Quantum Optical Temporal Phase Estimation by Homodyne Phase-Locked Loops

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Abstract: Using classical estimation techniques, we design homodyne phase-locked loops for optical temporal phase and instantaneous frequency measurements at the quantum limit.

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1. Introduction

Accurate measurements of a temporally varying optical phase and instantaneous frequency are important for coherent optical sensing, metrology, and communication applications. Such measurements are ultimately limited by quantum uncertainties. Although extensive research concerning the quantum limits for single-mode phase estimation has been done, few have considered the more general and practical problem of temporal phase estimation.

In this paper, we propose a general state-variable approach to the design of homodyne phase-locked loops (PLLs) for quantum-limited temporal phase and instantaneous frequency estimation. Berry and Wiseman have previously considered a simple homodyne PLL scheme when the phase to be estimated is a Wiener random process [1], while we have recently proposed a frequency-domain approach to homodyne PLL design when the phase is any stationary Gaussian random process [2]. The state-variable approach described in this paper unifies these two distinct approaches and can be applied to a much wider class of Gaussian random processes.

2. Homodyne phase-locked loop design by Kalman-Bucy filtering

Let the mean optical phase that we wish to estimate be a solution of stochastic differential equations,

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t), \qquad \bar{\phi}(t) = \mathbf{C}(t)\mathbf{x}(t), \qquad \langle \mathbf{u}(t)\otimes\mathbf{u}(t')\rangle = \mathbf{U}\delta(t-t'). \tag{1}$$

Upon homodyne detection, a coherent state can be regarded as a classical signal with additive white Gaussian noise,

$$p(t) = \sin[\bar{\phi}(t) - \tilde{\phi}(t)] + w(t), \qquad \langle w(t)w(t')\rangle = W(t)\delta(t-t'), \qquad W = \frac{\hbar\omega_0}{4\mathscr{P}}. \tag{2}$$

where $\tilde{\phi}$ is the local-oscillator phase and $\mathscr{P}$ is the average optical power. For other quantum states, $w(t)$ cannot be regarded as an additive white Gaussian noise, but it may still be desirable to approximate it as one to take advantage of existing classical estimation techniques. Linearizing $p \approx \bar{\phi} - \phi' + w$, one can use the Kalman-Bucy filtering theory to design the homodyne PLL, as shown in Fig. 1.

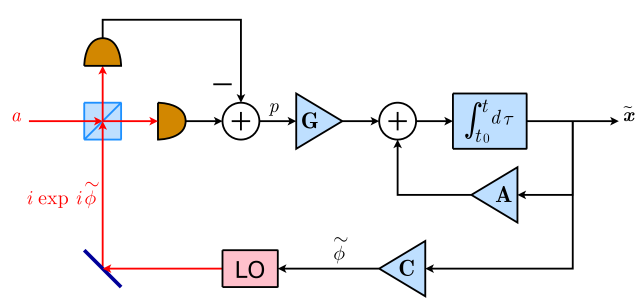

Fig. 1. A homodyne phase-locked loop that implements Kalman-Bucy filtering.

The optimal estimate of $\mathbf{x}$, defined as $\tilde{\mathbf{x}}$, the Kalman-Bucy gain $\mathbf{G}$, and the covariance matrix $\mathbf{P}$ are given by

$$\frac{d\tilde{\mathbf{x}}}{dt} = \mathbf{A}\tilde{\mathbf{x}} + \mathbf{G}\mathbf{p}, \quad \mathbf{G} = \mathbf{P}\mathbf{C}^T\mathbf{W}^{-1}, \quad \mathbf{P} \equiv \langle(\mathbf{x}-\tilde{\mathbf{x}})\otimes(\mathbf{x}-\tilde{\mathbf{x}})\rangle, \quad \frac{d\mathbf{P}}{dt} = \mathbf{A}\mathbf{P} + \mathbf{P}\mathbf{A}^T - \mathbf{P}\mathbf{C}^T\mathbf{W}^{-1}\mathbf{C}\mathbf{P} + \mathbf{B}\mathbf{U}\mathbf{B}^T, \tag{3}$$

where $\mathbf{p} = p$ is the homodyne output and $\mathbf{W} = W$. $\langle(\bar{\phi}-\tilde{\phi})^2\rangle = \mathbf{C}\mathbf{P}\mathbf{C}^T \ll 1$ is required to ensure that the linearized analysis is self-consistent. For instantaneous frequency estimation, we can define $\bar{\phi} \propto x_1$ and $d\bar{\phi}/dt \propto x_2 \propto dx_1/dt$ and estimate x_2. For time-invariant systems, Kalman-Bucy filtering at steady state is equivalent to the Wiener filtering proposed in [2]. For example, if $\bar{\phi}$ is an Ornstein-Uhlenbeck random process, $d\bar{\phi}/dt = -k\bar{\phi} + Bu$ and let $\kappa \equiv B^2U$, the steady-state phase variance is given by

$$\langle(\bar{\phi}-\tilde{\phi})^2\rangle = \frac{\hbar\omega_0 k}{4\mathscr{P}}\left[\left(\frac{4\kappa\mathscr{P}}{\hbar\omega_0 k^2}+1\right)^{1/2}-1\right]. \tag{4}$$

In the limit of $k \to 0$, $\bar{\phi}$ becomes a Wiener random process and our result agrees with that derived by Berry and Wiseman [1].

3. Smoothing

If delay is permitted in the estimation, one can improve upon Kalman-Bucy filtering by taking into account more advanced measurements and employing smoothing techniques [3, 4]. This can be done by taking the Kalman-Bucy estimates $\tilde{\mathbf{x}}$ and covariance matrix $\mathbf{P}$ and solving the following differential equations backward in time,

$$\frac{d\mathring{\mathbf{x}}}{dt} = \mathbf{A}\mathring{\mathbf{x}} + \mathbf{B}\mathbf{U}\mathbf{B}^T\mathbf{P}^{-1}(\mathring{\mathbf{x}}-\tilde{\mathbf{x}}), \qquad \frac{d\mathbf{Q}}{dt} = (\mathbf{A}+\mathbf{B}\mathbf{U}\mathbf{B}^T\mathbf{P}^{-1})\mathbf{Q} + \mathbf{Q}(\mathbf{A}+\mathbf{B}\mathbf{U}\mathbf{B}^T\mathbf{P}^{-1})^T - \mathbf{B}\mathbf{U}\mathbf{B}^T, \tag{5}$$

where $\mathring{\mathbf{x}}$ is the optimal smoothing estimates, $\mathbf{Q} \equiv \langle(\mathbf{x}-\mathring{\mathbf{x}})\otimes(\mathbf{x}-\mathring{\mathbf{x}})\rangle$ is the smoothing covariance matrix, and the final conditions are $\mathring{\mathbf{x}}(T) = \tilde{\mathbf{x}}(T)$ and $\mathbf{Q}(T) = \mathbf{P}(T)$. For the Ornstein-Uhlenbeck random process, the smoothing error at steady state can be shown to be

$$\left\langle(\bar{\phi}-\mathring{\phi})^2\right\rangle = \mathbf{C}\mathbf{Q}\mathbf{C}^T = \frac{\kappa}{2k(4\kappa\mathscr{P}/\hbar\omega_0 k^2+1)^{1/2}}, \tag{6}$$

which agrees with that derived in Ref. [2] and is approximately 3dB smaller than the Kalman-Bucy filtering error. In the limit of $k \to 0$, the error is exactly 3dB smaller than that derived by Berry and Wiseman [1].

4. Conclusion

In conclusion, we have used a general state-variable approach to design homodyne PLLs for quantum-limited phase estimation. Since the use of homodyne PLL for single-mode phase estimation has already been experimentally demonstrated [5, 6], we expect our proposed measurement schemes to be experimentally realizable using current technology and useful for future metrology and communication applications.

5. References

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