Quantum Theory of Optical Temporal Phase in the Continuous Time Limit

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We consider our recently proposed quantum theory of optical temporal phase in the continuous time limit.

We have recently introduced a quantum theory of optical temporal phase and instantaneous frequency for slowly varying signals in the discrete time domain [1]. In this paper, we consider the theory in the continuous time limit. The frequency-domain field operators satisfy the bosonic commutation relation:

$$[\hat{a}(f), \hat{a}^\dagger(f')] = \delta(f - f').$$

(1)

The time-domain envelope operators are defined as

$$\hat{A}(t) \equiv \int_{-\infty}^{\infty} df \ \hat{a}(f) \exp(-i2\pi ft), \quad [\hat{A}(t), \hat{A}^\dagger(t')] = \delta(t - t').$$

(2)

The Hilbert space can be spanned by Fock states. Let $dn(t)$ be a continuous-time discrete-photon-number process, and $t_j$ be the times at which $dn(t_j) > 0$. The photon-number flux is then

$$I(t) \equiv \frac{dn(t)}{dt} = \sum_j dn(t_j)\delta(t - t_j).$$

(3)

A Fock state is defined as

$$|dn(t)\rangle \equiv \left\{ \prod_j \frac{1}{\sqrt{dn(t_j)!}} \left[ \hat{A}^\dagger(t_j) \sqrt{dt} \right]^{dn(t_j)} \right\} |0\rangle, \quad \hat{1} = \sum_{dn(t)} |dn(t)\rangle\langle dn(t)|.$$ 

(4)

For pure states, the photon-number representation is $C[dn(t)] \equiv \langle dn(t)|\Psi\rangle$, related to the $N$-photon temporal wavefunction by

$$\psi_N(\tau_1, \ldots, \tau_N) = \sum_{dn(t)} C[dn(t)] \Phi_{dn(t)}(\tau_1, \ldots, \tau_N), \quad N = \int_{-\infty}^{\infty} dt \ I(t) = \sum_j dn(t_j),$$

$$\Phi_{dn(t)}(\tau_1, \ldots, \tau_N) \equiv \left[ \frac{\prod_j dn(t_j)!}{N!} \right]^{1/2} \sum_P \prod_j \delta^2(\tau_{P(j,1)} - t_j) \times \cdots \times \delta^2(\tau_{P(j,dn(t_j))} - t_j).$$

(5)
where the summation over all permutations symmetrizes $\Phi_{dn(t)}$. The inverse relation is

$$C[dn(t)] = \left[ \frac{N!dt^N}{\prod_t dn(t)!} \right]^{1/2} \psi_N(\ldots, t, \ldots, t, \ldots).$$

(6)

For example, a multimode coherent state can be written as

$$|\alpha(t)\rangle = \exp \left[ -\frac{\bar{N}}{2} + \int_{-\infty}^{\infty} dt\ \alpha(t)\hat{A}^\dagger(t) \right]|0\rangle, \quad \bar{N} \equiv \int_{-\infty}^{\infty} dt\ |\alpha(t)|^2,$$

(7)

$$C[dn(t)] = \prod_t \exp \left[ -\frac{1}{2} dt|\alpha(t)|^2 \right] \frac{\alpha(t)\sqrt{dt}}{\sqrt{dn(t)!}}.$$

(8)

and the photon-number probability density $|C[dn(t)]|^2$ describes a Poisson process,

$$[dn(t)]^2 = dn(t), \quad \langle dn(t) \rangle = |\alpha(t)|^2 dt,$$

(9)

$$\langle I(t)I(\tau) \rangle = \langle I(t) \rangle \delta(t - \tau), \quad \langle I(t) \rangle = |\alpha(t)|^2 = \frac{\mathcal{P}(t)}{h f_0},$$

(10)

$\mathcal{P}(t)$ being the optical power and $f_0$ the carrier frequency.

The Susskind-Glogower phase state is defined as the functional Fourier transform of Fock states,

$$|\phi(t)\rangle \equiv \sum_{dn(t)} \exp \left[ i \int_{-\infty}^{\infty} dt\ I(t)\phi(t) \right] |dn(t)\rangle = \sum_{dn(t)} \exp \left[ i \sum_j dn(t_j)\phi(t_j) \right] |dn(t)\rangle.$$

(11)

A temporal-phase POVM can then be written as

$$\hat{\Pi}[\phi(t)] \equiv |\phi(t)\rangle\langle\phi(t)|, \quad \int_{\phi_0(t)}^{\phi_0(t)+2\pi} D\phi(t)\ \hat{\Pi}[\phi(t)] = \hat{1}, \quad D\phi(t) \equiv \prod_t \frac{d\phi(t)}{2\pi}.$$

(12)

To measure the temporal phase in practice, in Ref. [1] we have designed homodyne phase-locked loops (PLLs) using the Wigner distribution. In the continuous time domain, the Wigner distribution of a squeezed state is

$$\Delta X_1(t) \equiv A(t) + A^\dagger(t) - \langle A(t) + A^\dagger(t) \rangle,$$

(13)

$$\Delta X_2(t) \equiv -i[A(t) - A^\dagger(t)] - \langle -i[A(t) - A^\dagger(t)] \rangle,$$

(14)

$$W_0[\Delta X_1(t), \Delta X_2(t)] \propto \exp \left[ -\frac{1}{2} \sum_{k=1,2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau \ \Delta X_k(t) K_k^{-1}(t, \tau) \Delta X_k(\tau) \right],$$

(15)

$$K_{1,2}(t, \tau) \equiv \langle \Delta X_{1,2}(t)\Delta X_{1,2}(\tau) \rangle, \quad K_{1,2}(t, \tau) = K_{2,1}^{-1}(t, \tau),$$

(16)

$$\int_{-\infty}^{\infty} d\tau\ K_{1,2}(t, \tau) K_{1,2}^{-1}(\tau, u) = \delta(t - u).$$

(17)
Assuming that the squeezed state has time-invariant statistics, $K_{1,2}(t, \tau) = K_{1,2}(t - \tau)$, we can define the quadrature power spectral densities as

$$S_{1,2}(f) \equiv \int_{-\infty}^{\infty} dt \ K_{1,2}(t) \exp(i2\pi ft), \quad S_1(f)S_2(f) = 1. \quad (18)$$

For example, the coherent state has the covariance functions $K_{1,2}(t, \tau) = \delta(t - \tau)$, which means that it can be regarded as a classical signal with additive white Gaussian noise upon homodyne detection.

Assume that a message $m(t)$ is linearly encoded in the optical phase,

$$\bar{\phi}(t) = \int_{-\infty}^{\infty} d\tau \ h(t - \tau)m(\tau). \quad (19)$$

For example, for phase modulation $h(t - \tau) = \beta \delta(t - \tau)$ and for frequency modulation $h(t - \tau) = -2\pi F \int_{-\infty}^{t} du \ \delta(u - \tau)$. The a priori probability density for the message is assumed to be

$$P[m(t)] \propto \exp\left[-\frac{1}{2} \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\tau \ m(t)K_m^{-1}(t - \tau)m(\tau)\right], \quad K_m(t - \tau) \equiv \langle m(t)m(\tau) \rangle. \quad (20)$$

The conditional Wigner distribution, upon phase modulation, is

$$W[X_1(t), X_2(t)|m(t)] = W_0[Y_1(t), Y_2(t)], \quad (21)$$

$$Y_1(t) = X_1(t) \cos \bar{\phi}(t) + X_2(t) \sin \bar{\phi}(t) - 2|\alpha(t)|, \quad (22)$$

$$Y_2(t) = -X_1(t) \sin \bar{\phi}(t) + X_2(t) \cos \bar{\phi}(t). \quad (23)$$

The maximum a posteriori (MAP) estimate equation becomes

$$\frac{\delta}{\delta m(t)} \ln P[m(t)|X_1(t), X_2(t)] = \frac{\delta}{\delta m(t)} \left\{ \ln W[X_1(t), X_2(t)|m(t)] + \ln P[m(t)] \right\} = 0. \quad (24)$$

The solution of this equation gives the MAP estimate $\tilde{m}(t)$. For coherent states the MAP equation is

$$\tilde{m}(t) = 4|\alpha|^2 \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} du \ K_m(t - \tau)h(u - \tau) \left\{ \sin[\bar{\phi}(u) - \bar{\phi}(u)] + Z(u) \right\}, \quad (25)$$

$$\langle Z(t)Z(\tau) \rangle = \frac{1}{4|\alpha|^2} \delta(t - \tau), \quad \bar{\phi}(t) \equiv \int_{-\infty}^{\infty} d\tau \ h(t - \tau)\tilde{m}(\tau). \quad (26)$$

Linearizing the MAP equation by assuming $\sin(\bar{\phi} - \bar{\phi}) \approx \bar{\phi} - \bar{\phi}$,

$$\tilde{m}(t) = \int_{-\infty}^{\infty} d\tau \ g(t - \tau)\phi(\tau), \quad \phi(t) \equiv \bar{\phi}(t) + Z(t), \quad (27)$$
where \( g(t - \tau) \) is the optimal linear filter,
\[
\int_{-\infty}^{\infty} d\tau \ g(t - \tau) \left[ K_\phi(\tau - u) + \frac{1}{4|\alpha|^2} \delta(\tau - u) \right] = \int_{-\infty}^{\infty} dv \ K_m(t - v) h(u - v), \tag{28}
\]
and \( K_\phi \) is
\[
K_\phi(t - \tau) \equiv \langle \bar{\phi}(t) \bar{\phi}(\tau) \rangle = \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} dv \ h(t - u) K_m(u - v) h(\tau - v). \tag{29}
\]
In the frequency domain, defining
\[
G(f) \equiv \int_{-\infty}^{\infty} dt \ g(t) \exp(i2\pi ft), \quad S_m(f) \equiv \int_{-\infty}^{\infty} dt \ K_m(t) \exp(i2\pi ft),
H(f) \equiv \int_{-\infty}^{\infty} dt \ h(t) \exp(i2\pi ft), \tag{30}
\]
we can derive the irreducible error,
\[
G(f) = \frac{S_m(f)H^*(f)}{S_m(f)|H(f)|^2 + 1/4|\alpha|^2}, \quad \langle [m(t) - \bar{m}(t)]^2 \rangle = \int_{-\infty}^{\infty} df \ \frac{S_m(f)}{4|\alpha|^2 S_m(f)|H(f)|^2 + 1}. \tag{31}
\]
To implement the PLLs, we also need the realizable estimate of the phase in terms of the Wiener-Hopf filter,
\[
\phi'(t) = \int_{-\infty}^{t} d\tau \ g'(t - \tau) \phi(\tau), \quad \int_{-\infty}^{\infty} d\tau \ g'(\tau) \left[ K_\phi(t - \tau) + \frac{1}{4|\alpha|^2} \delta(t - \tau) \right] = K_\phi(t). \tag{32}
\]
The error of Wiener filtering is
\[
\langle [\bar{\phi}(t) - \phi'(t)]^2 \rangle = \frac{1}{4|\alpha|^2} \int_{-\infty}^{\infty} df \ \ln \left[ 1 + 4|\alpha|^2 S_m(f)|H(f)|^2 \right], \tag{33}
\]
which is required to be much smaller than 1 for the linear analysis to be valid. Under this constraint, the quantum limits to temporal phase and instantaneous frequency measurements can be derived and are given in [1]. For broadband phase-squeezed states, we can replace \( |\alpha|^2 \) with \( |\alpha|^2 \exp(2\pi r) \) if the phase-locking is tight enough so that the anti-squeezed quadrature can be neglected.

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