Optical Hydrodynamics

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Abstract: The propagation of light can be studied in a hydrodynamic picture, which is especially useful in nonlinear optics. Classical and quantum formulations of optical hydrodynamics are discussed.

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Newton considered light as a stream of particles. This particle picture of light can be described mathematically by the eikonal equation in ray optics $\nabla S(\mathbf{r})\cdot\nabla S(\mathbf{r}) = n^2(\mathbf{r})$ [1], where $S(\mathbf{r})$ is called the optical path and n is the refractive index. The Poynting vector is proportional to $\nabla S(\mathbf{r})$, and optical rays are trajectories that follow the Poynting vector. These trajectories obey Fermat's principle. Connections with fluid dynamics can be made by invoking the paraxial approximation, $k_0 S(\mathbf{r}) = k_0 z + \theta(\mathbf{r})$, and $\frac{\partial\theta}{\partial z} \ll k_0$. Defining the transverse ray vector as $\nabla_\perp\theta$, and assuming $n = n_0 + \Delta n$, $\Delta n \ll 1$, we obtain an equation identical to the Euler fluid equation of motion,

$$k_0 \frac{\partial \mathbf{u}}{\partial z} + \mathbf{u}\cdot\nabla_\perp \mathbf{u} = k_0^2 n_0 \nabla_\perp(\Delta n), \tag{1}$$

with $\mathbf{u}$ playing the role of fluid velocity and z playing the role of time. The optical intensity, on the other hand, obeys the continuity equation in paraxial ray optics,

$$k_0 \frac{\partial \rho}{\partial z} + \nabla_\perp\cdot(\rho\mathbf{u}) = 0, \tag{2}$$

which is identical to the fluid continuity equation that governs the evolution of fluid density. If Δn depends on the optical intensity, in a Kerr nonlinear medium for example, it becomes analogous to a negative fluid pressure that depends on fluid density. Self-defocusing nonlinearity corresponds to a physical fluid pressure that increases with density.

The particle picture of light is, of course, an approximation of the wave picture. In the paraxial regime, the hydrodynamic variables (ρ,θ) can be related to the wave envelope $A(\mathbf{r})$ by the Madelung transformation $A(\mathbf{r}) = \sqrt{\rho}\exp(i\theta)$ [2]. The optical envelope obeys the nonlinear Schrödinger equation. The wave picture adds a "quantum pressure" term $\nabla_\perp[(2\sqrt{\rho})^{-1}\nabla_\perp^2\sqrt{\rho}]$ to the right-hand side of Eq. (2), but a strong self-defocusing nonlinearity can dominate over the quantum pressure effect, and the fluidic picture can still be useful.

The optical phase $\theta(\mathbf{r})$ takes the center stage in the fluid picture of light, but a quantum description of the optical phase is, unfortunately, not trivial even for one discrete optical mode [3]. Previous studies of single-mode quantum optical phase can be generalized to the continuous space with the aid of the sampling theorem [4]. By imposing cutoffs to the momentum space, one can discretize space into discrete wave packet modes, and an optical phase operator can be subsequently defined for each mode using the Pegg-Barnett theory [4,5]. The continuous optical phase operator $\hat{\theta}(\mathbf{r})$ can then be reconstructed from the discrete operators using the sampling theorem [4], and a quantum formulation of optical hydrodynamics based on the intensity operator $\hat{\rho}(\mathbf{r}) = \hat{A}^\dagger(\mathbf{r})\hat{A}(\mathbf{r})$ and a fluid velocity operator $\hat{\mathbf{u}}(\mathbf{r}) = \nabla_\perp\hat{\theta}(\mathbf{r})$ can be envisaged.

1. References

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