Inverse design mechanism of cylindrical cloaks without knowledge of the required coordinate transformation

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An inverse way to define the parameters of cylindrical cloaks is developed, in which the cloaking parameters can be independently obtained without any knowledge of the corresponding coordinate transformation. The required parameters are derived in terms of the integral form of *cloaking generators*, which are very general and allow us to examine the significance of the parametric profiles. The validity of such inverse way and the invisibility characteristics are presented in full-wave numerical simulation of plane wave scattering by cloaked cylinders. © 2010 Optical Society of America

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1. INTRODUCTION

Designing invisibility cloaks has received increasing interest from both engineering and scientific communities. Coordinate transformation is commonly used to control electromagnetic fields and render an object invisible to electromagnetic radiation [1,2]. This approach is generalized from the cloaking in terms of the conductivity [3], and further applied to acoustics and electromagnetics [4,5].

Given a virtual space with unit parameters, coordinate transformation performs the transition from the nonscattering virtual space to the realistic physical space, which in turn does not interact with the incident radiation. This is because the coordinate transformation method relies on the invariance of Maxwell's equations throughout the spatial transformation. This method can thus be applied to other classical waves, e.g., pressure waves [6] and surface liquid waves [7], excluding elastic waves [8], provided that the symmetry invariance of the corresponding wave equation is maintained under the coordinate transformation. Therefore, traditional cloaks need to be anisotropic and inhomogeneous. The anisotropy and the inhomogeneity of the cloak, as the deformation of the space, bend the wavefront around the cloaked object and enable waves to emerge on the other side along the propagation direction without any disturbance. Based on this transformation technique, cylindrical cloaks with circular [9], elliptical [10], and arbitrary cross-sections [11,12] have been studied, and practical attempts to realize the cylindrical cloak have been made with promising experimental results in microwave [13] and optical [14] regimes. The three-dimensional spherical cloak has been studied by extended Mie theory [15,16] and Von Neumann's method [17] for first- and higher-order transformations, respectively. The realization of spherical cloaks by multilayered isotropic coatings has been proposed [18].

Traditionally, in order to derive the anisotropic parameters for the cloaks, one must know the coordinate transformation beforehand. However, here we present an inverse methodology to determine the required cloaking parameters for cylindrical cloaks without knowing specific coordinate transformations. Once any parameter is given, the remaining parameters can be derived in integral form associated with the introduced cloaking generator. In this method, it is not even necessary to know the exact profile of the first given parameter. The cloaking generator, together with the boundary conditions, in fact, replaces the corresponding spatial transformation. It provides us an appropriate tool with promising applications in designing non-ideal cloaks and finding optimal performance under the same discretization criteria [19]. Note that for non-ideal discretized cloak models, the selection of a proper set of cloaking parameters (associated with a certain transformation function) does matter, especially when the discretization is not fine because of the practical costs in fabrication. However, it would be quite inefficient to examine one set of parameters by specifying one transformation function. Instead of considering the coordinate transformation, our inverse design mechanism enables us to directly envisage the importance of the profile of just one parameter, on the basis of which the remaining can be analytically and uniquely determined. Therefore, we can easily optimize groups and groups of parameter sets without turning to coordinate transformations. Full-wave simulation results are provided for verification. This work reverses the traditional design procedures and may be a step forward in the search of desirable cylindrical cloaks via parametric profiles in non-ideal situations.

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2. INVERSE DESIGN OF CLOAKING PARAMETERS

For impedance matching purposes, both the relative permittivity and permeability tensors of the cylindrical cloak are assumed to be equal, i.e., $\bar{\boldsymbol{\epsilon}}(r)=\bar{\boldsymbol{\mu}}(r)=\bar{\boldsymbol{\zeta}}(r)=\zeta_r(r)\hat{\mathbf{r}}\hat{\mathbf{r}}+\zeta_\varphi(r)\hat{\boldsymbol{\varphi}}\hat{\boldsymbol{\varphi}}+\zeta_z(r)\hat{\mathbf{z}}\hat{\mathbf{z}}$. First of all, we know only that a cylindrical cloak is designed by compressing the virtual region ($\Omega'(r')$ for $r'<b$) into the physical region ($\Omega(r)$ for $a<r<b$) via an *unknown* transformation, as Fig. 1 shows, where the prime corresponds to the virtual space.

The variables of a and b are the inner and outer radius of the cylindrical cloak in the physical space, respectively. It is assumed that the spatial compression is only with respect to the radial direction, and thus the Jacobian matrix is diagonal, though we still have no information of the specific form of the coordinate transformation. As a result, one can obtain [20] (an earlier precursor in coordinate transformation [1])

$$\bar{\boldsymbol{\zeta}}(r)=\begin{bmatrix}\zeta_r(r) & 0 & 0\\ 0 & \zeta_\varphi(r) & 0\\ 0 & 0 & \zeta_z(r)\end{bmatrix}=\begin{bmatrix}\lambda_r/(\lambda_\varphi\lambda_z) & 0 & 0\\ 0 & \lambda_\varphi/(\lambda_r\lambda_z) & 0\\ 0 & 0 & \lambda_z/(\lambda_r\lambda_\varphi)\end{bmatrix}, \tag{1}$$

where

$$\lambda_r=\frac{dr}{dr'},$$

$$\lambda_\varphi=\frac{r}{r'},$$

$$\lambda_z=1, \tag{2}$$

denote three principal stretches of the Jacobian matrix. The detailed derivations of the above equations are suppressed for compactness: they can be found in [21]. Then three equations can be derived:

$$\zeta_r(r)\zeta_\varphi(r)=1,$$

$$\zeta_r(r)\zeta_z(r)=\frac{r'^2}{r^2},$$

$$\sqrt{\zeta_\varphi(r)\zeta_z(r)}=\frac{dr'}{dr}. \tag{3}$$

Fig. 1. (Color online) Scheme of the proposed inverse design mechanism for cylindrical cloaks. The virtual space denoted by $\Omega'(r')$ ($r'<b$) is compressed into the physical space $\Omega(r)$ denoted by the shell on the right (blue online) ($a<r<b$), in which the required coordinate transformation is not specified. Based on the cloaking generator and the properties of the inverse mechanism, all the cloaking parameters in Eq. (1) can be determined uniquely. More importantly, the initially "indefinite" transformation can be revealed in turn.

By manipulating Eqs. (3) and eliminating the term $\zeta_\varphi(r)$, we derive the differential equation regarding radial and transverse parameters:

$$r\sqrt{\zeta_r(r)\zeta_z(r)}\frac{d[r\sqrt{\zeta_r(r)\zeta_z(r)}]}{dr}=r\zeta_z(r). \tag{4}$$

Integrating Eq. (4), one has

$$r^2\zeta_z(r)\zeta_r(r)=C+\int_a^r 2r_1\zeta_z(r_1)dr_1, \tag{5}$$

where C is the integration constant. Due to the spatial compression ($r'=0$ when $r=a$), it can be seen in Eq. (3) that $\zeta_r(a)\zeta_z(a)=0$, resulting in $C=0$. Another requirement ($r'=b$ when $r=b$) leads to the normalization condition for the z-component parameter,

$$b^2=\int_a^b 2r_1\zeta_z(r_1)dr_1, \tag{6}$$

which plays an important role in finding cloaking parameters. Here, we introduce *cloaking generator* $g(r)$ proportional to $\zeta_z(r)$, i.e., $g(r)=C_0\zeta_z(r)$, where C_0 is an arbitrary constant. Then ζ_z can be represented in the form

$$\zeta_z(r)=\frac{b^2g(r)}{2\int_a^b r_1g(r_1)dr_1}. \tag{7}$$

Radial and azimuthal parameters can be expressed as

$$\zeta_r(r)=\frac{2\int_a^r r_1g(r_1)dr_1}{r^2g(r)},$$

$$\zeta_\varphi(r)=\frac{r^2g(r)}{2\int_a^r r_1g(r_1)dr_1}. \tag{8}$$

Only after all parameters are determined can the corresponding coordinate transformation for such cylindrical cloaks be found, which is in contrast to the existing design approaches of transformation based cloaks:

$$r'=b\sqrt{\frac{\int_a^r r_1g(r_1)dr_1}{\int_a^b r_1g(r_1)dr_1}}. \tag{9}$$

One may consider some interesting situations. When two permittivities out of three are equal (i.e., $\zeta_r=\zeta_\varphi$, or $\zeta_z=\zeta_\varphi$, or $\zeta_r=\zeta_z$), it can be shown that in each situation only the trivial cloak ($r'=r$) is possible. Thus, an ideal cylindrical cloak has to be realized for three different permittivities: $\epsilon_r \neq \epsilon_\varphi \neq \epsilon_z$ [13,22]. This is different from the case of spherical cloaks with only two different parameters [1,18].

3. IMPLEMENTATION AND NUMERICAL VERIFICATION

To demonstrate the effectiveness of the cloaking generator and the proposed inverse design method, we consider and focus on just three different profiles of the power generators, as given in Table 1. Certainly, one may design other profiles for the cloaking generator (e.g., parabolic, periodic, exponential, and even sinh profiles), but these are out of the scope of the current paper.

When $n=0$, g_1 and g_2 power cloak generators are identical, and their ζ_z components are constant. As a numerical example, we present two sets of near-field patterns for the power generators in Table 1, i.e., $n=1$ and $n=20$. We use different meshes for distinct n, but identical meshes for the power generators g_1, g_2, and g_3 with the same n.

In Fig. 2, it is obvious that at $n=1$, the g_1 power cloak exhibits a larger disturbance in the near-field pattern not only in the wavefront but also in the variation of magnitude, and both g_2 and g_3 outperform g_1 under the same condition (the meshing of $n=1$ is the same for g_1, g_2, and g_3) simulated in COMSOL Multiphysics. If the meshing can be extremely fine, there will be no disturbance for all, but this would require too much computer memory in simulation. This non-ideality of the simulation conditions can be considered as a non-ideal cloak consisting of a number of discrete cylindrical layers.

Also, at extremely high order (e.g., $n=20$), g_1 outperforms g_2 and g_3, since it results in smaller forward scattering. The field distribution oscillates severely near the outer radius in g_2 and g_3 power cloaks ($n=20$), implying that the meshing has to be finer in this region. It is noted that the field inside the cloak is more homogeneous at $n=20$ because at larger n the rays inside the cloaking shell will propagate very close to the outer radius.

The phenomenon of both g_2 and g_3 outperforming g_1 at $n=1$ can be explained by noting that the surface current effect on a perfect Pendry's cloak (i.e., $n=1$ [23,24]) will be lost, which all introduce additional scattering, because in numerical simulation it is impossible to implement extremely large values in the parameters. Therefore a transformation that can squeeze more space close to the outer boundary is better than other transformations. However, because of the discretization limit, the performance of the power cloaks g_2 and g_3 deteriorate much faster than that of the power cloak g_1 with increasing values of n, because of too much space squeezed close to the outer boundary (which in turn results in extremely finer meshing in simulation beyond the reach of a traditional computer's memory). Therefore g_1 outperforms g_2 and g_3 at $n=20$.

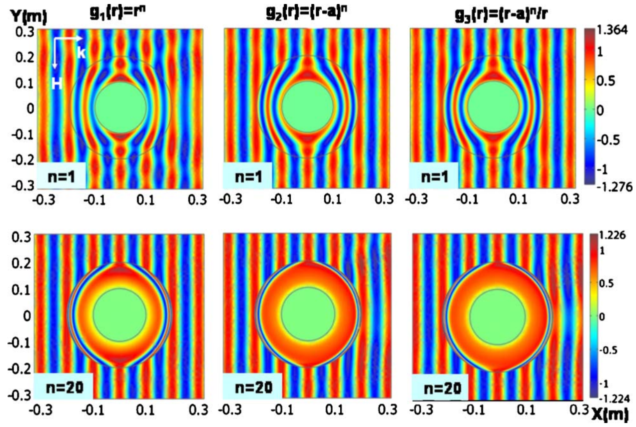

Fig. 2. (Color online) Total electric field Re$[E_z]$ on the x-y plane when $n=1$ (upper row) and $n=20$ (lower row) for the power cloak generators. The plane wave is propagating along the x-axis and its electric field is polarized along the z-axis. The working frequency is 3 GHz. The inner rod is a perfect electrical conductor of radius $a=\lambda$, and the outer radius is $b=2\lambda$.

From the transform point of view, if the order of the power cloak generator becomes larger, more portions of the original virtual space are transformed into the region close to the outer boundary of the physical space. In theory, the cloaking performance remains, while the field distribution inside the cloak is squeezed more into the outer radius. Thus, much finer meshing is needed close to the outer radius, and the cases of $n=20$ all adopt the same fine mesh. However, we cannot make the mesh extremely fine because of the limitation of computer memory. Moreover, considering that the order n can be any arbitrary value, there may exist an optimal range of n in each power cloak in which the invisibility performance will be further improved when the optimization is applied to the discretized model [25]. In other cloak designs other than power cloaks, it is possible to find such optimal cases as well. Although these characteristics are of technical importance, in our paper, we focus just on the proposed inverse method, which is practically useful to design cloak generators and determine the optimal cloak.

Table 1. Cloaking Parameters under Different Power Cloak Generators for $\zeta_z(r)$ Profile[a]

Generator Profile	$\zeta_z(r)$	$\zeta_r(r)$	$\zeta_\varphi(r)$	Implied Transformation
Power $g_1(r)=r^n$	$\frac{b^2 r^n(n+2)}{2(b^{n+2}-a^{n+2})}$	$\frac{2(r^{n+2}-a^{n+2})}{r^{n+2}(n+2)}$	$\frac{r^{n+2}(n+2)}{2(r^{n+2}-a^{n+2})}$	$b\sqrt{\frac{r^{n+2}-a^{n+2}}{b^{n+2}-a^{n+2}}}$
Power $g_2(r)=(r-a)^n$	$\frac{b^2(r-a)^n}{2(b-a)^n T_n(b)}$	$\frac{2T_n(r)}{r^2}$	$\frac{r^2}{2T_n(r)}$	$b(\frac{r-a}{b-a})^{n/2}\sqrt{\frac{T_n(r)}{T_n(b)}}$
[b]Power $g_3(r)=(r-a)^n/r$	$\frac{b^2(n+1)(r-a)^n}{2r(b-a)^{n+1}}$	$\frac{2(r-a)}{(n+1)r}$	$\frac{(n+1)r}{2(r-a)}$	$b(\frac{r-a}{b-a})^{(n+1)/2}$

[a]The symbol n is an arbitrarily positive constant. We introduce $T_n(r)=(r-a)(a+r+nr)/(2+3n+n^2)$ to simplify the expressions corresponding to the second power profile.

[b]In $g_3(r)$, it is simply Pendry's classic cylindrical cloak when $n=1$.

4. SUMMARY

In this paper, we report a method for designing cylindrical cloaks based on an inverse mechanism to derive the cloaking parameters without knowing the required coordinate transformation first. The numerical result confirms the validity of the proposed approach, and also brings the community's attention to an alternative design method for various cylindrical cloaks. In theory, the proposed inverse method provides a new perspective to design cylindrical cloaks. Moreover, it is particularly useful in determining the optimized cloaking parameters in non-ideal discretized models. Theoretically, all profiles of parameters obtained are equally perfect, while in non-ideal cases they are not, since the discretization has to be finite and the layer number cannot be too large, considering fabrication costs. This method can lead us to optimize profiles of cloaking parameters providing minimal radar cross-sections instead of examining one specific set of parameters calculated from a given coordinate transformation each time. A modified inverse method to avoid the singularity at the inner boundary is under further investigation.

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