Quantum Backaction Noise Cancellation for Linear Systems

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Abstract. We show that it is always possible to convert a dissipationless linear sensor system under quantum nondemolition measurements into a backaction-evading one using the technique of quantum noise cancellation. This result generalizes our earlier work on optomechanical sensors [Tsang and Caves, Phys. Rev. Lett. **105**, 123601 (2010)].

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INTRODUCTION

Quantum mechanics mandates the presence of measurement-induced disturbance. For future quantum sensors, such as optomechanical force sensors, gravitational wave detectors, and atomic magnetometry, it will be important to isolate this measurement backaction noise from the sensor output in order to optimize their sensitivities [1]. In a recent work [2], we have shown how an optomechanical force sensor can be converted to a backaction-evading sensor using a technique called quantum noise cancellation (QNC). Prior to our work, Eugene Polzik's group have applied a very similar technique to their experiments with atomic spin ensembles [3]. In this paper, we wish to generalize the QNC technique to arbitrary linear systems. Assuming that the linear sensor system is under quantum nondemolition (QND) measurements and experiences no dissipation, we shall show that the sensor can be made backaction-evading by introducing a physically realizable auxiliary system and performing collective QND measurements of both systems.

QUANTUM BACKACTION NOISE CANCELLATION FOR LINEAR SYSTEMS

Let $Q \equiv (q_1,\ldots,q_n,p_1,\ldots,p_n)^T$ be a vector of $2n$ phase-space coordinates, $q \equiv (q_1,q_2,\ldots,q_n)^T$ be the position vector, and $p \equiv (p_1,p_2,\ldots,p_n)^T$ be the momentum vector. The Heisenberg equations of motion for a dissipationless linear system under measurements can be written as

$$\frac{dQ(t)}{dt} = A(t)Q(t) + B(t)\xi(t) + C(t)x(t), \tag{1}$$

where A is the drift matrix, ξ is the measurement backaction noise, x is the unknown classical signal to be estimated from the sensor, and B and C are matrices that relate the noise and signal inputs to the system. Let $Z(t,t')$ be the impulse response of the system, given by

$$\frac{dZ(t,t_0)}{dt} = A(t)Z(t,t_0), \qquad Z(t_0,t_0) = I, \tag{2}$$

I being the identity matrix, so $Q(t)$ is given by

$$Q(t) = Z(t,t_0)Q(t_0) + \int_{t_0}^{t} dt' Z(t,t') \left[B(t')\xi(t') + C(t')x(t')\right]. \tag{3}$$

Without loss of generality, assume that the measurements are performed on the positions q only:

$$y(t) = D(t)Q(t) + \eta(t) = D_q(t)q(t) + \eta(t), \qquad D = \begin{pmatrix} D_q & 0 \end{pmatrix}, \tag{4}$$

where y is a vector of m observation processes, D is an m-by-$2n$ matrix, η is the observation noise vector, D_q is an m-by-n matrix, and the 0 denotes an m-by-n zero matrix. If some of the measurements are performed on the momenta, one can always redefine those momenta as position coordinates via a symplectic transformation of the phase-space coordinates. For QND position measurements, the backaction noise acts on momenta, so the B matrix can be expressed as

$$B = \begin{pmatrix} 0 \\ B_p \end{pmatrix}, \tag{5}$$

B_p being an n-by-m matrix and the 0 denoting an n-by-m zero matrix.

Consider the inhomogeneous solution of $y(t)$ due to $\xi(t)$:

$$y_\xi(t) = D(t)\int_{t_0}^{t} dt' Z(t,t')B(t')\xi(t'). \tag{6}$$

If we split $Z(t,t')$ into four n-by-n matrices:

$$Z = \begin{pmatrix} Z_{qq} & Z_{qp} \\ Z_{pq} & Z_{pp} \end{pmatrix}, \tag{7}$$

y_ξ would be given by

$$y_\xi(t) = D_q(t)\int_{t_0}^{t} dt' Z_{qp}(t,t')B_p(t')\xi(t'). \tag{8}$$

The Z_{qp} component of the impulse response thus causes the backaction noise on the momenta to be coupled to the observations of the positions.

To perform noise cancellation, let us add an auxiliary system with coordinates $Q'(t)$, drift matrix $A'(t)$, and impulse response $Z'(t,t')$, all with the same dimensions as the original system. If we measure

$$y(t) = D(t)[Q(t) + Q'(t)] + \eta(t), \tag{9}$$

the original system will have the same dynamics, while the dynamics of the auxiliary system is

$$\frac{dQ'}{dt} = A'Q' + B\xi\,, \tag{10}$$

which has the same backaction noise acting on it. y_ξ becomes

$$y_\xi(t) = D_q(t)\int_{t_0}^{t} dt'\left[Z_{qp}(t,t') + Z'_{qp}(t,t')\right] B_p(t')\xi(t'). \tag{11}$$

The backaction noise in the observations can be eliminated if we make

$$Z'_{qp} = -Z_{qp}. \tag{12}$$

The question is then whether this Z' is physically realizable.

For a linear quantum system with no dissipation, $Z(t,t')$ is a symplectic matrix [4]. This is equivalent to the conditions

$$Z_{qq}^T Z_{pq} - Z_{pq}^T Z_{qq} = 0, \qquad Z_{qp}^T Z_{pp} - Z_{pp}^T Z_{qp} = 0, \qquad Z_{qq}^T Z_{pp} - Z_{pq}^T Z_{qp} = I. \tag{13}$$

We can satisfy the same conditions for Z' and $Z'_{qp} = -Z_{qp}$ if we make

$$Z' = \left(\begin{array}{cc} Z_{qq} & -Z_{qp} \\ -Z_{pq} & Z_{pp} \end{array}\right). \tag{14}$$

This guarantees that Z' is also a symplectic matrix, which means that Z' and A' are physically realizable. If

$$A = \left(\begin{array}{cc} A_{qq} & A_{qp} \\ A_{pq} & A_{pp} \end{array}\right), \tag{15}$$

it is not difficult to show that the desired A' is

$$A' = \left(\begin{array}{cc} A_{qq} & -A_{qp} \\ -A_{pq} & A_{pp} \end{array}\right). \tag{16}$$

If the original system is dissipative and $Z(t,t')$ is nonsymplectic, the system will see additional noise. In that case, even if we can find an auxiliary system with $Z_{qp} = -Z'_{qp}$ such that the backaction noise is cancelled, the auxiliary system is in general also dissipative and would introduce even more noise uncorrelated with the other noise sources to the observations, rendering the QNC technique less effective.

We can better understand the principle of QNC by writing down the following equations of motion for the collective position and momentum vectors with the A' given by

Eq. (16):

$$\frac{d\,(q+q')}{dt} = A_{qq}(q+q') + A_{qp}(p-p') + C_q x, \tag{17}$$

$$\frac{d\,(p-p')}{dt} = A_{pq}(q+q') + A_{pp}(p-p') + C_p x, \tag{18}$$

$$\frac{d\,(q-q')}{dt} = A_{qq}(q-q') + A_{qp}(p+p') + C_q x, \tag{19}$$

$$\frac{d\,(p+p')}{dt} = A_{pq}(q-q') + A_{pp}(p+p') + 2B_p\xi + C_p x. \tag{20}$$

We see that the backaction noise is decoupled from the dynamics of $q+q'$ and $p-p'$. Writing a new collective coordinate vector as

$$\tilde{Q} = \begin{pmatrix} q+q' \\ p-p' \end{pmatrix}, \tag{21}$$

we have

$$\frac{d\tilde{Q}}{dt} = A\tilde{Q} + Cx, \qquad y = D\tilde{Q} + \eta, \tag{22}$$

which has the same dynamics as the original system, except that the backaction noise is completely removed from the dynamics and the observations. It remains an open question whether the QNC technique combined with quantum smoothing [5] will be sufficient to enable arbitrary linear sensors to reach the fundamental sensitivity limit set by the quantum Cramér-Rao bound [1].

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