Quantum Sensing and Imaging

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Outline

Super-resolution Quantum Imaging
- Tsang, PRA 75, 043813 (2007).
- Tsang, PRL 102, 253601 (2009).

Quantum Smoothing Theory for Optimal Quantum Sensing
- Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008).
- Tsang, Shapiro, and Lloyd, PRA 79, 053843 (2009).
- Tsang, PRL 102, 250403 (2009).
- Tsang, PRA 80, 033840 (2009).
- Tsang, PRA 81, 013824 (2010).

For path-entangled photons, fringe period $\sim \lambda/N$
Multiphoton-Absorption Rate of NOON State

\[ |\Psi\rangle \propto |N\rangle_k |0\rangle_{-k} + |0\rangle_k |N\rangle_{-k} \]

- Positively correlated in momentum, **anti-correlated in space**
- Photons don’t arrive at the same absorber as likely as uncorrelated photons.
Configuration-Space Picture

with paraxial approximation:

\[
\phi(k_1, \ldots, k_N) = \frac{1}{\sqrt{N!}} \langle 0|\hat{a}(k_1) \cdots \hat{a}(k_N)|\Psi\rangle, \quad [\hat{a}(k), \hat{a}^\dagger(k')] = \delta(k - k'),
\]

(1)

\[
\psi(x_1, \ldots, x_N) \equiv \int \frac{d^N k}{(2\pi)^{N/2}} \phi(k_1, \ldots, k_N) \exp(ik_1 x_1 + \cdots + ik_N x_N),
\]

(2)

\[
\left\langle :\hat{I}^N(x):\right\rangle = \eta^N N!|\psi(x, x, \ldots, x)|^2
\]

(3)

Peak multi-photon absorption rate of NOON state is a factor of \(2^{N-1}\) lower than that of \(N\) independent photons [Tsang, PRA 75, 043813 (2007)].

Coherent fields achieve the highest peak multi-photon absorption rate [Tsang, PRL 101, 033602 (2008)].
Tsang, PRL 102, 253601 (2009); Anisimov and Dowling, Physics 2, 52 (2009).


Figure 2: STORM imaging of cells with photo-switchable dyes and fluorescent proteins. a) STORM image of microtubules in a B cell. The microtubules are immunostained with antibodies that are labeled with photo-switchable Alexa Fluor 488, a conventional (N) and STORM (m) images correspond to the boxed region in a. b) Conventional (4) and STORM (m) images of microtubules in a B cell. The microtubules are labeled with a photo-switchable fluorescent protein, mcherry. A moderate aggregation of the microtubules is observed, potentially caused by the fluorescent protein tags.

Atomic Force Microscopy

Squeezing of transverse laser beam position:

Centroid measurements allow detection of non-Gaussian signatures of light and the cantilever.
Tracking an Aircraft

or submarine, terrorist, criminal, mosquito, cancer cell, nanoparticle, . . .

Use Bayes theorem:

\[
P(x_t|y_t) = \frac{P(y_t|x_t)P(x_t)}{\int dx_t \text{(numerator)}},
\]

\(P(y_t|x_t)\) from observation noise, \(P(x_t)\) from a priori information.
Assume $x_t$ is a Markov process, use Chapman-Kolmogorov equation:

$$P(x_{t+\delta t}|y_t) = \int dx_t P(x_{t+\delta t}|x_t) P(x_t|y_t)$$  \hspace{1cm} (5)
Filtering: Real-Time Estimation

Applying Bayes theorem and Chapman-Kolmogorov equation repeatedly, we can obtain

$$P(x_t | y_{t-\delta t}, \cdots, y_{t+\delta t}, y_{t_0})$$

(6)

Useful for control, weather and finance forecast, etc.

Wiener, Stratonovich, Kalman, Kushner, etc.
Quantum Filtering

\[ \hat{\rho}_t (|y_t) = \frac{\hat{M}(y_t)\hat{\rho}_t \hat{M}^+(y_t)}{\text{tr}(\text{numerator})} \]  \hspace{1cm} (7)

Use "quantum Bayes theorem,"

Use a completely positive map to evolve the system state (analogous to Chapman-Kolmogorov),

\[ \hat{\rho}_{t+\delta t} (|y_t) = \sum_{\mu} \hat{K}_\mu \hat{\rho}_t (|y_t) \hat{K}^\dagger_\mu \]  \hspace{1cm} (8)

Belavkin, Barchielli, Carmichael, Caves, Milburn, Wiseman, Mabuchi, etc.

Useful for cavity QED, quantum optics, etc.
Define hybrid density operator $\hat{\rho}(x_t)$, $P(x_t) = \text{tr}[\hat{\rho}(x_t)]$, $\hat{\rho}_t = \int dx_t \hat{\rho}(x_t)$

Use generalized quantum Bayes theorem and positive map + Chapman-Kolmogorov:

$$\hat{\rho}(x_t|y_t) = \frac{\hat{M}(y_t|x_t)\hat{\rho}(x_t)\hat{M}^\dagger(y_t|x_t)}{\int dx_t \text{tr}(\text{numerator})},$$

$$\hat{\rho}(x_{t+\delta t}|y_t) = \int dx_t P(x_{t+\delta t}|x_t) \sum_\mu \hat{K}_\mu(x_t)\hat{\rho}(x_t|y_t)\hat{K}^\dagger_\mu(x_t)$$
Hybrid Filtering Equations

Hybrid Belavkin equation (analogous to Kushner-Stratonovich):

\[
d\hat{\rho}(x, t) = dt \left[ \mathcal{L}_0 + \mathcal{L}(x) - \frac{\partial}{\partial x_\mu} A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu} B_{\mu\nu} \right] \hat{\rho}(x, t) \\
+ \frac{dt}{8} \left[ 2\hat{C}^T R^{-1} \hat{\rho}(x, t) \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{\rho}(x, t) - \hat{\rho}(x, t) \hat{C}^\dagger T R^{-1} \hat{C} \right] \\
+ \frac{1}{2} \left( dy_t - \frac{dt}{2} \langle \hat{C} + \hat{C}^\dagger \rangle \right)^T R^{-1} \left[ (\hat{C} - \langle \hat{C} \rangle) \hat{\rho}(x, t) + \text{H.c.} \right]
\] (11)

Linear version (analogous to Duncan-Mortensen-Zakai):

\[
d\hat{f}(x, t) = dt \left[ \mathcal{L}_0 + \mathcal{L}(x) - \frac{\partial}{\partial x_\mu} A_\mu + \frac{\partial}{\partial x_\mu \partial x_\nu} B_{\mu\nu} \right] \hat{f}(x, t) \\
+ \frac{dt}{8} \left[ 2\hat{C}^T R^{-1} \hat{f}(x, t) \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{C} \hat{\rho}(x, t) - \hat{f}(x, t) \hat{C}^\dagger T R^{-1} \hat{C} \right] \\
+ \frac{1}{2} dy_t T R^{-1} \left[ \hat{C} \hat{f}(x, t) + \text{H.c.} \right], \quad \hat{\rho}(x, t) = \frac{\hat{f}(x, t)}{\int dx \, \text{tr} [\hat{f}(x, t)]}.
\] (12)
Phase-Locked Loop Design


Berry and Wiseman, PRA 65, 043803 (2002); 73, 063824 (2006).

Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008); 79, 053843 (2009); Tsang, PRA 80, 033840 (2009).

Smoothing: Estimation with Delay

More accurate than filtering

Sensing, analog communication, astronomy, crime investigation, . . .

Delay
Quantum Smoothing?

Conventional quantum theory is a predictive theory

Quantum state described by $|\Psi_t\rangle$ or $\hat{\rho}_t$ can only be conditioned only upon past observations

“Weak values” by Aharonov, Vaidman, et al.

“Quantum retrodiction” by Barnett, Pegg, Yanagisawa, etc.
Use two operators to describe system: density operator $\hat{\rho}(x_t|y_{\text{past}})$ and a retrodictive likelihood operator $\hat{E}(y_{\text{future}}|x_t)$

$$P(x_t|y_{\text{past}}, y_{\text{future}}) = \frac{\text{tr} \left[ \hat{E}(y_{\text{future}}|x_t) \hat{\rho}(x_t|y_{\text{past}}) \right]}{\int dx \text{(numerator)}}$$  \hspace{1cm} (13)
Equations

Solve the predictive equation from $t_0$ to $t$ using a priori $\hat{f}$ as initial condition, and solve the retrodictive equation from $T$ to $t$ using $\hat{g}(x, T) \propto \hat{1}$ as the final condition

\[
d\hat{f} = dt \mathcal{L}(x)\hat{f} + \frac{dt}{8} \left( 2\hat{C}^T R^{-1} \hat{f} \hat{C}^\dagger - \hat{C}^\dagger T R^{-1} \hat{f} \hat{C} - \hat{f} \hat{C}^\dagger T R^{-1} \hat{C} \right) + \frac{1}{2} dy_t T R^{-1} \left( \hat{C} \hat{f} + \hat{f} \hat{C}^\dagger \right)
\]

\[-d\hat{g} = dt \mathcal{L}^*(x)\hat{g} + \frac{dt}{8} \left( 2\hat{C}^\dagger T R^{-1} \hat{g} \hat{C} - \hat{C}^\dagger T R^{-1} \hat{g} \hat{C} - \hat{g} \hat{C}^\dagger T R^{-1} \hat{C} \right) + \frac{1}{2} dy T R^{-1} \left( \hat{C}^\dagger \hat{g} + \hat{g} \hat{C} \right)\]

\[h(x, t) = P(x_t = x | y_{\text{past}}, y_{\text{future}}) = \frac{\text{tr} \left[ \hat{g}(x, t) \hat{f}(x, t) \right]}{\int dx \ (\text{numerator})}\]

Tsang, PRL 102, 250403 (2009); PRA 80, 033840 (2009).

Phase-Space Smoothing

Convert to Wigner distributions:

\[
\text{tr}[\hat{g}(x, t)\hat{f}(x, t)] \propto \int dq dp \ g(q, p, x, t) f(q, p, x, t)
\]

\[
h(x, t) = \frac{\int dq dp \ g(q, p, x, t) f(q, p, x, t)}{\int dx \text{ (numerator)}}
\]

If \( f(q, p, x, t) \) and \( g(q, p, x, t) \) are non-negative, equivalent to classical smoothing:

\[
h(q, p, x, t) = \frac{g(q, p, x, t) f(q, p, x, t)}{\int dx dq dp \text{ (numerator)}}, \quad h(x, t) = \int dq dp \ h(q, p, x, t)
\]

\( q \) and \( p \) can be regarded as classical, with \( h(q, p, x, t) \) the classical smoothing probability distribution

If \( f(q, p, x, t) \) and \( g(q, p, x, t) \) are Gaussian, equivalent to linear smoothing, use two Kalman filters (Mayne-Fraser-Potter smoother)
Phase-Locked Loop: Post-Loop Smoother

Tsang, Shapiro, and Lloyd, PRA 78, 053820 (2008); 79, 053843 (2009); Tsang, PRA 80, 033840 (2009).

Assume force is an Ornstein-Uhlenbeck process:

\[ dx_t = -ax_t dt + bdW_t \]  \hspace{1cm} (17)

Hamiltonian:

\[ \hat{H}(x_t) = \frac{\hat{p}^2}{2m} - x_t \hat{q} \]  \hspace{1cm} (18)

Caves et al., RMP 52, 341 (1980); Braginsky and Khalili, Quantum Measurements (Cambridge University Press, Cambridge, 1992); Mabuchi, PRA 58, 123 (1998); Verstraete et al., PRA 64, 032111 (2001).
Filtering equation:

\[ d\hat{f}(x, t) = -\frac{i}{\hbar} dt \left[ \hat{H}(x), \hat{f}(x, t) \right] + dt \left( a \frac{\partial}{\partial x} + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} \right) \hat{f}(x, t) \]

\[ + \frac{\gamma}{8} dt \left[ 2\hat{q}\hat{f}(x, t)\hat{q} - \hat{q}^2 \hat{f}(x, t) - \hat{f}(x, t)\hat{q}^2 \right] + \frac{\gamma}{2} dy_t \left[ \hat{q}\hat{f}(x, t) + \hat{f}(x, t)\hat{q} \right], \]

In terms of \( f(q, p, x, t) \):

\[ df = dt \left( -\frac{p}{m} \frac{\partial}{\partial q} - x \frac{\partial}{\partial p} + a \frac{\partial}{\partial x} + \frac{b^2}{2} \frac{\partial^2}{\partial x^2} + \frac{\hbar^2 \gamma}{8} \frac{\partial^2}{\partial p^2} \right) f + \gamma dy_t q f. \]

Kalman filter:

\[ d\mu = A\mu dt + \Sigma C^T \gamma d\eta, \quad \frac{d\Sigma}{dt} = A\Sigma + \Sigma A^T - \Sigma C^T \gamma C \Sigma + Q, \]

\[ A = \begin{pmatrix} 0 & 1/m & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -a \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hbar^2 \gamma/4 & 0 \\ 0 & 0 & b^2 \end{pmatrix}. \]
Steady-State Filtering

\[ \mu_q, \mu_p, \Sigma_{qq}, \Sigma_{qp}, \text{and } \Sigma_{pp} \text{ determine conditional sensor quantum state } \hat{\rho}, \Sigma_{xx} \text{ is mean-square force estimation error} \]

At steady state, let \( d\Sigma/dt = 0 \) (solve numerically)

\[ x_t = 0 \text{ limit (analytic):} \]

\[ \Sigma_{qq} = \sqrt{\frac{\hbar}{m\gamma}}, \quad \Sigma_{qp} = \frac{\hbar}{2}, \quad \Sigma_{pp} = \frac{\hbar\sqrt{\hbar m \gamma}}{2}, \quad \Sigma_{qq}\Sigma_{pp} - \Sigma_{qp}^2 = \frac{1}{4} \quad (19) \]

Smoothing

- Write retrodictive equation for $\hat{g}(x, t)$
- Convert $\hat{g}(x, t)$ to $g(q, p, x, t)$
- Solve for mean vector $\nu$ and covariance matrix $\Xi$ of $g(q, p, x, t)$ using a retrodictive Kalman filter
- Define

$$h(q, p, x, t) = \frac{g(q, p, x, t) f(q, p, x, t)}{\int dq dp dx g(q, p, x, t) f(q, p, x, t)} \quad (20)$$

- Mean $\xi$ and covariance $\Pi$ of $h(q, p, x, t)$:

$$\xi = \begin{pmatrix} \xi_q \\
\xi_p \\
\xi_x \end{pmatrix} = \Pi \left( \Sigma^{-1} \mu + \Xi^{-1} \nu \right) \quad \Pi = \begin{pmatrix} \Pi_{qq} & \Pi_{qp} & \Pi_{qx} \\
\Pi_{pq} & \Pi_{pp} & \Pi_{px} \\
\Pi_{xq} & \Pi_{xp} & \Pi_{xx} \end{pmatrix} = \left( \Sigma^{-1} + \Xi^{-1} \right)^{-1}$$

- $\xi_x$ is smoothing estimate of force, $\Pi_{xx}$ is smoothing error
- But what are $\xi_q, \xi_p, \Pi_{qq}, \Pi_{qp}, \Pi_{pp}$, and $h(q, p, x, t)$ in general?
Filtering vs Smoothing at Steady State

\[ a/\sqrt{b} = 0.01/(\hbar m)^{1/4}, \ G = \hbar^{3/2}\gamma/m^{1/2}b, \ s_{qq} = \sqrt{m\gamma/\hbar}\Sigma_{qq}, \ s_{qp} = \Sigma_{qp}/\hbar, \ s_{pp} = \Sigma_{pp}/\sqrt{\hbar^{3}m\gamma}, \] blue: filtering, green: \( x_t = 0 \), red: smoothing

Signal-to-Noise Ratio can be further improved by frequency-dependent squeezing and coherent quantum filtering [Kimble et al., PRD 65, 022002 (2001)].
Quantum Smoothing

Can we use

\[ h(q, p, x, t) = \frac{g(q, p, x, t) f(q, p, x, t)}{\int dqdpdx} \]  

(21)

to estimate \( q \) and \( p \)?

- when \( g \) and \( f \) are non-negative, problem becomes classical

\[ h(x, t) = \int dqdph(q, p, x, t) \]  

(22)

- \( \xi_q, \xi_p \equiv \) real part of weak values
- \( h(q, p, x, t) \) arises from statistics of weak measurements
- \( h(q, p, x, t) \) can go negative, many versions of Wigner distributions for discrete degrees of freedom
Beyond Heisenberg?

Bayesian view: shouldn't we be able to learn more about a system, whether classical or quantum, in retrospect?

Everett view: Need two wavefunctions or two density operators to solve problems.
Summary

- Quantum Imaging
- Bayesian Quantum Estimation
- Hybrid Classical-Quantum Filtering
  - Phase-Locked Loop
- Time-Symmetric Quantum Smoothing
  - Force Sensing
  - Smoothing for Quantum Systems?
- Other applications:
  - Atomic Magnetometry
  - Hardy’s Paradox
  - Tsang, PRA 81, 013824 (2010).
  - Epistemology: Einstein vs Bohr
  - Novel way of doing quantum mechanics? Aharonov, Vaidman et al.; Wheeler and Feynman; Gell-Mann and Hartle

http://sites.google.com/site/mankeitsang/ or Google “Mankei Tsang”

Life can only be understood backwards; but it must be lived forwards. – Soren Kierkegaard
Time-Symmetric Smoothing

\[ P(x_t | y_{t_0}, \ldots, y_{T}) = \frac{P(y_{\text{future}} | x_t) P(x_t | y_{\text{past}})}{\int dx_t \text{ (numerator)}} \]  

(23)

Mayne, Fraser, Potter, Pardoux

Unnormalized versions of \( P(x_t | \delta y_{\text{past}}) \) and \( P(\delta y_{\text{future}} | x_t) \) (retrodictive likelihood function) obey a pair of adjoint equations, one to be solved forward in time and one backward in time.
Phase-Locked Loop: Post-Loop Smoother

\[ df = dt \left\{ -\sum_{\mu} \frac{\partial}{\partial x_\mu} (A_\mu f) + \frac{1}{2} \sum_{\mu,\nu} \frac{\partial^2}{\partial x_\mu \partial x_\nu} \left[ (BQB^T)_{\mu\nu} f \right] \right. \]

\[ - \left[ \left( \chi - \frac{\gamma}{2} \right) \frac{\partial}{\partial q} (qf) + \left( -\chi - \frac{\gamma}{2} \right) \frac{\partial}{\partial p} (pf) \right] + \frac{\gamma}{4} \left( \frac{\partial^2 f}{\partial q^2} + \frac{\partial^2 f}{\partial p^2} \right) \}

\[ + dy_t \left[ \sin(\phi - \phi'_t) \left( 2b + \sqrt{2\gamma} q + \sqrt{2\gamma} \frac{\partial}{\partial q} \right) + \cos(\phi - \phi'_t) \left( \sqrt{2\gamma} p + \sqrt{2\gamma} \frac{\partial}{\partial p} \right) \right] f. \]  

(24)

\[ - dg = dt \left\{ \sum_{\mu} A_\mu \frac{\partial g}{\partial x_\mu} + \frac{1}{2} \sum_{\mu,\nu} (BQB^T)_{\mu\nu} \frac{\partial^2 g}{\partial x_\mu \partial x_\nu} \right. \]

\[ + \left[ \left( \chi - \frac{\gamma}{2} \right) q \frac{\partial g}{\partial q} + \left( -\chi - \frac{\gamma}{2} \right) p \frac{\partial g}{\partial p} \right] + \frac{\gamma}{4} \left( \frac{\partial^2 g}{\partial q^2} + \frac{\partial^2 g}{\partial p^2} \right) \}

\[ + dy_t \left[ \sin(\phi - \phi'_t) \left( 2b + \sqrt{2\gamma} q - \sqrt{2\gamma} \frac{\partial}{\partial q} \right) + \cos(\phi - \phi'_t) \left( \sqrt{2\gamma} p - \sqrt{2\gamma} \frac{\partial}{\partial p} \right) \right] g. \]  

(25)
Weak Measurements

The smoothing estimates $\xi_q$ and $\xi_p$ are equivalent to “weak values”:

$$\xi_q = \text{Re} \frac{\int dx \text{tr}(\hat{g}\hat{q}\hat{f})}{\int dx \text{tr}(\hat{g}\hat{f})}, \quad \xi_p = \text{Re} \frac{\int dx \text{tr}(\hat{g}\hat{p}\hat{f})}{\int dx \text{tr}(\hat{g}\hat{f})}. \quad (26)$$


Make weak Gaussian and backaction evading measurements of $q$ and $p$:

$$\hat{M}(y_q) \propto \int dq \exp \left[ -\frac{\epsilon (y_q - q)^2}{4} \right] |q\rangle \langle q|, \text{ same for } y_p, \quad (27)$$

$$P(y_q, y_p | y_{\text{past}}, y_{\text{future}}) \propto \int dq dp \exp \left\{ -\frac{\epsilon}{2} \left[ (y_q - q)^2 + (y_p - p)^2 \right] \right\} h(q, p, t) \quad (28)$$

in the limit of $\epsilon \to 0$ (opposite limit to von Neumann measurements)

$h(q, p, t)$ describes the apparent statistics of $q$ and $p$ in the weak measurement limit