Time-Symmetric Quantum Smoothing: A General Theory of Optimal Quantum Sensing

Mankei Tsang
Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
mankei@mit.edu

Abstract: The important problem of optimal waveform estimation for quantum sensing is solved using a time-symmetric approach. The theory generalizes prior work in classical and quantum estimation and can significantly out-perform previously proposed techniques.

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1. Introduction

As optical sensors become more sensitive, more compact, and more energy-efficient, quantum noise will eventually become prominent and hamper the sensing accuracy. Important optical sensing systems, such as gravitational wave detection and atomic magnetometry, are expected to reach their quantum limits in the near future. The optimal estimation of a signal coupled to an optical sensor in the presence of quantum noise is thus an important problem in quantum optics. Currently, the most popular approach to the problem is called quantum filtering or quantum trajectory theory, pioneered by Belavkin, Barchielli, Carmichael, and others [1]. The theory may be regarded as a generalization of the classical Bayesian filtering technique widely used in aeronautics, machine learning, quantitative finance, adaptive optics, image processing, and astronomy. Quantum filtering enables one to determine the quantum state of a sensor, such as a mechanical oscillator or an atomic spin ensemble, based on the outcomes of optical measurements, such as homodyne detection and polarimetry. From the conditional quantum state, one can then estimate the signal, such as a gravitational wave or a magnetic field, that shifts the sensor quantum state.

While quantum filtering has been tremendously successful in explaining quantum optics experiments and useful for detection, parameter estimation and control applications, it is well known in classical estimation theory that filtering is not the optimal way of estimating a signal. Instead of using real-time observations in the case of filtering, one can often obtain a more accurate estimate by delaying the estimation and taking the more advanced observations into account via a technique called smoothing. Smoothing is more accurate than filtering when the signal is a random process, and this is certainly the case for most optical sensors, where the signal usually evolves in time in a non-deterministic manner. It seems that a quantum generalization of smoothing is needed for optimal quantum sensing, but quantum mechanics is fundamentally a predictive theory and does not permit one to define a quantum state based on prior as well as posterior observations. While the superior performance of delayed estimation has been shown in the Gaussian case for quantum optical phase measurements with phase-locked loops [2, 3] and also discovered by Petersen and Mølmer for atomic magnetometry [4], a general non-Gaussian theory is crucial for the accurate modeling of quantum sensors. It is also unclear how one can incorporate the stochastic evolution of the signal into the estimation procedure in general. All these difficulties have been solved by the recently proposed quantum smoothing theory [5–7].

2. Time-symmetric quantum smoothing

The proposed theory defines the classical signal and the quantum sensor as one hybrid super-system and uses not one but *two* density operators to describe the system: the usual predictive one conditioned upon prior observations and also a *retrodictive* one conditioned upon posterior observations. Interestingly, the operators obey a pair of adjoint equations, one to be solved forward in time and one backward in time [5–7]. The forward equation for the unnormalized predictive density operator is a generalization of the Duncan-Mortensen-Zakai equation in classical filtering and the Belavkin quantum filtering equation [1], while the backward equation is simply the adjoint of the forward one. Both Gaussian measurements [5, 6], such as homodyne detection, and Poissonian measurements [7], such as photon counting, can be taken into account. The final smoothing estimates and the associated errors can then be determined by combining the two operators. This quantum smoothing theory may be regarded as a generalization of the classical nonlinear smoothing theory proposed by Pardoux [8] and degenerates to the classical theory in appropriate limits.

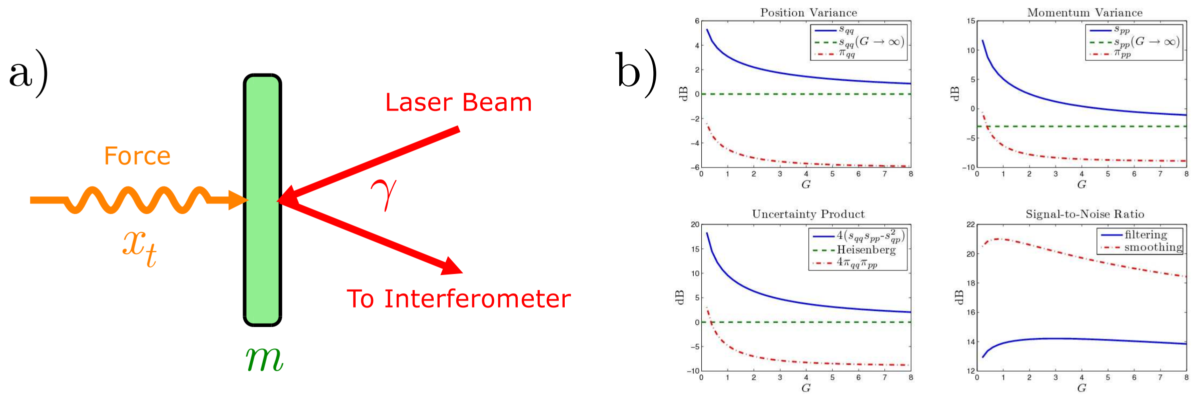

Fig. 1. a) Schematic of the force estimation problem. The force obeys the Itō equation $dx_t = ax_t dt + bdW_t$, a and b being known parameters and dW_t being a Wiener increment. b) Numerically calculated position variances (top-left), momentum variances (top-right), and force estimation signal-to-noise ratios (bottom-right) for filtering (solid lines) and smoothing (dash-dot) at steady state in normalized logarithmic scale for $a/\sqrt{b} = 0.01/(\hbar m)^{1/4}$ with respect to a normalized measurement strength parameter $G \equiv \hbar^{3/2}\gamma/(m^{1/2}b)$. The bottom-left figure plots the position-momentum uncertainty product. Dash lines are zero-force limits for filtering.

As an example, consider the well studied problem of force estimation, as shown in Fig. 1a). The force is assumed to be an Ornstein-Uhlenbeck process, acting on a free, one-dimensional quantum mechanical object. The object position is continuously monitored using a coherent laser beam and an interferometer or a phase-locked loop. The measurement strength is characterized by a parameter γ, which is proportional to the optical power. The problem can be converted to an equivalent classical Gaussian smoothing problem via the use of Wigner distributions, in which case well known linear smoothers can be used. Figure 1b) plots the numerically calculated position and momentum variances and force estimation signal-to-noise ratios at steady state in normalized logarithmic scale. As expected, the smoothing variances are much lower than the filtering ones, while the smoothing signal-to-noise ratio is at most 7 dB higher in this example. Intriguingly, the product of the smoothing position and momentum variances can go below that allowed by the Heisenberg uncertainty relation. This counter-intuitive phenomenon has been discussed at length in Refs. [3, 6, 7]. Regardless of how one interprets the violation of the Heisenberg relation in an epistemological context, the force estimation accuracy enhancement is real and can be confirmed experimentally in future quantum force sensors.

3. Conclusion and acknowledgments

The use of two density operators neatly solves the problem of optimal estimation for quantum sensing and completes the intriguing correspondence between classical Bayesian estimation theory and quantum mechanics. The theory is expected to be useful not only for optical sensing applications, such as gravitational wave sensing, atomic magnetometry, and optical phase-locked loop design, but also for all quantum sensors in general. Discussions with Jeff Shapiro, Seth Lloyd, Carl Caves, JM Geremia, Ivan Deustch, and Andrew Landahl are gratefully acknowledged. This work was financially supported by the Keck Foundation Center for Extreme Quantum Information Theory.

4. References

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