Quantum Theory of Optical Temporal Phase in the Continuous Time Limit

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We consider our recently proposed quantum theory of optical temporal phase in the continuous time limit.

We have recently introduced a quantum theory of optical temporal phase and instantaneous frequency for slowly varying signals in the discrete time domain [1]. In this paper, we consider the theory in the continuous time limit. The frequency-domain field operators satisfy the bosonic commutation relation:

$$[\hat{a}(f), \hat{a}^\dagger(f')] = \delta(f - f'). \tag{1}$$

The time-domain envelope operators are defined as

$$\hat{A}(t) \equiv \int_{-\infty}^{\infty} df \;\hat{a}(f)\exp(-i2\pi f t), \qquad [\hat{A}(t), \hat{A}^\dagger(t')] = \delta(t - t'). \tag{2}$$

The Hilbert space can be spanned by Fock states. Let $dn(t)$ be a continuous-time discrete-photon-number process, and t_j be the times at which $dn(t_j) > 0$. The photon-number flux is then

$$I(t) \equiv \frac{dn(t)}{dt} = \sum_j dn(t_j)\delta(t - t_j). \tag{3}$$

A Fock state is defined as

$$|dn(t)\rangle \equiv \left\{\prod_j \frac{1}{\sqrt{dn(t_j)!}}\left[\hat{A}^\dagger(t_j)\sqrt{dt}\right]^{dn(t_j)}\right\}|0\rangle, \qquad \hat{1} = \sum_{dn(t)} |dn(t)\rangle\langle dn(t)|. \tag{4}$$

For pure states, the photon-number representation is $C[dn(t)] \equiv \langle dn(t)|\Psi\rangle$, related to the N-photon temporal wavefunction by

$$\psi_N(\tau_1, \dots, \tau_N) = \sum_{dn(t)} C[dn(t)]\Phi_{dn(t)}(\tau_1, \dots, \tau_N), \quad N = \int_{-\infty}^{\infty} dt\; I(t) = \sum_j dn(t_j),$$

$$\Phi_{dn(t)}(\tau_1, \dots, \tau_N) \equiv \left[\frac{\prod_j dn(t_j)!}{N!}\right]^{1/2} \sum_P \prod_j \delta^{\frac{1}{2}}(\tau_{P(j,1)} - t_j) \times \dots \times \delta^{\frac{1}{2}}(\tau_{P(j,dn(t_j))} - t_j), \tag{5}$$

where the summation over all permutations symmetrizes $\Phi_{dn(t)}$. The inverse relation is

$$C[dn(t)] = \left[\frac{N! dt^N}{\prod_t dn(t)!}\right]^{1/2} \psi_N(\ldots, \underbrace{t, \ldots, t}_{dn(t) \text{ terms}}, \ldots). \tag{6}$$

For example, a multimode coherent state can be written as

$$|\alpha(t)\rangle = \exp\left[-\frac{\bar{N}}{2} + \int_{-\infty}^{\infty} dt\ \alpha(t)\hat{A}^\dagger(t)\right]|0\rangle, \quad \bar{N} \equiv \int_{-\infty}^{\infty} dt\ |\alpha(t)|^2, \tag{7}$$

$$C[dn(t)] = \prod_t \exp\left[-\frac{1}{2}dt|\alpha(t)|^2\right] \frac{\left[\alpha(t)\sqrt{dt}\right]^{dn(t)}}{\sqrt{dn(t)!}}, \tag{8}$$

and the photon-number probability density $|C[dn(t)]|^2$ describes a Poisson process,

$$[dn(t)]^2 = dn(t), \qquad \langle dn(t)\rangle = |\alpha(t)|^2 dt, \tag{9}$$

$$\langle I(t)I(\tau)\rangle = \langle I(t)\rangle\,\delta(t-\tau), \qquad \langle I(t)\rangle = |\alpha(t)|^2 = \frac{\mathcal{P}(t)}{hf_0}, \tag{10}$$

$\mathcal{P}(t)$ being the optical power and f_0 the carrier frequency.

The Susskind-Glogower phase state is defined as the functional Fourier transform of Fock states,

$$|\phi(t)\rangle \equiv \sum_{dn(t)} \exp\left[i\int_{-\infty}^{\infty} dt\ I(t)\phi(t)\right]|dn(t)\rangle = \sum_{dn(t)} \exp\left[i\sum_j dn(t_j)\phi(t_j)\right]|dn(t)\rangle. \tag{11}$$

A temporal-phase POVM can then be written as

$$\hat{\Pi}[\phi(t)] \equiv |\phi(t)\rangle\langle\phi(t)|, \qquad \int_{\phi_0(t)}^{\phi_0(t)+2\pi} D\phi(t)\ \hat{\Pi}[\phi(t)] = \hat{1}, \qquad D\phi(t) \equiv \prod_t \frac{d\phi(t)}{2\pi}. \tag{12}$$

To measure the temporal phase in practice, in Ref. [1] we have designed homodyne phase-locked loops (PLLs) using the Wigner distribution. In the continuous time domain, the Wigner distribution of a squeezed state is

$$\Delta X_1(t) \equiv A(t) + A^*(t) - \langle A(t) + A^*(t)\rangle, \tag{13}$$

$$\Delta X_2(t) \equiv -i[A(t) - A^*(t)] - \langle -i[A(t) - A^*(t)]\rangle, \tag{14}$$

$$W_0[\Delta X_1(t), \Delta X_2(t)] \propto \exp\left[-\frac{1}{2}\sum_{k=1,2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau\ \Delta X_k(t) K_k^{-1}(t,\tau)\Delta X_k(\tau)\right], \tag{15}$$

$$K_{1,2}(t,\tau) \equiv \langle \Delta X_{1,2}(t)\Delta X_{1,2}(\tau)\rangle, \quad K_{1,2}(t,\tau) = K_{2,1}^{-1}(t,\tau), \tag{16}$$

$$\int_{-\infty}^{\infty} d\tau\ K_{1,2}(t,\tau)K_{1,2}^{-1}(\tau,u) = \delta(t-u). \tag{17}$$

Assuming that the squeezed state has time-invariant statistics, $K_{1,2}(t,\tau) = K_{1,2}(t-\tau)$, we can define the quadrature power spectral densities as

$$S_{1,2}(f) \equiv \int_{-\infty}^{\infty} dt\ K_{1,2}(t)\exp(i2\pi f t), \qquad S_1(f)S_2(f) = 1. \tag{18}$$

For example, the coherent state has the covariance functions $K_{1,2}(t,\tau) = \delta(t-\tau)$, which means that it can be regarded as a classical signal with additive white Gaussian noise upon homodyne detection.

Assume that a message $m(t)$ is linearly encoded in the optical phase,

$$\bar{\phi}(t) = \int_{-\infty}^{\infty} d\tau\ h(t-\tau)m(\tau). \tag{19}$$

For example, for phase modulation $h(t-\tau) = \beta\delta(t-\tau)$ and for frequency modulation $h(t-\tau) = -2\pi\mathcal{F}\int_{-\infty}^{t} du\ \delta(u-\tau)$. The a priori probability density for the message is assumed to be

$$P[m(t)] \propto \exp\left[-\frac{1}{2}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau\ m(t)K_m^{-1}(t-\tau)m(\tau)\right], \quad K_m(t-\tau) \equiv \langle m(t)m(\tau)\rangle. \tag{20}$$

The conditional Wigner distribution, upon phase modulation, is

$$W[X_1(t), X_2(t)|m(t)] = W_0[Y_1(t), Y_2(t)], \tag{21}$$

$$Y_1(t) = X_1(t)\cos\bar{\phi}(t) + X_2(t)\sin\bar{\phi}(t) - 2|\alpha(t)|, \tag{22}$$

$$Y_2(t) = -X_1(t)\sin\bar{\phi}(t) + X_2(t)\cos\bar{\phi}(t). \tag{23}$$

The maximum a posteriori (MAP) estimate equation becomes

$$\frac{\delta}{\delta m(t)}\ln P[m(t)|X_1(t), X_2(t)] = \frac{\delta}{\delta m(t)}\Big\{\ln W[X_1(t), X_2(t)|m(t)] + \ln P[m(t)]\Big\} = 0. \tag{24}$$

The solution of this equation gives the MAP estimate $\widetilde{m}(t)$. For coherent states the MAP equation is

$$\widetilde{m}(t) = 4|\alpha|^2\int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du\ K_m(t-\tau)h(u-\tau)\Big\{\sin[\bar{\phi}(u) - \widetilde{\phi}(u)] + Z(u)\Big\}, \tag{25}$$

$$\langle Z(t)Z(\tau)\rangle = \frac{1}{4|\alpha|^2}\delta(t-\tau), \quad \widetilde{\phi}(t) \equiv \int_{-\infty}^{\infty} d\tau\ h(t-\tau)\widetilde{m}(\tau). \tag{26}$$

Linearizing the MAP equation by assuming $\sin(\bar{\phi} - \widetilde{\phi}) \approx \bar{\phi} - \widetilde{\phi}$,

$$\widetilde{m}(t) = \int_{-\infty}^{\infty} d\tau\ g(t-\tau)\phi(\tau), \quad \phi(t) \equiv \bar{\phi}(t) + Z(t), \tag{27}$$

where $g(t-\tau)$ is the optimal linear filter,

$$\int_{-\infty}^{\infty} d\tau\ g(t-\tau)\left[K_{\bar{\phi}}(\tau-u)+\frac{1}{4|\alpha|^2}\delta(\tau-u)\right]=\int_{-\infty}^{\infty} dv\ K_m(t-v)h(u-v), \tag{28}$$

and $K_{\bar{\phi}}$ is

$$K_{\bar{\phi}}(t-\tau)\equiv\left\langle\bar{\phi}(t)\bar{\phi}(\tau)\right\rangle=\int_{-\infty}^{\infty} du\int_{-\infty}^{\infty} dv\ h(t-u)K_m(u-v)h(\tau-v). \tag{29}$$

In the frequency domain, defining

$$G(f)\equiv\int_{-\infty}^{\infty} dt\ g(t)\exp(i2\pi ft), \qquad S_m(f)\equiv\int_{-\infty}^{\infty} dt\ K_m(t)\exp(i2\pi ft),$$
$$H(f)\equiv\int_{-\infty}^{\infty} dt\ h(t)\exp(i2\pi ft), \tag{30}$$

we can derive the irreducible error,

$$G(f)=\frac{S_m(f)H^*(f)}{S_m(f)|H(f)|^2+1/4|\alpha|^2}, \quad \left\langle[m(t)-\widetilde{m}(t)]^2\right\rangle=\int_{-\infty}^{\infty} df\ \frac{S_m(f)}{4|\alpha|^2 S_m(f)|H(f)|^2+1}. \tag{31}$$

To implement the PLLs, we also need the realizable estimate of the phase in terms of the Wiener-Hopf filter,

$$\phi'(t)=\int_{-\infty}^{t} d\tau\ g'(t-\tau)\phi(\tau), \quad \int_{0}^{\infty} d\tau\ g'(\tau)\left[K_{\bar{\phi}}(t-\tau)+\frac{1}{4|\alpha|^2}\delta(t-\tau)\right]=K_{\bar{\phi}}(t). \tag{32}$$

The error of Wiener filtering is

$$\left\langle[\bar{\phi}(t)-\phi'(t)]^2\right\rangle=\frac{1}{4|\alpha|^2}\int_{-\infty}^{\infty} df\ \ \ln\left[1+4|\alpha|^2 S_m(f)|H(f)|^2\right], \tag{33}$$

which is required to be much smaller than 1 for the linear analysis to be valid. Under this constraint, the quantum limits to temporal phase and instantaneous frequency measurements can be derived and are given in [1]. For broadband phase-squeezed states, we can replace $|\alpha|^2$ with $|\alpha|^2\exp(2r)$ if the phase-locking is tight enough so that the anti-squeezed quadrature can be neglected.

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[1] M. Tsang, J. H. Shapiro, and S. Lloyd, e-print arXiv:0804.0463 (Phys. Rev. A, in press).