Quantum Optical Phase and Instantaneous Frequency in the Time Domain

Mankei Tsang, Jeffrey H. Shapiro, and Seth Lloyd
Keck Foundation Center for Extreme Quantum Information Theory, MIT
http://xqit.mit.edu/ mankei@mit.edu
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Massachusetts Institute of Technology

Motivation

- Optical phase measurement is important for
  - Gravitational wave detection
  - Biological sensing and imaging
  - Coherent optical communication
- Instantaneous frequency measurement is important for
  - Doppler velocimetry (e.g. blood flow diagnostics)
  - FM communication
- We need a quantum continuous-parameter estimation theory to determine fundamental limits to measurements of rapidly varying phase and instantaneous frequency.

Discretizing Time

- Consider only bandlimited signals with bandwidth B

$$\hat{A}(t) = \frac{1}{\sqrt{2\pi}}\int_{-B/2}^{B/2} df\,\hat{a}(f)\exp(-i2\pi f t), \qquad (1)$$

$$[\hat{A}(t), \hat{A}^\dagger(t')] = B\,\mathrm{sinc}\,B(t-t'). \qquad (2)$$

- Discretize time:

$$t_j \equiv t_0 + j\delta t, \qquad \delta t \equiv \frac{1}{B}. \qquad (3)$$

- Redefine discrete wave-packet mode operators:

$$\hat{a}_j \equiv \hat{A}(t_j)\sqrt{\delta t}, \qquad [\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}. \qquad (4)$$

Temporal-Phase POVM

- Discrete-time Fock states:

$$|\boldsymbol{n}\rangle = |\ldots, n_j, n_{j+1}, \ldots\rangle. \qquad (5)$$

- Susskind-Glogower operator:

$$\hat{\mathcal{E}}_j \equiv \frac{1}{\sqrt{\hat{a}_j\hat{a}_j^\dagger}}\hat{a}_j. \qquad (6)$$

- Eigenstates:

$$|\boldsymbol{\phi}\rangle = \sum_{\boldsymbol{n}} \exp(i\boldsymbol{n}\cdot\boldsymbol{\phi})|\boldsymbol{n}\rangle. \qquad (7)$$

- POVM:

$$\hat{\Pi}[\phi] = D\phi|\phi\rangle\langle\phi|, \quad D\phi \equiv \prod_j \frac{d\phi_j}{2\pi}. \qquad (8)$$

Pegg-Barnett Formalism

- Impose upper bound on photon number of each wave-packet mode,

$$\exp(i\hat{\phi}_j) = \sum_{n=0}^{s}|n-1\rangle_j\langle n| + \exp\left[i(s+1)\phi_{0j}\right]|s\rangle_j\langle 0|. \qquad (9)$$

- Pegg and Barnett, PRA 39, 1665 (1989).

Instantaneous Frequency Operator

- Classical definition:

$$F(t) = -\frac{1}{2\pi}\frac{d}{dt}\phi(t). \qquad (10)$$

- To avoid multivalued ambiguity in $\phi(t)$,

$$F(t) = \frac{1}{2\pi}\frac{d}{dt'}\sin\left[\phi(t)-\phi(t')\right]_{t'=t}. \qquad (11)$$

- Discrete-time domain:

$$\hat{F}(t_j) = \frac{1}{2\pi}\sum_k d_{j-k}\sin(\hat{\phi}_j - \hat{\phi}_k). \qquad (12)$$

- Can be generalized to 3D to become Landau's fluid velocity operator.

Maximum A Posteriori (MAP) Estimation

- Temporal-phase POVM is difficult. Can homodyne detection achieve quantum-limited temporal-phase measurements?
- Consider MAP estimation, which maximizes the a posteriori probability:

$$P[\boldsymbol{m}|\boldsymbol{x},\boldsymbol{y}] = \frac{P[\boldsymbol{x},\boldsymbol{y}|\boldsymbol{m}]P[\boldsymbol{m}]}{P[\boldsymbol{x},\boldsymbol{y}]}. \qquad (13)$$

- $\boldsymbol{x}$ and $\boldsymbol{y}$ are quadratures.
- $\boldsymbol{m} = \{\ldots, m_j, m_{j+1}, \ldots\}$ is the message.
- MAP estimation is asymptotically efficient.
- For homodyne detection and squeezed states, use the Wigner distribution as $P[\boldsymbol{x},\boldsymbol{y}|\boldsymbol{m}]$.
- Vectorial MAP equation:

$$\nabla_{\boldsymbol{m}}\Big\{\ln P[\boldsymbol{x},\boldsymbol{y}|\boldsymbol{m}] + \ln P[\boldsymbol{m}]\Big\} = \boldsymbol{0}. \qquad (14)$$

- Assume the message is a zero-mean Gaussian,

$$P[\boldsymbol{m}] \propto \exp\left(-\frac{1}{2}\boldsymbol{m}\cdot\boldsymbol{K}_{\boldsymbol{m}}^{-1}\cdot\boldsymbol{m}\right). \qquad (15)$$

MAP Estimation for Coherent States

- Let

$$\bar{\boldsymbol{\phi}} = \boldsymbol{H}\cdot\boldsymbol{m}, \quad H_{jk} = \beta\delta_{jk} \quad \text{for PM}, \qquad (16)$$

$$H_{jk} = -2\pi\mathcal{F}\int_{-\infty}^{t_j} dt'\,\mathrm{sinc}\,B(t'-t_k) \quad \text{for FM}. \qquad (17)$$

- For coherent states,

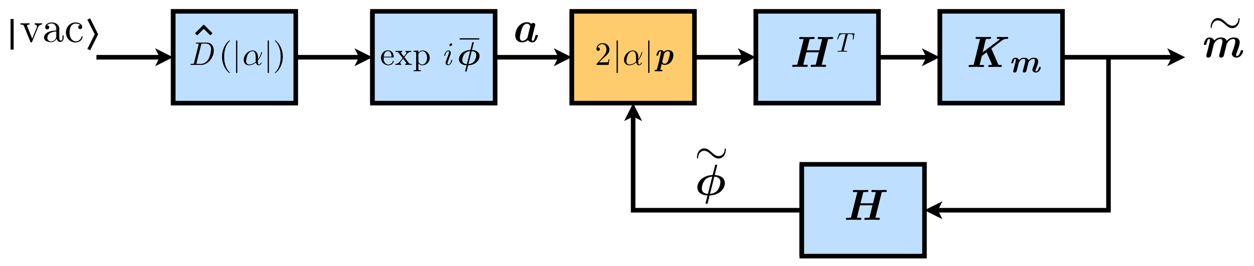
$$P[\boldsymbol{x},\boldsymbol{y}|\bar{\boldsymbol{\phi}}] \propto \exp\left(-\frac{1}{2}\boldsymbol{x}_0\cdot\boldsymbol{x}_0 - \frac{1}{2}\boldsymbol{y}_0\cdot\boldsymbol{y}_0\right), \qquad (18)$$

$$\boldsymbol{x}_0 = \boldsymbol{x}\cos\bar{\boldsymbol{\phi}} - \boldsymbol{y}\sin\bar{\boldsymbol{\phi}} - 2|\alpha|, \qquad (19)$$

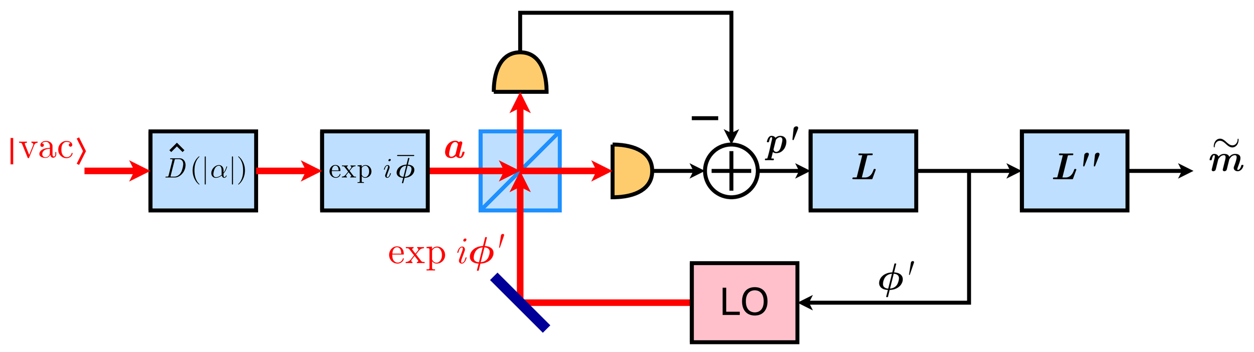
$$\boldsymbol{y}_0 = \boldsymbol{x}\sin\bar{\boldsymbol{\phi}} + \boldsymbol{y}\cos\bar{\boldsymbol{\phi}}. \qquad (20)$$

- MAP equation:

$$\tilde{\boldsymbol{m}} = 2|\alpha|\boldsymbol{K}_{\boldsymbol{m}}\cdot\boldsymbol{H}^T\cdot\Big\{-i\big[\boldsymbol{a}\exp(-i\tilde{\boldsymbol{\phi}}) - \boldsymbol{a}^*\exp(i\tilde{\boldsymbol{\phi}})\big]\Big\}. \qquad (21)$$



Phase-Locked Loops



- Homodyne phase-locked loops can solve the linearized MAP equation [Viterbi, *Principles of Coherent Communication*; Van Trees, *Nonlinear Modulation Theory*].
- Somewhat similar to Berry and Wiseman [Berry and Wiseman, PRA 65 043803 (2002)], but our scheme works with arbitrary stationary messages.

Standard Quantum Limits

- Bandlimited message spectrum with bandwidth b:

$$\mathrm{SNR}_{\mathrm{PM}} \approx \frac{1}{3}\mathrm{SNR}_{\mathrm{FM}} \approx 4\beta^2\mathcal{N}, \qquad \mathcal{N} = \frac{\mathcal{P}}{hf_0 b}. \qquad (22)$$

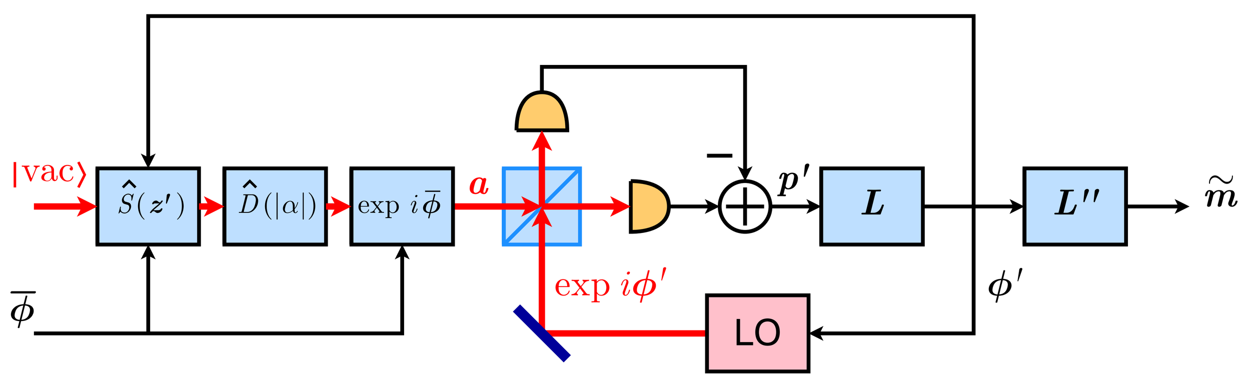
- Lorentzian message spectrum:

$$\mathrm{SNR}_{\mathrm{PM}} \approx \frac{1}{\sqrt{2\pi}}\sqrt{4\beta^2\mathcal{N}}. \qquad (23)$$

- Require threshold constraint $\langle(\bar{\phi}_j - \phi_j')^2\rangle \ll 1$.

Beating the SQL

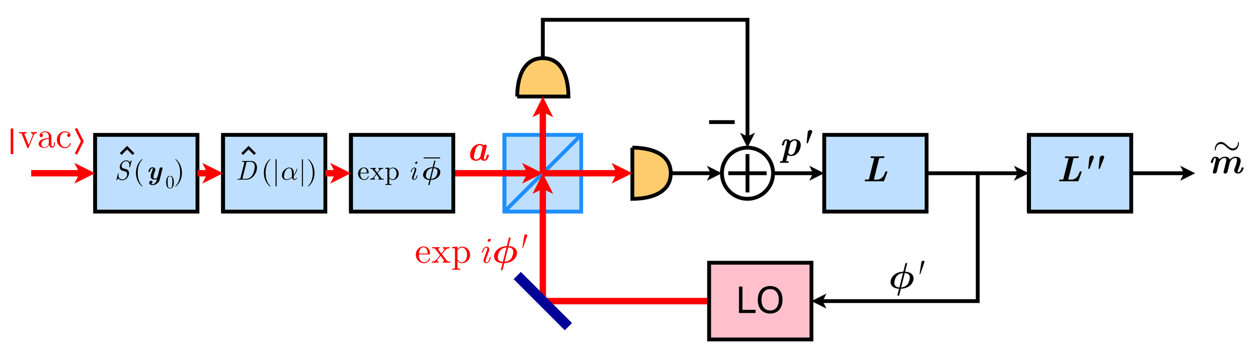
- Squeeze the optimum quadrature:



- Heisenberg limit (bandlimited message):

$$\mathrm{SNR}_{\mathrm{PM}} \approx \frac{1}{3}\mathrm{SNR}_{\mathrm{FM}} \approx 4\beta^2\mathcal{N}^2 \qquad (24)$$

- Squeeze the phase quadrature:



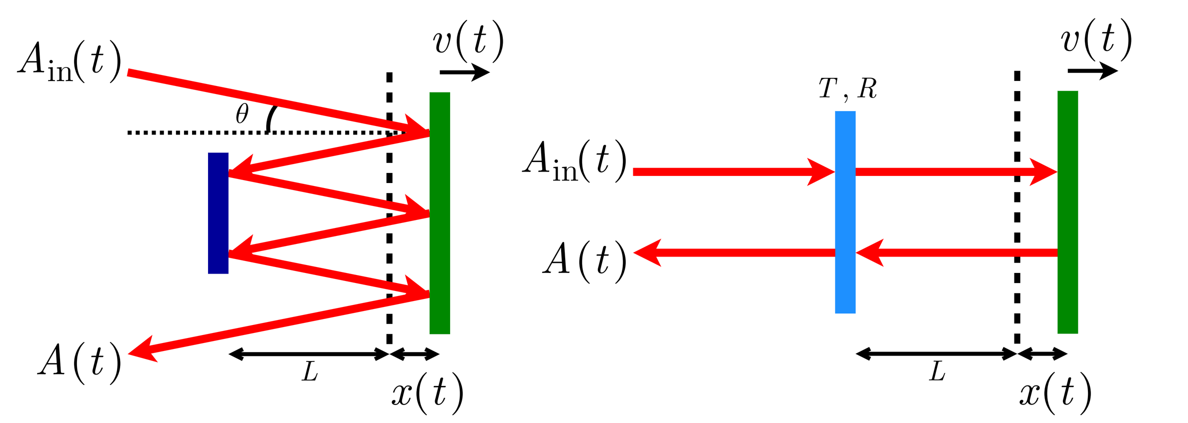
- Require anti-squeezed quadrature to be negligible:

$$\langle(\bar{\phi}_j - \phi_j')^2\rangle \ll \exp(-4r). \qquad (25)$$

- Cannot reach Heisenberg limit:

$$\mathrm{SNR}_{\mathrm{PM}} \approx \frac{1}{3}\mathrm{SNR}_{\mathrm{FM}} \ll \frac{8\beta^2\mathcal{N}^2}{\ln\mathcal{N}}. \qquad (26)$$

Multipass Position Sensing and Doppler Velocimetry



- Phase modulation or instantaneous frequency modulation:

$$\bar{\phi}(t) = (2M\cos\theta)\frac{2\pi}{\lambda_0}x(t) = (2M\cos\theta)\frac{2\pi f_0}{c}\int_{-\infty}^{t}dt'v(t') \qquad (27)$$

$$\beta = (2M\cos\theta)\frac{2\pi\sqrt{\langle x^2\rangle}}{\lambda_0} = (2M\cos\theta)\frac{2f_0\sqrt{\langle v^2\rangle}}{bc}. \qquad (28)$$

- Increasing M, the number of times the target is interrogated, increases the SNR quadratically.
- For Fabry-Pérot, $M \approx (1+\sqrt{R})/(1-\sqrt{R})$ if $\beta \ll 1$.