 (2009); Tsang, PRA **80**, 033840 (2009).
- Wheatley, *et al.*, arXiv:0912.1162.

Smoothing: Estimation with Delay



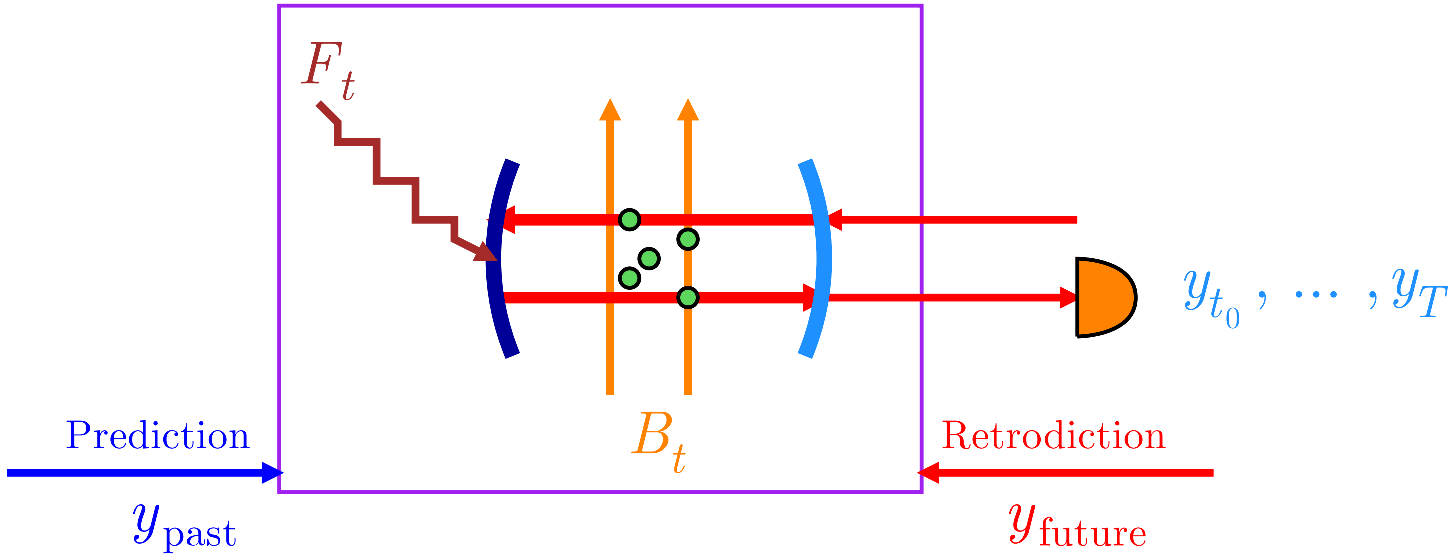
- More accurate than filtering
- Sensing, analog communication, astronomy, crime investigation, . . .
- Delay

Quantum Smoothing?



- Conventional quantum theory is a predictive theory
- Quantum state described by $|\Psi_t\rangle$ or $\hat{\rho}_t$ can only be conditioned only upon past observations
- "Weak values" by Aharonov, Vaidman, *et al.*
- "Quantum retrodiction" by Barnett, Pegg, Yanagisawa, etc.

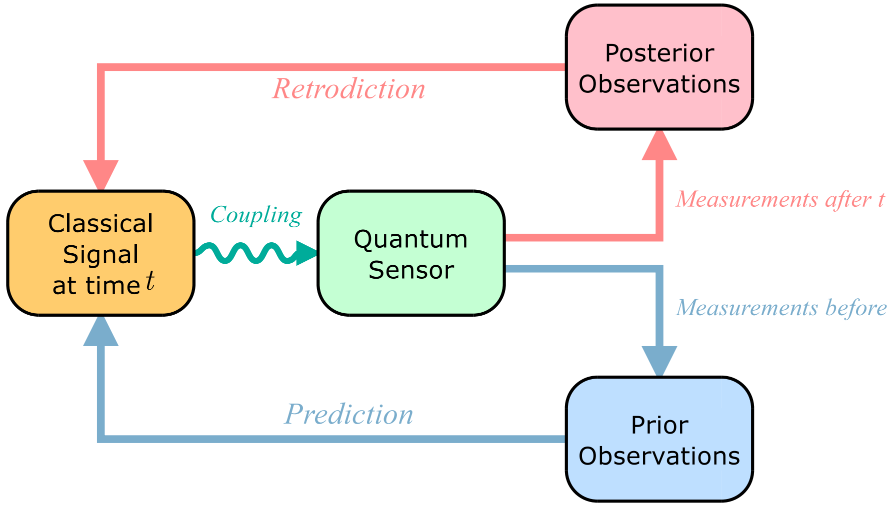
Hybrid Time-Symmetric Smoothing



- Use two operators to describe system: density operator $\hat{\rho}(x_t|y_{\text{past}})$ and a retrodictive likelihood operator $\hat{E}(y_{\text{future}}|x_t)$

$$P(x_t|y_{\text{past}}, y_{\text{future}}) = \frac{\mathrm{tr}\left[\hat{E}(y_{\text{future}}|x_t)\hat{\rho}(x_t|y_{\text{past}})\right]}{\int dx(\text{numerator})} \tag{13}$$

Equations



- Solve the predictive equation from t_0 to t using *a priori* $\hat{f}$ as initial condition, and solve the retrodictive equation from T to t using $\hat{g}(x, T) \propto \hat{1}$ as the final condition

$$d\hat{f} = dt\mathcal{L}(x)\hat{f} + \frac{dt}{8}\left(2\hat{C}^T R^{-1}\hat{f}\hat{C}^\dagger - \hat{C}^{\dagger T} R^{-1}\hat{C}\hat{f} - \hat{f}\hat{C}^{\dagger T} R^{-1}\hat{C}\right) + \frac{1}{2}dy_t^T R^{-1}\left(\hat{C}\hat{f} + \hat{f}\hat{C}^\dagger\right)$$

$$-d\hat{g} = dt\mathcal{L}^*(x)\hat{g} + \frac{dt}{8}\left(2\hat{C}^{\dagger T} R^{-1}\hat{g}\hat{C} - \hat{C}^{\dagger T} R^{-1}\hat{C}\hat{g} - \hat{g}\hat{C}^{\dagger T} R^{-1}\hat{C}\right) + \frac{1}{2}dy^T R^{-1}\left(\hat{C}^\dagger\hat{g} + \hat{g}\hat{C}\right)$$

$$h(x, t) = P(x_t = x|y_{\text{past}}, y_{\text{future}}) = \frac{\mathrm{tr}\left[\hat{g}(x, t)\hat{f}(x, t)\right]}{\int dx\,(\text{numerator})}$$

- Tsang, PRL **102**, 250403 (2009); PRA **80**, 033840 (2009).
- Classical: Pardoux, Stochastics **6**, 193 (1982).

Phase-Space Smoothing

- Convert to Wigner distributions:

$$\mathrm{tr}[\hat{g}(x,t)\hat{f}(x,t)] \propto \int dq dp \, g(q,p,x,t) f(q,p,x,t) \tag{14}$$

$$h(x,t) = \frac{\int dq dp \, g(q,p,x,t) f(q,p,x,t)}{\int dx \, (\text{numerator})} \tag{15}$$

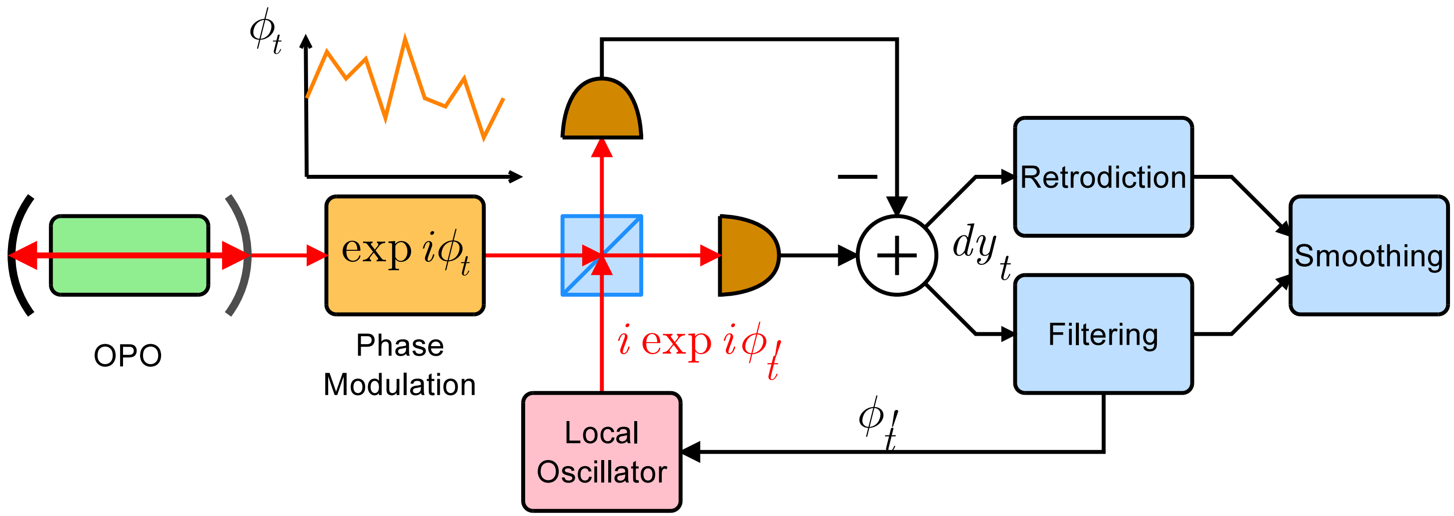
- If $f(q,p,x,t)$ and $g(q,p,x,t)$ are non-negative, equivalent to classical smoothing:

$$h(q,p,x,t) = \frac{g(q,p,x,t) f(q,p,x,t)}{\int dx dq dp \, (\text{numerator})}, \qquad h(x,t) = \int dq dp \, h(q,p,x,t) \tag{16}$$

  q and p can be regarded as classical, with $h(q,p,x,t)$ the classical smoothing probability distribution

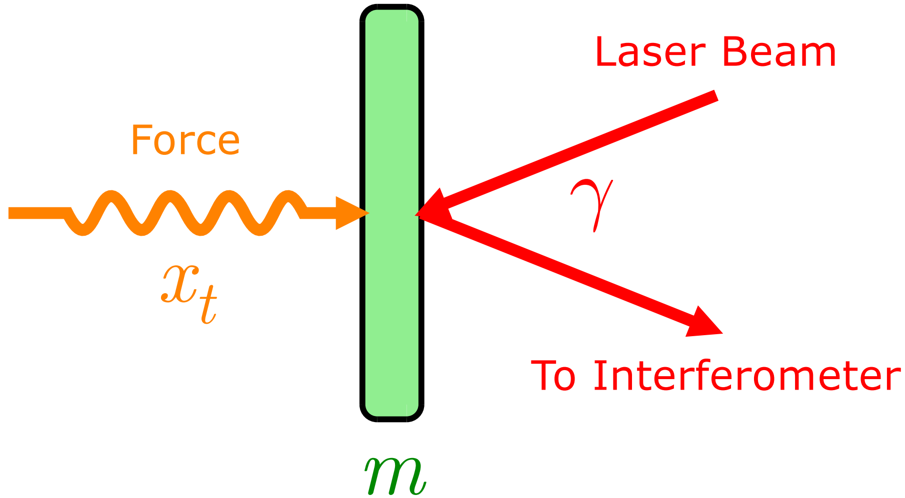
- If $f(q,p,x,t)$ and $g(q,p,x,t)$ are Gaussian, equivalent to linear smoothing, use two Kalman filters (Mayne-Fraser-Potter smoother)

Phase-Locked Loop: Post-Loop Smoother



- Tsang, Shapiro, and Lloyd, PRA **78**, 053820 (2008); **79**, 053843 (2009); Tsang, PRA **80**, 033840 (2009).
- Wheatley *et al.*, "Adaptive Optical Phase Estimation Using Time-Symmetric Quantum Smoothing," arXiv:0912.1162.

Opto-mechanical Force Sensor



- Assume force is an Ornstein-Uhlenbeck process:

$$dx_t = -ax_t dt + bdW_t \tag{17}$$

- Hamiltonian:

$$\hat{H}(x_t) = \frac{\hat{p}^2}{2m} - x_t \hat{q} \tag{18}$$

- Caves *et al.*, RMP **52**, 341 (1980); Braginsky and Khalili, *Quantum Measurements* (Cambridge University Press, Cambridge, 1992); Mabuchi, PRA **58**, 123 (1998); Verstraete *et al.*, PRA **64**, 032111 (2001).

Filtering

- Filtering equation:

$$d\hat{f}(x,t) = -\frac{i}{\hbar}dt\left[\hat{H}(x), \hat{f}(x,t)\right] + dt\left(a\frac{\partial}{\partial x} + \frac{b^2}{2}\frac{\partial^2}{\partial x^2}\right)\hat{f}(x,t)$$
$$+ \frac{\gamma}{8}dt\left[2\hat{q}\hat{f}(x,t)\hat{q} - \hat{q}^2\hat{f}(x,t) - \hat{f}(x,t)\hat{q}^2\right] + \frac{\gamma}{2}dy_t\left[\hat{q}\hat{f}(x,t) + \hat{f}(x,t)\hat{q}\right],$$

- In terms of $f(q,p,x,t)$:

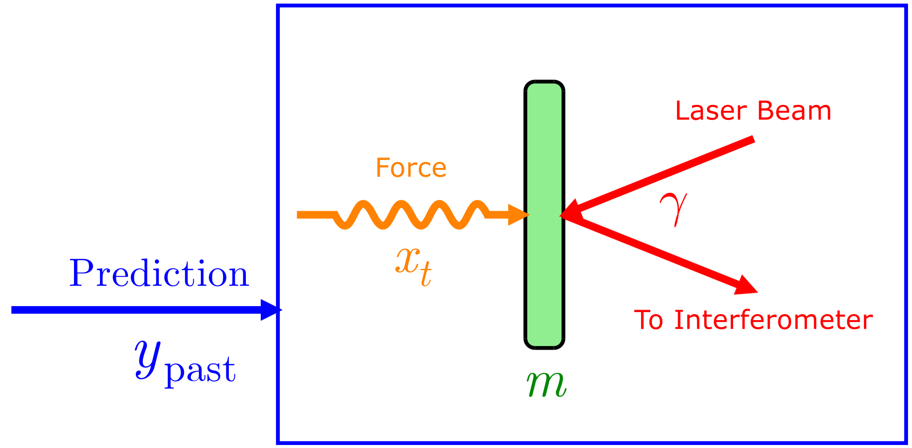
$$df = dt\left(-\frac{p}{m}\frac{\partial}{\partial q} - x\frac{\partial}{\partial p} + a\frac{\partial}{\partial x} + \frac{b^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\hbar^2\gamma}{8}\frac{\partial^2}{\partial p^2}\right)f + \gamma dy_t q f.$$

- Kalman filter:

$$d\mu = A\mu dt + \Sigma C^T\gamma d\eta, \qquad \frac{d\Sigma}{dt} = A\Sigma + \Sigma A^T - \Sigma C^T\gamma C\Sigma + Q,$$

$$A = \begin{pmatrix} 0 & 1/m & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -a \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar^2\gamma/4 & 0 \\ 0 & 0 & b^2 \end{pmatrix}.$$

Steady-State Filtering



- μ_q, μ_p, Σ_{qq}, Σ_{qp}, and Σ_{pp} determine conditional sensor quantum state $\hat{\rho}$, Σ_{xx} is mean-square force estimation error
- At steady state, let $d\Sigma/dt = 0$ (solve numerically)
- $x_t = 0$ limit (analytic):

$$\Sigma_{qq} = \sqrt{\frac{\hbar}{m\gamma}}, \qquad \Sigma_{qp} = \frac{\hbar}{2}, \qquad \Sigma_{pp} = \frac{\hbar\sqrt{\hbar m\gamma}}{2}, \qquad \Sigma_{qq}\Sigma_{pp} - \Sigma_{qp}^2 = \frac{1}{4} \tag{19}$$

Mensky, *Continuous Quantum Measurements and Path Integrals* (IOP, Bristol, 1993); Barchielli, Lanz, and Prosperi, Nuovo Cimento **72B**, 79 (1982); Caves and Milburn, PRA **36**, 5543 (1987); Belavkin and Staszewski, PLA **140**, 359 (1989); PRA **45**, 1347 (1992).

Smoothing

- Write retrodictive equation for $\hat{g}(x, t)$
- Convert $\hat{g}(x, t)$ to $g(q, p, x, t)$
- solve for mean vector ν and covariance matrix Ξ of $g(q, p, x, t)$ using a retrodictive Kalman filter
- Define

$$h(q, p, x, t) = \frac{g(q, p, x, t) f(q, p, x, t)}{\int dqdpdx\; g(q, p, x, t) f(q, p, x, t)} \tag{20}$$

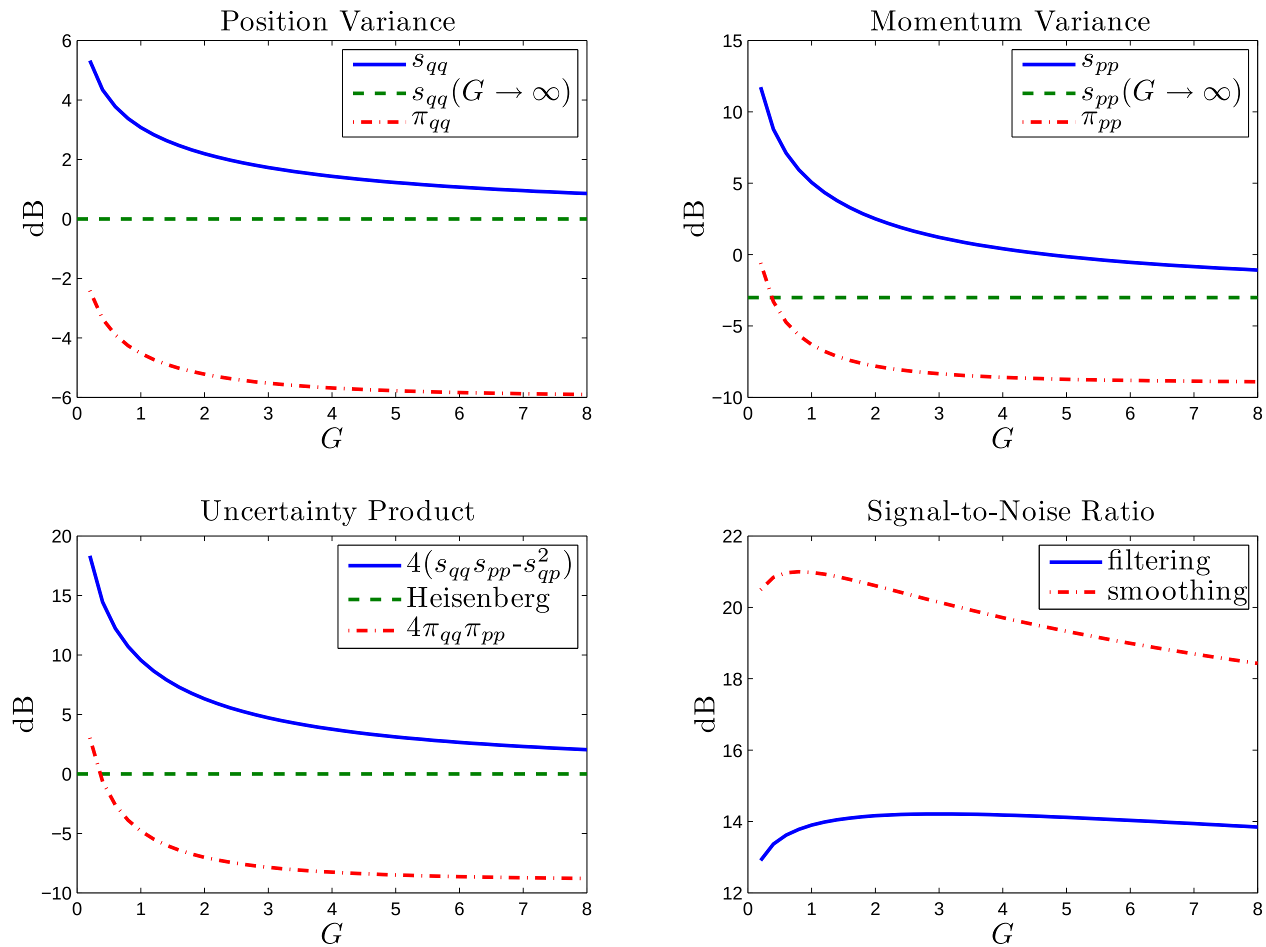
- Mean ξ and covariance Π of $h(q, p, x, t)$:

$$\xi = \begin{pmatrix} \xi_q \\ \xi_p \\ \xi_x \end{pmatrix} = \Pi \left(\Sigma^{-1}\mu + \Xi^{-1}\nu\right) \quad \Pi = \begin{pmatrix} \Pi_{qq} & \Pi_{qp} & \Pi_{qx} \\ \Pi_{pq} & \Pi_{pp} & \Pi_{px} \\ \Pi_{xq} & \Pi_{xp} & \Pi_{xx} \end{pmatrix} = \left(\Sigma^{-1} + \Xi^{-1}\right)^{-1}$$

- ξ_x is smoothing estimate of force, Π_{xx} is smoothing error
- but what are ξ_q, ξ_p, Π_{qq}, Π_{qp}, Π_{pp}, and $h(q, p, x, t)$ in general?

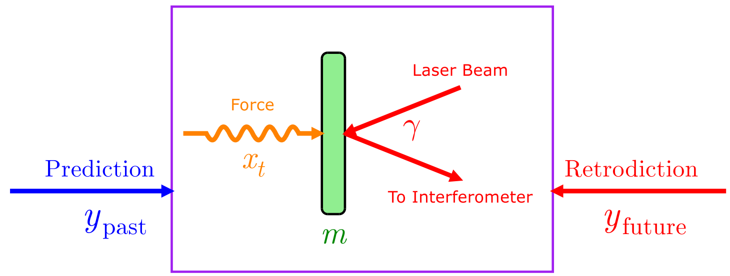
Filtering vs Smoothing at Steady State

- $a/\sqrt{b} = 0.01/(\hbar m)^{1/4}$, $G = \hbar^{3/2}\gamma/m^{1/2}b$, $s_{qq} = \sqrt{m\gamma/\hbar}\Sigma_{qq}$, $s_{qp} = \Sigma_{qp}/\hbar$, $s_{pp} = \Sigma_{pp}/\sqrt{\hbar^3 m\gamma}$, blue: filtering, green: $x_t = 0$, red: smoothing



- Signal-to-Noise Ratio can be further improved by frequency-dependent squeezing and coherent quantum filtering [Kimble *et al.*, PRD **65**, 022002 (2001)].

Quantum Smoothing



- Can we use

$$h(q,p,x,t) = \frac{g(q,p,x,t)f(q,p,x,t)}{\int dqdpdx\ (\text{numerator})} \tag{21}$$

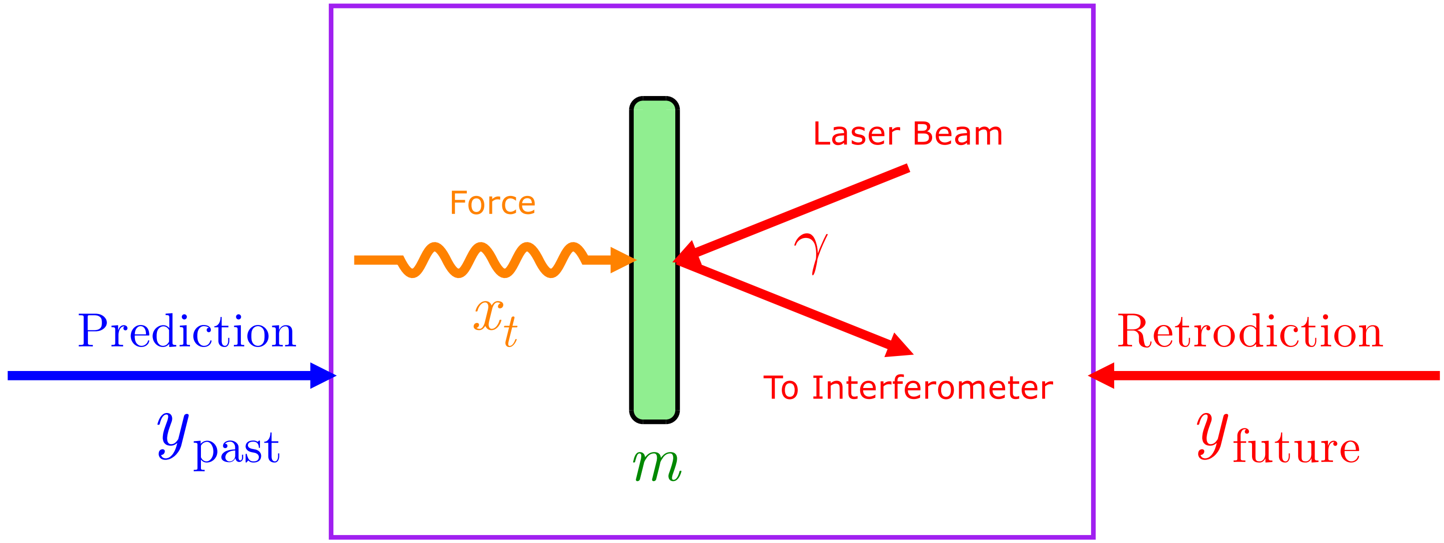
to estimate q and p?

  - when g and f are non-negative, problem becomes classical
  -

$$h(x,t) = \int dqdph(q,p,x,t) \tag{22}$$

  - $\xi_q, \xi_p \equiv$ real part of weak values
  - $h(q,p,x,t)$ arises from statistics of weak measurements
- $h(q,p,x,t)$ can go negative, many versions of Wigner distributions for discrete degrees of freedom

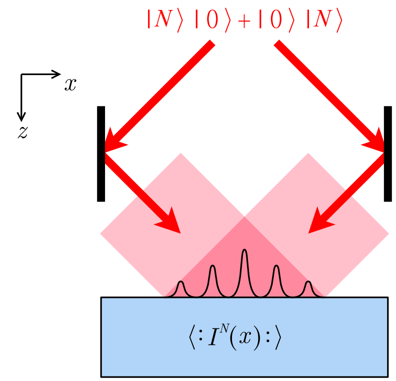
Beyond Heisenberg?

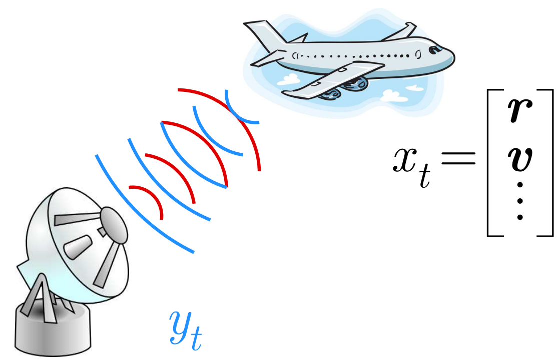


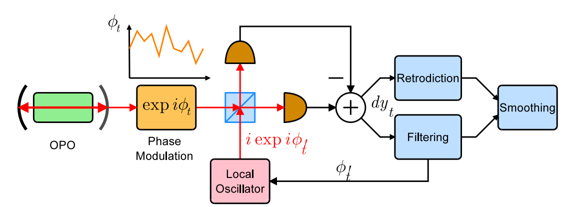
- Bayesian view: shouldn't we be able to learn more about a system, whether classical or quantum, in retrospect?
- Everett view: Need two wavefunctions or two density operators to solve problems.

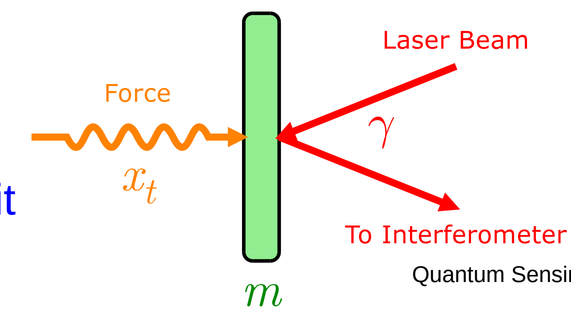
Summary

- Quantum Imaging
- Bayesian Quantum Estimation
- Hybrid Classical-Quantum Filtering
  - Phase-Locked Loop
- Time-Symmetric Quantum Smoothing
  - Force Sensing
  - Smoothing for Quantum Systems?
- Other applications:
  - Atomic Magnetometry
  - Hardy's Paradox
  - Tsang, PRA **81**, 013824 (2010).
  - Epistemology: Einstein vs Bohr
  - Novel way of doing quantum mechanics? Aharonov, Vaidman *et al.*; Wheeler and Feynman; Gell-Mann and Hartle
- `http://sites.google.com/site/mankeitsang/` or Google "Mankei Tsang"
- Life can only be understood backwards; but it must be lived forwards. – Soren Kierkegaard

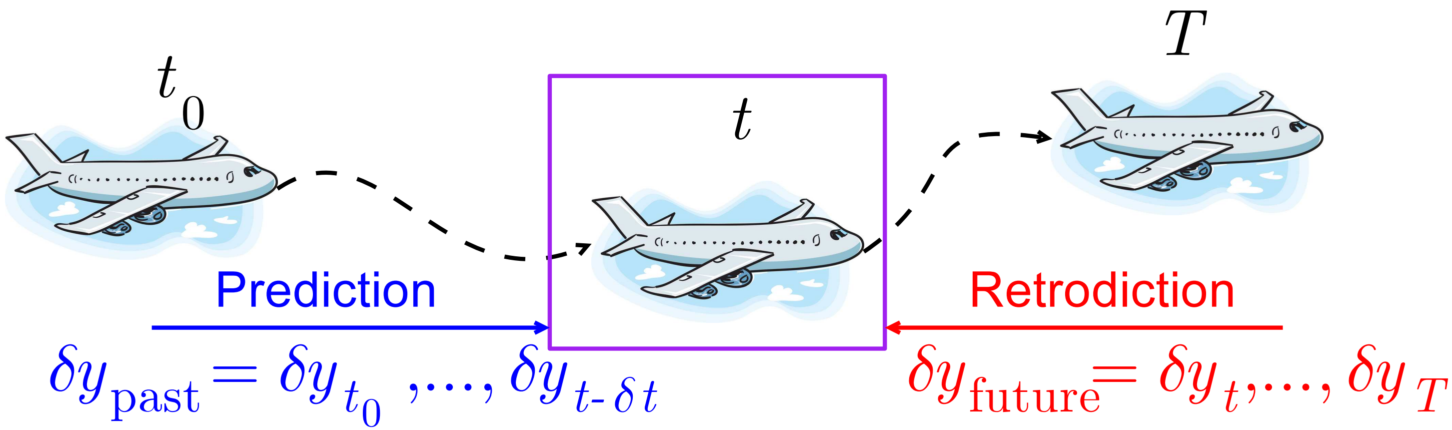








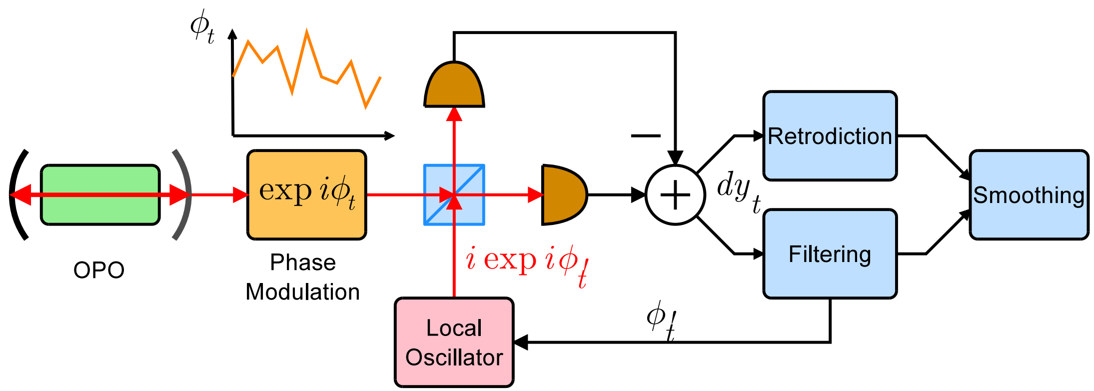
Time-Symmetric Smoothing



$$P(x_t|y_{t_0},\ldots,y_T) = \frac{P(y_{\text{future}}|x_t)P(x_t|y_{\text{past}})}{\int dx_t(\text{numerator})} \tag{23}$$

- Mayne, Fraser, Potter, Pardoux
- Unnormalized versions of $P(x_t|\delta y_{\text{past}})$ and $P(\delta y_{\text{future}}|x_t)$ (retrodictive likelihood function) obey a pair of adjoint equations, one to be solved forward in time and one backward in time

Phase-Locked Loop: Post-Loop Smoother



$$df = dt\Big\{ -\sum_{\mu} \frac{\partial}{\partial x_\mu}(A_\mu f) + \frac{1}{2}\sum_{\mu,\nu}\frac{\partial^2}{\partial x_\mu \partial x_\nu}\Big[\Big(BQB^T\Big)_{\mu\nu} f\Big]$$
$$-\Big[\Big(\chi - \frac{\gamma}{2}\Big)\frac{\partial}{\partial q}(qf) + \Big(-\chi - \frac{\gamma}{2}\Big)\frac{\partial}{\partial p}(pf)\Big] + \frac{\gamma}{4}\Big(\frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial p^2}\Big)\Big\}$$
$$+dy_t\Big[\sin(\phi - \phi'_t)\Big(2b + \sqrt{2\gamma}q + \sqrt{\frac{\gamma}{2}}\frac{\partial}{\partial q}\Big) + \cos(\phi - \phi'_t)\Big(\sqrt{2\gamma}p + \sqrt{\frac{\gamma}{2}}\frac{\partial}{\partial p}\Big)\Big]f. \tag{24}$$

$$-dg = dt\Big\{\sum_{\mu} A_\mu \frac{\partial g}{\partial x_\mu} + \frac{1}{2}\sum_{\mu,\nu}\Big(BQB^T\Big)_{\mu\nu}\frac{\partial^2 g}{\partial x_\mu \partial x_\nu}$$
$$+\Big[\Big(\chi - \frac{\gamma}{2}\Big)q\frac{\partial g}{\partial q} + \Big(-\chi - \frac{\gamma}{2}\Big)p\frac{\partial g}{\partial p}\Big] + \frac{\gamma}{4}\Big(\frac{\partial^2 g}{\partial q^2} + \frac{\partial^2 g}{\partial p^2}\Big)\Big\}$$
$$+dy_t\Big[\sin(\phi - \phi'_t)\Big(2b + \sqrt{2\gamma}q - \sqrt{\frac{\gamma}{2}}\frac{\partial}{\partial q}\Big) + \cos(\phi - \phi'_t)\Big(\sqrt{2\gamma}p - \sqrt{\frac{\gamma}{2}}\frac{\partial}{\partial p}\Big)\Big]g. \tag{25}$$

Weak Measurements

- The smoothing estimates ξ_q and ξ_p are equivalent to "weak values":

$$\xi_q = \text{Re}\frac{\int dx\, \mathrm{tr}(\hat{g}\hat{q}\hat{f})}{\int dx\, \mathrm{tr}(\hat{g}\hat{f})}, \qquad \xi_p = \text{Re}\frac{\int dx\, \mathrm{tr}(\hat{g}\hat{p}\hat{f})}{\int dx\, \mathrm{tr}(\hat{g}\hat{f})}. \tag{26}$$

- Aharonov, Albert, and Vaidman, PRL 60, 1351 (1988); Wiseman, PRA 65, 032111 (2002).
- Make weak Gaussian and backaction evading measurements of q and p:

$$\hat{M}(y_q) \propto \int dq \exp\left[-\frac{\epsilon\,(y_q - q)^2}{4}\right] |q\rangle\langle q|, \text{ same for } y_p, \tag{27}$$

$$P(y_q, y_p|y_{\text{past}}, y_{\text{future}}) \propto \int dqdp \exp\left\{-\frac{\epsilon}{2}\left[(y_q - q)^2 + (y_p - p)^2\right]\right\} h(q,p,t) \tag{28}$$

in the limit of $\epsilon \to 0$ (opposite limit to von Neumann measurements)

- $h(q,p,t)$ describes the apparent statistics of q and p in the weak measurement limit