Properties of Faraday chiral media: Green dyadics and negative refraction

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Selected properties of generalized Faraday chiral media are thoroughly studied in this paper where Green's dyadics are formulated for unbounded and layered structures, and the possibility of negative refractive index, the backward eigenwaves, and quantum vacuum are also investigated. After a general representation of the Green's dyadics is obtained, the scattering coefficients of the Green's dyadics are determined from the boundary conditions at each interface and are expressed in a greatly compact form of recurrence matrices. In the formulation of the Green's dyadics and their scattering coefficients, three cases are considered, i.e., the current source is immersed in (i) the intermediate, (ii) the first, and (iii) the last regions, respectively. We present here layered dyadic Green's functions for generalized Faraday chiral media. This kind of Faraday chiral media can also be manipulated to achieve negative refraction and possible backward wave propagation is presented as well. As compared to the existing results, the present work mainly contributes: (1) the exact representation of the dyadic Green's functions, with irrotational part extracted out, for the gyrotropic Faraday chiral medium in multilayered geometry; (2) the general DGFs and scattering coefficients which can be reduced to either layered chiroferrite, chiroplasma or other simpler cases; and (3) negative refractive index and backward waves achieved with less restriction and more advantages compared to chiral media.

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I. INTRODUCTION

Recently, composite materials have attracted considerable attention in various areas.[1–4] Among these materials, a double negative material (DNG)[5] exhibits a left-handedness ruling the polarizations of electric and magnetic fields which is referred to as left-handed materials.[6–10] Those materials can possess negative refraction of the waves and thus are considered to be negative-index media (NIM) which open new avenues to achieving unprecedented physical properties and functionality unattainable with natural materials.[11–14] Metamaterials with negative refraction in microwave region and related applications have been studied, including metamaterial waveguides,[15] SRR and spiral resonators,[16,17] leaky wave antennas,[18] and subwavelength cavity resonators.[19] Since the negative refraction by the artificial NIM was experimentally verified by Shelby,[20] more studies on metamaterials have been carried out such as tensor-parameter retrieval using quasistatic Lorentz theory,[21] *S*-parameter retrieval using the plane wave incidence,[22] and constitutive relation retrieval using the transmission line method.[23,24]

In this paper, we propose a different way to achieve negative refraction and backward waves by using Faraday chiral media. Although some recent works[25,26] have shown that chiral media can also exhibit negative refraction, this chiral route still has many limitations (e.g., pure chirality, both permittivity and permeability tending zero and chiral nihility only valid at or near a resonant frequency). Herein, in particular, we examine the Faraday chiral medium with gyrotropy, which is characterized in a rectangular coordinate system by chirality ξ_c, $\overline{\boldsymbol{\epsilon}}$, and $\overline{\boldsymbol{\mu}}$ as follows:

$$\boldsymbol{D} = \overline{\boldsymbol{\epsilon}} \cdot \boldsymbol{E} + i\xi_c B, \tag{1}$$

$$\boldsymbol{H} = i\xi_c E + \overline{\boldsymbol{\mu}}^{-1} \cdot B, \tag{2}$$

where

$$\overline{\boldsymbol{\epsilon}} = \begin{bmatrix} \epsilon & -ig & 0 \\ ig & \epsilon & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \tag{3}$$

$$\overline{\boldsymbol{\mu}} = \begin{bmatrix} \mu & -iw & 0 \\ iw & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix}. \tag{4}$$

It is found that these gyrotropic Faraday chiral media have advantages over normal chiral media: (i) negative index of refraction in a gyrotropic chiral medium can be realized with less restrictions, since the refractive indices are greatly reduced by those gyrotropic parameters; (ii) two backward eigenwaves are found in certain frequency bands; and (iii) the parameters in permittivity and permeability tensors and chirality admittance can be positive even when negative refraction occurs. The negative-index effects and eigenwaves propagating in a backward direction can be realized from gyrotropic Faraday chiral media as shown in this paper. Further, we will consider potential applications in NIM, phase conservation, and quantum fields.

Since the interaction between materials and electromagnetic waves is another important aspect in material characterization, Green dyadics are of particular interest for Faraday chiral media, which can describe the wave interaction in a macroscopic view. Dyadic Green's functions,[27] which relate directly the radiated electromagnetic fields and the source distribution, provide a good way to characterize the macroscopic performance of artificial complex media including metamaterials. DGFs play an important role in solving both source-free and source-incorporated boundary value problems for electromagnetic scattering, radiation, and propagation.[28] However, DGFs in complex media like gyrotropic or metamaterials have never been studied especially in multilayered structures, though the DGFs for some isotropic,[29] chiral,[30] anisotropic,[31] chiroplasma,[32] and bianisotropic[33–35] media have been formulated over the last three decades. The technique of eigenfunctional expansion provides a systematic approach in electromagnetic theory for interpreting various electromagnetic representations;[36] most importantly, it is applicable in almost all the fundamental coordinates. Even in the cylindrical structure considered in detail in this paper, the eigenfunctional expansion technique can provide an explicit form of the dyadic Green's functions, so that it becomes easy and convenient when the source distribution is independent from the azimuth directions or when the far-zone fields are computed.

Different from the existing work, this paper aims at two important points, namely, Green dyadics and negative refraction index. Physical properties of materials are often described by using these two concepts. For the part of DGFs, we achieve (i) the direct development of the unbounded dyadic Green's functions with the irrotational part extracted out in an unbounded gyrotropic Faraday chiral medium where the eigenfunctional expansion technique is employed, and (ii) the formulations of the scattering dyadic Green's functions and their coefficients in layered gyrotropic Faraday chiral media where each layer can be stratified with arbitrary thickness and material parameters. For the negative-index part, after DGFs have been obtained, we also realize some characteristics of metamaterials (e.g., backward waves and negative refraction) by using this kind of material with a number of advantages and less limitations. Some potential applications are suggested.

Throughout the paper, a time dependence $e^{-i\omega t}$ is assumed but always suppressed.

II. DGFS FOR UNBOUNDED GYROTROPIC FARADAY CHIRAL MEDIA

A homogeneous gyrotropic Faraday chiral medium has been characterized as in Eq. (1). Experimentally, there might be some problems of effectively controlling the gyroelectric (g) and the gyromagnetic (w) parameters simultaneously. However, gyrotropic Faraday chiral materials may occur unintentionally in the fabrication process of chiroplasma and chiroferrite, and theoretical physics goes ahead often. Due to the generality of the material discussed in this paper, it is valuable to investigate the DGFs for this material in its multilayered structure as well as its potential ways to achieve negative refraction and characteristics of left-handed materials.

Substituting Eq. (1) into the source incorporated Maxwell's equations, we have

$$\nabla \times [\bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \boldsymbol{E}] - 2\omega\xi_c \nabla \times \boldsymbol{E} - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \boldsymbol{E} = i\omega \boldsymbol{J}. \quad (5)$$

The electric field can thus be expressed in terms of the DGF and electric source distribution as follows:

$$\boldsymbol{E}(\boldsymbol{r}) = i\omega \int_{V'} \bar{\boldsymbol{G}}_e(\boldsymbol{r},\boldsymbol{r}') \cdot \boldsymbol{J}(\boldsymbol{r}') dV', \quad (6)$$

where V' denotes the volume occupied by the source. Substituting Eq. (6) into Eq. (5) leads to

$$\nabla \times [\bar{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \bar{\boldsymbol{G}}_e] - 2\omega\xi_c \nabla \times \bar{\boldsymbol{G}}_e - \omega^2 \bar{\boldsymbol{\epsilon}} \cdot \bar{\boldsymbol{G}}_e = \bar{\boldsymbol{I}}\delta(\boldsymbol{r}-\boldsymbol{r}'), \quad (7)$$

where $\bar{\boldsymbol{I}}$ and $\delta(\boldsymbol{r}-\boldsymbol{r}')$ denotes the identity dyadic and Dirac delta function, respectively.

According to the well-known Ohm-Rayleigh method, the source term in Eq. (7) can be expanded in terms of the solenoidal and irrotational cylindrical vector wave functions in cylindrical coordinates. Thus, we obtain

$$\bar{\boldsymbol{I}}\delta(\boldsymbol{r}-\boldsymbol{r}') = \int_0^\infty d\lambda \int_{-\infty}^{\infty} dh \sum_{n=-\infty}^{\infty} [\boldsymbol{M}_n(h,\lambda)\boldsymbol{A}_n(h,\lambda) + \boldsymbol{N}_n(h,\lambda)\boldsymbol{B}_n(h,\lambda) + \boldsymbol{L}_n(h,\lambda)\boldsymbol{C}_n(h,\lambda)], \quad (8)$$

where the vector wave functions $\boldsymbol{M}$, $\boldsymbol{N}$, and $\boldsymbol{L}$ in a cylindrical coordinate system are defined

$$\boldsymbol{M}_n(h,\lambda) = \nabla \times [\Psi_n(h,\lambda)\hat{z}], \quad (9)$$

$$\boldsymbol{N}_n(h,\lambda) = \frac{1}{k_\lambda} \nabla \times \boldsymbol{M}_n(h,\lambda), \quad (10)$$

$$\boldsymbol{L}_n(h,\lambda) = \nabla[\Psi_n(h,\lambda)], \quad (11)$$

with $k_\lambda = \sqrt{\lambda^2+h^2}$, and the generating function given by $\Psi_n(h,\lambda) = J_n(\lambda\rho)e^{i(n\phi+hz)}$. The coefficients $\boldsymbol{A}_n(h,\lambda)$, $\boldsymbol{B}_n(h,\lambda)$, and $\boldsymbol{C}_n(h,\lambda)$ in Eq. (8) are to be determined from the orthogonality relations among the cylindrical vector wave functions. Therefore, scalar-dot multiplying both sides of Eq. (8) with $\boldsymbol{M}_{-n'}(-h',-\lambda')$, $\boldsymbol{N}_{-n'}(-h',-\lambda')$, and $\boldsymbol{L}_{-n'}(-h',-\lambda')$ each at a time and integrating them over the entire source volume, we obtain from the orthogonality that

$$\boldsymbol{A}_n(h,\lambda) = \frac{1}{4\pi^2\lambda}\boldsymbol{M}'_{-n}(-h,-\lambda), \quad (12)$$

$$\boldsymbol{B}_n(h,\lambda) = \frac{1}{4\pi^2\lambda}\boldsymbol{N}'_{-n}(-h,-\lambda), \quad (13)$$

$$\boldsymbol{C}_n(h,\lambda) = \frac{\lambda}{4\pi^2(\lambda^2+h^2)}\boldsymbol{L}'_{-n}(-h,-\lambda). \quad (14)$$

The unbounded dyadic Green's function can thus be expanded as follows:

$$\bar{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}') = \int_0^\infty d\lambda \int_{-\infty}^{\infty} dh \sum_{n=-\infty}^{\infty} [\boldsymbol{M}_n(h,\lambda)\boldsymbol{a}_n(h,\lambda) + \boldsymbol{N}_n(h,\lambda)\boldsymbol{b}_n(h,\lambda) + \boldsymbol{L}_n(h,\lambda)\boldsymbol{c}_n(h,\lambda)], \quad (15)$$

where the vector expansion coefficients $\boldsymbol{a}_n(h,\lambda)$, $\boldsymbol{b}_n(h,\lambda)$, and $\boldsymbol{c}_n(h,\lambda)$ are unknown vector coefficients to be determined from the orthogonality and permittivity and permeability tensors' properties. To obtain these unknown vectors, we substitute Eq. (15) and Eq. (8) into Eq. (7), noting the instinct properties of the vector wave functions of

$$\nabla \times \boldsymbol{N}_n(h,\lambda) = k_\lambda \boldsymbol{M}_n(h,\lambda), \tag{16}$$

$$\nabla \times \boldsymbol{M}_n(h,\lambda) = k_\lambda \boldsymbol{N}_n(h,\lambda), \tag{17}$$

$$\nabla \times \boldsymbol{L}_n(h,\lambda) = 0. \tag{18}$$

For the compactness of the subsequent manipulation, we define

$$\bar{\boldsymbol{\alpha}} = \bar{\boldsymbol{\mu}}^{-1} = \begin{bmatrix} \alpha_t & -\alpha_a & 0 \\ \alpha_a & \alpha_t & 0 \\ 0 & 0 & \alpha_z \end{bmatrix}, \tag{19}$$

where

$$\alpha_t = \frac{\mu}{\mu^2 - w^2}, \quad \alpha_a = \frac{-iw}{\mu^2 - w^2}, \quad \alpha_z = \frac{1}{\mu_z}. \tag{20}$$

By substituting Eq. (15) into Eq. (7), taking, respectively, the anterior scalar product with the vector wave equations, and performing the integration over the entire source volume, we can formulate the equations satisfied by the unknown vectors and the known scalar and vector parameters in a matrix form as given by

$$[\Phi][X] = [\Theta], \tag{21}$$

where

$$[\Phi] = \begin{bmatrix} h^2\alpha_t + \lambda^2\alpha_z - \omega^2\epsilon & -\left(\frac{\omega^2 hg}{k_\lambda} + ik_\lambda h\alpha_a + 2\omega\xi_c k_\lambda\right) & i\omega^2 g \\ -\left(\frac{\omega^2 hg}{k_\lambda} + ik_\lambda h\alpha_a + 2\omega\xi_c k_\lambda\right) & k_\lambda^2\alpha_t - \omega^2\frac{h^2\epsilon + \lambda^2\epsilon_z}{k_\lambda^2} & -\omega^2\frac{ih}{k_\lambda}(\epsilon_z - \epsilon) \\ -i\omega^2\frac{\lambda^2}{k_\lambda^2}g & \omega^2\frac{ih\lambda^2}{k_\lambda^3}(\epsilon_z - \epsilon) & -\omega^2\frac{h^2\epsilon_z + \lambda^2\epsilon}{k_\lambda^2} \end{bmatrix}. \tag{22}$$

The quantities $[X]$ and $[\Theta]$ are known and parameter column vectors are given, respectively, by

$$[X] = [\boldsymbol{a}_n(h,\lambda), \boldsymbol{b}_n(h,\lambda), \boldsymbol{c}_n(h,\lambda)]^T,$$

$$[\Theta] = [\boldsymbol{A}_n(h,\lambda), \boldsymbol{B}_n(h,\lambda), \boldsymbol{C}_n(h,\lambda)]^T.$$

Solving Eq. (21), we have the solutions to $\boldsymbol{a}_n(h,\lambda)$, $\boldsymbol{b}_n(h,\lambda)$, and $\boldsymbol{c}_n(h,\lambda)$ as follows:

$$\boldsymbol{a}_n(h,\lambda) = \frac{1}{\Gamma}[\alpha_1\boldsymbol{A}_n(h,\lambda) + \beta_1\boldsymbol{B}_n(h,\lambda) + \gamma_1\boldsymbol{C}_n(h,\lambda)],$$

$$\boldsymbol{b}_n(h,\lambda) = \frac{1}{\Gamma}[\alpha_2\boldsymbol{A}_n(h,\lambda) + \beta_2\boldsymbol{B}_n(h,\lambda) + \gamma_2\boldsymbol{C}_n(h,\lambda)],$$

$$\boldsymbol{c}_n(h,\lambda) = \frac{1}{\Gamma}[\alpha_3\boldsymbol{A}_n(h,\lambda) + \beta_3\boldsymbol{B}_n(h,\lambda) + \gamma_3\boldsymbol{C}_n(h,\lambda)],$$

where

$$\Gamma = \epsilon_z\alpha_t(k_\lambda^2 - k_1^2)(k_\lambda^2 - k_2^2)/\alpha_z \tag{23}$$

and

$$k_{1,2}^2 = \frac{1}{2\epsilon_z\alpha_t/\alpha_z}[-p_\lambda \pm \sqrt{p_\lambda^2 + 4\epsilon_z\alpha_t/\alpha_z q_\lambda}] \tag{24}$$

with p_λ and q_λ given, respectively, below:

$$p_\lambda = \frac{1}{\alpha_z^2}\{(\alpha_t^2 + \alpha_a^2)h^2\epsilon - 4ih\alpha_a\epsilon\xi_c\omega - (4\epsilon\xi_c^2 + \epsilon\alpha_z\epsilon_z)\omega^2 + \alpha_t(g^2 - \epsilon^2)\omega^2 - \alpha_t\alpha_z h^2\epsilon_z\}, \tag{25}$$

$$q_\lambda = \frac{1}{\alpha_z^2}\{-(\alpha_t^2 + \alpha_a^2)h^4\epsilon_z + 4ih^2\alpha_a(2h\xi_c + g\omega)\epsilon_z\omega + \epsilon_z[4h^2\xi_c^2 + 4gh\xi_c\omega + (g^2 - \epsilon^2)\omega^2 + 2\alpha_t h^2\epsilon\epsilon_z]\omega^2\}. \tag{26}$$

It should be noted that the coupling coefficients β_1, γ_1, α_2, γ_2, α_3, and β_3 were assumed to be zero in Ref. 32. Here it is proved that those coupling coefficients must be considered in the formulation since they are not always zero, and the coupling coefficients $\alpha_{1,2,3}$, $\beta_{1,2,3}$, and $\gamma_{1,2,3}$ are given in detail below

$$\alpha_1 = \frac{\alpha_t}{\alpha_z^2}(h^2\epsilon_z + \lambda^2\epsilon) - \frac{1}{\alpha_z^2}\omega^2\epsilon\epsilon_z, \tag{27}$$

$$\alpha_2 = \beta_1 = \frac{1}{k_\lambda\alpha_z^2}[ih\alpha_a(h^2\epsilon_z + \lambda^2\epsilon) + 2\xi_c(h^2\epsilon_z + \epsilon\lambda^2)\omega + hg\epsilon_z\omega^2], \tag{28}$$

$$\gamma_1 = -\frac{k_\lambda^2}{\lambda^2}\alpha_3 = \frac{i}{\alpha_z^2}[gk_\lambda^2\alpha_t + ih^2\alpha_a(\epsilon - \epsilon_z) + 2h\xi_c(\epsilon - \epsilon_z)\omega - g\epsilon_z\omega^2], \tag{29}$$

$$\gamma_2 = -\frac{k_\lambda^2}{\lambda^2}\beta_3 = \frac{i}{k_\lambda}\left[h\left(h^2\frac{\alpha_t}{\alpha_z} + \lambda^2\right)(\epsilon - \epsilon_z)/\alpha_z + ighk_\lambda^2\alpha_a/\alpha_z^2 + 2g\xi_c k_\lambda^2\omega/\alpha_z^2 - h(\epsilon^2 - \epsilon\epsilon_z - g^2)\omega^2/\alpha_z^2\right], \tag{30}$$

$$\beta_2 = \frac{1}{k_\lambda^2}\left[\left(h^2\frac{\alpha_t}{\alpha_z} - \lambda^2\right)(h^2\epsilon_z + \lambda^2\epsilon)/\alpha_z - (h^2\epsilon\epsilon_z + \lambda^2(\epsilon^2 - g^2))\omega^2/\alpha_z^2\right], \tag{31}$$

$$\gamma_3 = \frac{1}{\omega^2}\left\{-k_\lambda^2\left(h^2\frac{\alpha_t^2 + \alpha_a^2}{\alpha_z^2} + \lambda^2\frac{\alpha_t}{\alpha_z}\right) + 4ihk_\lambda^2\alpha_a\xi_c\omega/\alpha_z^2 + \left[k_\lambda^2\epsilon\alpha_t + \frac{h^2\alpha_t + \lambda^2\alpha_z}{k_\lambda^2}(h^2\epsilon + \lambda^2\epsilon_z) + 2ih^2g\alpha_a + 4k_\lambda^2\xi_c^2\right]\omega^2/\alpha_z^2 + 4gh\xi_c\omega^3/\alpha_z^2 + \frac{1}{k_\lambda^2}[h^2(g^2 - \epsilon^2) - \lambda^2\epsilon\epsilon_z]\omega^4/\alpha_z^2\right\}. \tag{32}$$

Note that there are some special relations between vectors $\boldsymbol{L}$ and $\boldsymbol{N}$ as shown below:

$$\boldsymbol{L}_n(h,\lambda) = \boldsymbol{L}_{nt}(h,\lambda) + \boldsymbol{L}_{nz}(h,\lambda), \tag{33}$$

$$\boldsymbol{L}'_{-n}(-h,-\lambda) = \boldsymbol{L}'_{-nt}(-h,-\lambda) + \boldsymbol{L}'_{-nz}(-h,-\lambda), \tag{34}$$

$$\boldsymbol{N}_n(h,\lambda) = \boldsymbol{N}_{nt}(h,\lambda) + \boldsymbol{N}_{nz}(h,\lambda), \tag{35}$$

$$\boldsymbol{N}'_{-n}(-h,-\lambda) = \boldsymbol{N}'_{-nt}(-h,-\lambda) + \boldsymbol{N}'_{-nz}(-h,-\lambda), \tag{36}$$

$$\boldsymbol{L}_{nt}(h,\lambda) = \frac{-ik_\lambda}{h}\boldsymbol{N}_{nt}(h,\lambda), \tag{37}$$

$$\boldsymbol{L}'_{-nt}(-h,-\lambda) = \frac{ik_\lambda}{h}\boldsymbol{N}'_{-nt}(-h,-\lambda), \tag{38}$$

$$\boldsymbol{L}_{nz}(h,\lambda) = \frac{ihk_\lambda}{\lambda^2}\boldsymbol{N}_{nz}(h,\lambda), \tag{39}$$

$$\boldsymbol{L}'_{-nz}(-h,-\lambda) = \frac{-ihk_\lambda}{\lambda^2}\boldsymbol{N}'_{-nz}(-h,-\lambda), \tag{40}$$

where the subscripts t and z denote, respectively, the transverse component and the longitude component and they apply similarly for the primed functions. Thus we can rewrite Eq. (15) as follows:

$$\bar{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}') = \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2\lambda\Gamma}\{\tau_1\boldsymbol{M}_n(h,\lambda)\boldsymbol{M}'_{-n}(-h,-\lambda) + \tau_2[\boldsymbol{M}_n(h,\lambda)\boldsymbol{N}'_{-nt}(-h,-\lambda) + \boldsymbol{N}_{nt}(h,\lambda)\boldsymbol{M}'_{-n}(-h,-\lambda)] + \tau_3[\boldsymbol{M}_n(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda) + \boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{M}'_{-n}(-h,-\lambda)] + \tau_4[\boldsymbol{N}_{nt}(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda) + \boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{N}'_{-nt}(-h,-\lambda)] + \tau_5\boldsymbol{N}_{nt}(h,\lambda)\boldsymbol{N}'_{-nt}(-h,-\lambda) + \tau_6\boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda)\}, \tag{41}$$

where the intermediates τ_1 to τ_6 are defined as follows:

$$\tau_1 = \alpha_1, \tag{42}$$

$$\tau_2 = \beta_1 + \frac{i\lambda^2}{k_\lambda h}\gamma_1, \tag{43}$$

$$\tau_3 = \beta_1 - \frac{ih}{k_\lambda}\gamma_1, \tag{44}$$

$$\tau_4 = \beta_2 - \frac{ih}{k_\lambda}\gamma_2 - \frac{ik_\lambda}{h}\beta_3 - \gamma_3, \tag{45}$$

$$\tau_5 = \beta_2 + \frac{i\lambda^2}{k_\lambda h}\gamma_2 - \frac{ik_\lambda}{h}\beta_3 + \frac{\lambda^2}{h^2}\gamma_3, \tag{46}$$

$$\tau_6 = \beta_2 - \frac{ih}{k_\lambda}\gamma_2 + \frac{ihk_\lambda}{\lambda^2}\beta_3 + \frac{h^2}{\lambda^2}\gamma_3. \tag{47}$$

By applying the idea of Tai[27] to obtain an exact expression of the irrotational term, we obtain from Eq. (8)

$$\hat{z}\hat{z}\delta(\boldsymbol{r}-\boldsymbol{r}') = \int_0^{\infty} d\lambda \int_{-\infty}^{\infty} dh \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2\lambda} \times \frac{k_\lambda^2}{\lambda^2}\boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda). \tag{48}$$

Apparently, the irrotational term of the unbounded DGF is contained in the $\boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda)$ dyadic hybrid modes.

After some lengthy but careful algebraic manipulations, we rewrite Eq. (41) in the following form

$$\bar{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}') = -\frac{\alpha_z}{\omega^2\epsilon_z\alpha_t}\hat{z}\hat{z}\delta(\boldsymbol{r}-\boldsymbol{r}') + \int_{-\infty}^{\infty} dh \int_0^{\infty} d\lambda \times \sum_{n=-\infty}^{\infty} \frac{1}{4\pi^2\lambda\Gamma}\{\tau_1 \boldsymbol{M}_n(h,\lambda)\boldsymbol{M}'_{-n}(-h,-\lambda) + \tau_2[\boldsymbol{M}_n(h,\lambda)\boldsymbol{N}'_{-nt}(-h,-\lambda) + \boldsymbol{N}_{nt}(h,\lambda)\boldsymbol{M}'_{-n}(-h,-\lambda)] + \tau_3[\boldsymbol{M}_n(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda) + \boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{M}'_{-n}(-h,-\lambda)] + \tau_4[\boldsymbol{N}_{nt}(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda) + \boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{N}'_{-nt}(-h,-\lambda)] + \tau_5 \boldsymbol{N}_{nt}(h,\lambda)\boldsymbol{N}'_{-nt}(-h,-\lambda) + \tau_7 \boldsymbol{N}_{nz}(h,\lambda)\boldsymbol{N}'_{-nz}(-h,-\lambda)\}, \tag{49}$$

where

$$\tau_7 = \beta_2 + \frac{1}{\omega^2\lambda^2}k_\lambda^2(k_\lambda^2 - k_1^2)(k_\lambda^2 - k_2^2) + \frac{h}{\lambda^2}(ik_\lambda\beta_3 + h\gamma_3) - \frac{ih}{k_\lambda}\gamma_2. \tag{50}$$

The first term of Eq. (49) is due to the contribution from the nonsolenoidal vector wave functions. The second integration term can be evaluated by making use of the residue theorem in the λ plane. This irrotational part of DGFs in a gyrotropic Faraday chiral medium is obtained for the first time when an eigenfunction expansion technique is applied. This irrotational part in specific cases agrees well with the previous solutions of a chiroplasma medium by letting $\alpha_z = \alpha_t = 1/\mu$ or an isotropic medium by letting $\epsilon_z = \epsilon$ further if we first set $g = w = 0$. The final expression of the unbounded DGFs is given after mathematical manipulations for $\rho \gtrless \rho'$,

$$\bar{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}') = -\frac{\alpha_z}{\omega^2\epsilon_z\alpha_t}\hat{z}\hat{z}\delta(\boldsymbol{r}-\boldsymbol{r}') + \frac{i}{4\pi}\int_{-\infty}^{\infty} dh \sum_{n=-\infty}^{\infty} \frac{\alpha_z}{\epsilon_z\alpha_t(k_1^2 - k_2^2)} \sum_{j=1}^{2} \frac{(-1)^{j+1}}{\lambda_j^2} \times \begin{cases} \boldsymbol{M}^{(1)}_{n,h}(\lambda_j)\boldsymbol{P}'_{-n,-h}(-\lambda_j) + \boldsymbol{Q}^{(1)}_{n,h}(\lambda_j)\boldsymbol{M}'_{-n,-h}(-\lambda_j) + \boldsymbol{U}^{(1)}_{n,h}(\lambda_j)\boldsymbol{N}'_{-nt,-h}(-\lambda_j) + \boldsymbol{V}^{(1)}_{n,h}(\lambda_j)\boldsymbol{N}'_{-nz,-h}(-\lambda_j); \\ \boldsymbol{M}_{n,h}(-\lambda_j)\boldsymbol{P}'^{(1)}_{-n,-h}(\lambda_j) + \boldsymbol{Q}_{n,h}(-\lambda_j)\boldsymbol{M}'^{(1)}_{-n,-h}(\lambda_j) + \boldsymbol{U}_{n,h}(-\lambda_j)\boldsymbol{N}'^{(1)}_{-nt,-h}(\lambda_j) + \boldsymbol{V}_{n,h}(-\lambda_j)\boldsymbol{N}'^{(1)}_{-nz,-h}(\lambda_j). \end{cases} \tag{51}$$

The vector functions $\boldsymbol{P}'_{-n,-h}(-\lambda_j)$, $\boldsymbol{Q}_{n,h}(\lambda_j)$, $\boldsymbol{U}_{n,h}(\lambda_j)$, and $\boldsymbol{V}_{n,h}(\lambda_j)$ in Eq. (51) are given, respectively, by

$$\boldsymbol{P}'_{-n,-h}(-\lambda_j) = \tau_1\boldsymbol{M}'_{-n,-h}(-\lambda_j) + \tau_2\boldsymbol{N}'_{-nt,-h}(-\lambda_j) + \tau_3\boldsymbol{N}'_{-nz,-h}(-\lambda_j), \tag{52}$$

$$\boldsymbol{Q}_{n,h}(\lambda_j) = \tau_2\boldsymbol{N}_{nt,h}(\lambda_j) + \tau_3\boldsymbol{N}_{nz,h}(\lambda_j), \tag{53}$$

$$\boldsymbol{U}_{n,h}(\lambda_j) = \tau_5\boldsymbol{N}_{nt,h}(\lambda_j) + \tau_4\boldsymbol{N}_{nz,h}(\lambda_j), \tag{54}$$

$$\boldsymbol{V}_{n,h}(\lambda_j) = \tau_4\boldsymbol{N}_{nt,h}(\lambda_j) + \tau_7\boldsymbol{N}_{nz,h}(\lambda_j). \tag{55}$$

Although the present results are more generalized and complicated, the present form of the DGFs expanded in terms of cylindrical vector wave functions is believed to be a correct one due to the rigorous formulation and multiple checks of the mathematical derivation processes. It can also be verified by reducing this generalized form to those forms for materials of simpler tensor forms such as gyroelectric and isotropic media. Also, this paper will contribute to both the unbounded Green's function and the scattering Green's functions due to the presence of the dielectric boundaries. Such problems have never been studied to our knowledge and the generalized results of DGFs with the application in left-handed materials (which will be discussed later) are reported.

III. FORMULATION OF SCATTERING DYADIC GREEN'S FUNCTIONS

In this section, we extend our theoretical analysis to derive scattering DGFs for the fth region assuming that the current source is located in the sth layer. As such that the scattering DGFs have a similar form as the unbounded DGF as given in Eq. (51), the expression for the scattering DGFs for each region of the layered gyrotropic Faraday chiral media can be constructed as follows:

$$\bar{\boldsymbol{G}}^{(fs)}_{es} = \bar{\boldsymbol{G}}_1 + \bar{\boldsymbol{G}}_2, \tag{56}$$

where the dyadics $\bar{\boldsymbol{G}}_j$ $(j=1,2)$ are given by Eq. (57),

$$\bar{\boldsymbol{G}}_j = \frac{i}{4\pi}\int_0^{\infty} dh \sum_{n=0}^{\infty} \frac{\alpha_{zs}(2-\delta_n^0)(-1)^{j+1}}{\epsilon_{zs}\alpha_{ts}(k_{1s}^2-k_{2s}^2)\lambda_{js}^2} \times \{(1-\delta_f^N)\boldsymbol{M}_{n,h}^{(1)}(\lambda_j^f)[(1-\delta_s^1)A_{Mj}^{fs}\boldsymbol{P}'_{-n,-h}(-\lambda_j^s) + (1-\delta_s^N)B_{Mj}^{fs}\boldsymbol{P}'^{(1)}_{-n,-h}(\lambda_j^s)] + (1-\delta_f^N)\boldsymbol{Q}_{n,h}^{(1)}(\lambda_j^f) \times[(1-\delta_s^1)A_{Qj}^{fs}\boldsymbol{M}'_{-n,-h}(-\lambda_j^s) + (1-\delta_s^N)B_{Qj}^{fs}\boldsymbol{M}'^{(1)}_{-n,-h}(\lambda_j^s)] + (1-\delta_f^N)\boldsymbol{U}_{n,h}^{(1)}(\lambda_j^f)[(1-\delta_s^1)A_{Uj}^{fs}\boldsymbol{N}'_{-nt,-h}(-\lambda_j^s) + (1-\delta_s^N)B_{Uj}^{fs}\boldsymbol{N}'^{(1)}_{-nt,-h} \times(\lambda_j^s)] + (1-\delta_f^N)\boldsymbol{V}_{n,h}^{(1)}(\lambda_j^f)[(1-\delta_s^1)A_{Vj}^{fs}\boldsymbol{N}'_{-nz,-h}(-\lambda_j^s) + (1-\delta_s^N)B_{Vj}^{fs}\boldsymbol{N}'^{(1)}_{-nz,-h}(\lambda_j^s)] + (1-\delta_f^1)\boldsymbol{M}_{n,h}(-\lambda_j^f)[(1-\delta_s^1)C_{Mj}^{fs}\boldsymbol{P}'_{-n,-h}(-\lambda_j^s) + (1-\delta_s^N)D_{Mj}^{fs}\boldsymbol{P}'^{(1)}_{-n,-h}(\lambda_j^s)] + (1-\delta_f^1)\boldsymbol{Q}_{n,h}(-\lambda_j^f)[(1-\delta_s^1)C_{Qj}^{fs}\boldsymbol{M}'_{-n,-h}(-\lambda_j^s) + (1-\delta_s^N)D_{Qj}^{fs}\boldsymbol{M}'^{(1)}_{-n,-h}(\lambda_j^s)] + (1-\delta_f^1)\boldsymbol{U}_{n,h}(-\lambda_j^f)[(1-\delta_s^1)C_{Uj}^{fs}\boldsymbol{N}'_{-nt,-h}(-\lambda_j^s) + (1-\delta_s^N)D_{Uj}^{fs}\boldsymbol{N}'^{(1)}_{-nt,-h}(\lambda_j^s)] + (1-\delta_f^1)\boldsymbol{V}_{n,h}(-\lambda_j^f)[(1-\delta_s^1)C_{Vj}^{fs}\boldsymbol{N}'_{-nz,-h}(-\lambda_j^s) + (1-\delta_s^N)D_{Vj}^{fs}\boldsymbol{N}'^{(1)}_{-nz,-h}(\lambda_j^s)]\}, \tag{57}$$

where multiple transmissions and reflections have been taken into account, $\lambda_{jf}=\sqrt{k_{jf}^2-h^2}$ and the subscript f means the fth region. The $ABCD$ coefficients where the superscripts and subscripts have been suppressed for compactness are scattering DGF coefficients to be determined from the boundary conditions. By considering the multiple transmissions and reflections, the scattering DGFs are thus constructed physically by inspecting Eq. (57) and taking into account all the possible physical modes in the presence of the multiple interfaces as shown in Fig. 1.

For instance, if the source is located in the first/last region (i.e., $1-\delta_s^1=0/1-\delta_s^N=0$), the wavelets in the scattering DGFs are only excited by inward-coming/outward-going wavelets with excitation functions $[\boldsymbol{P}'^{(1)}_{-n,-h}(\lambda_{js})$, $\boldsymbol{M}'^{(1)}_{-n,-h}(\lambda_{js})$, $\boldsymbol{N}'^{(1)}_{-nt,-h}(\lambda_{js})$, and $\boldsymbol{N}'^{(1)}_{-nz,-h}(\lambda_{js})]$ / $[\boldsymbol{P}'_{-n,-h}(-\lambda_{js})$, $\boldsymbol{M}'_{-n,-h}(-\lambda_{js})$, $\boldsymbol{N}'_{-nt,-h}(-\lambda_{js})$, and $\boldsymbol{N}'_{-nz,-h}(-\lambda_{js})]$. When the source point is located in any other layer, the excitation functions consist of both outward-going and inward-coming wavelets. If the observation point is in the first/last region (i.e., $1-\delta_f^1=0/1-\delta_f^N=0$), the field terms consist of only outward-going/inward-coming wavelets.

The layered structure is shown in Fig. 1.

Based on the principle of scattering superposition, we have

$$\bar{\boldsymbol{G}}_e^{(fs)}(\boldsymbol{r},\boldsymbol{r}') = \bar{\boldsymbol{G}}_0(\boldsymbol{r},\boldsymbol{r}')\,\delta_f^s + \bar{\boldsymbol{G}}_s^{(fs)}(\boldsymbol{r},\boldsymbol{r}'), \tag{58}$$

where $\bar{\boldsymbol{G}}_e$ and $\bar{\boldsymbol{G}}_0$ denote the total and unbounded electric DGFs, respectively, and superscripts f and s, respectively, denote the field point located in the fth region and source located in the sth region.

The boundary conditions that must be satisfied by the dyadic Green's function at the interface of regions f and $f+1$ at $\rho=\rho_f$ $(f=1,2,\ldots,N-1)$ are shown as follows:

$$\hat{\boldsymbol{\rho}} \times \bar{\boldsymbol{G}}_e^{(fs)}(\boldsymbol{r},\boldsymbol{r}') = \hat{\boldsymbol{\rho}} \times \bar{\boldsymbol{G}}_e^{[(f+1)s]}(\boldsymbol{r},\boldsymbol{r}'), \tag{59}$$

$$\hat{\boldsymbol{\rho}} \times [\bar{\boldsymbol{\alpha}}_f \cdot \boldsymbol{\nabla} \times \bar{\boldsymbol{G}}_e^{(fs)}(\boldsymbol{r},\boldsymbol{r}') - \omega\xi_{cf}\bar{\boldsymbol{G}}_e^{(fs)}(\boldsymbol{r},\boldsymbol{r}')] \tag{60}$$

$$= \hat{\boldsymbol{\rho}} \times [\bar{\boldsymbol{\alpha}}_{f+1} \cdot \boldsymbol{\nabla} \times \bar{\boldsymbol{G}}_e^{[(f+1)s]}(\boldsymbol{r},\boldsymbol{r}') - \omega\xi_{c(f+1)}\bar{\boldsymbol{G}}_e^{[(f+1)s]}(\boldsymbol{r},\boldsymbol{r}')].$$

A. Recurrence matrix equation of DGFs' scattering coefficients

To simplify the derivation of the general solution of these coefficients, we rewrite the boundary conditions in Eq. (3) into the matrix form subsequently. Now, it is clear that the equations to be obtained here for the layered Faraday chiral medium are different from those in all the previous work. By using the boundary conditions, a set of linear equations satisfied by scattering coefficients can be obtained and then represented by a series of compact matrices as follows:

$$[F_{lj(f+1)}] \cdot \{[\Upsilon_{lj(f+1)s}] + \delta_{f+1}^s[U_{(f+1)}]\} \tag{61}$$

$$= [F_{ljf}] \cdot \{[\Upsilon_{ljfs}] + \delta_f^s[D_f]\}.$$

The intermediate matrices in (61) are defined as follows:

$$[F_{Mjf}] = \begin{bmatrix} \partial\hbar_j & W_{M1} \\ \partial\hbar_j & W_{M2} \end{bmatrix}^T, \tag{62}$$

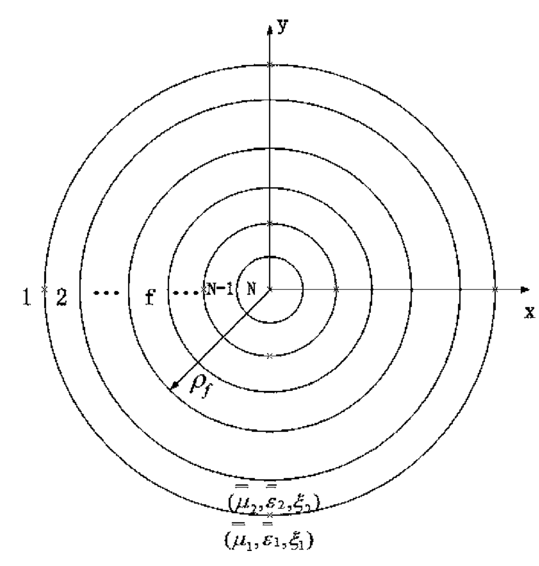

FIG. 1. geometry of cylindrically layered gyrotropic Faraday chiral media.

$$[F_{Ljf}] = \begin{bmatrix} \left(\frac{\tau_{pjf}nh}{\rho_f} + \tau_{qjf}\lambda_{jf}^2\right)\frac{\hbar_j}{k_{\lambda jf}} & W_{L1} \\ \left(\frac{\tau_{pjf}nh}{\rho_f} + \tau_{qjf}\lambda_{jf}^2\right)\frac{J_j}{k_{\lambda jf}} & W_{L2} \end{bmatrix}^T, \tag{63}$$

where

$$W_{M1} = \left(\frac{\alpha_{tf}nh}{\rho_f} + \alpha_{zf}\lambda_{jf}^2\right)\hbar_j - (\omega\xi_{cf} + ih\alpha_{af})\partial\hbar_j, \tag{64}$$

$$W_{M2} = \left(\frac{\alpha_{tf}nh}{\rho_f} + \alpha_{zf}\lambda_{jf}^2\right)J_j - (\omega\xi_{cf} + ih\alpha_{af})\partial J_j, \tag{65}$$

$$W_{L1} = \Delta_{qjf}^p\alpha_{tf}\partial\hbar_j + \left(\frac{ih\lambda_{jf}^2}{k_{\lambda jf}^2} - \frac{in}{\rho_f}\right)\Delta_{qjf}^p\alpha_{af}\hbar_j - \omega\xi_{cf}\left(\frac{hn\tau_{pjf}}{k_{\lambda jf}\rho_f} + \frac{\lambda_{jf}^2\tau_{qjf}}{k_{\lambda jf}}\right)\hbar_j, \tag{66}$$

$$W_{L2} = \Delta_{qjf}^p\alpha_{tf}\partial J_j + \left(\frac{ih\lambda_{jf}^2}{k_{\lambda jf}^2} - \frac{in}{\rho_f}\right)\Delta_{qjf}^p\alpha_{af}J_j - \omega\xi_{cf}\left(\frac{hn\tau_{pjf}}{k_{\lambda jf}\rho_f} + \frac{\lambda_{jf}^2\tau_{qjf}}{k_{\lambda jf}}\right)J_j. \tag{67}$$

As in the matrices $[F_{Ljf}]$, the subscript L denotes Q, U, or V, which come in pairs with Δ_{3jf}^2, Δ_{5jf}^4 or Δ_{7jf}^4, respectively, with the definition of

$$\Delta_{qjf}^p = \frac{h^2(\tau_{pjf} - \tau_{qjf}) + k_{\lambda jf}^2\tau_{qjf}}{k_{\lambda jf}}. \tag{68}$$

For simplicity, we define

$$\hbar_j = H_n^{(1)}(\lambda_{jf}\rho_f), \tag{69}$$

$$\partial\hbar_j = \left.\frac{d[H_n^{(1)}(\lambda_{jf}\rho)]}{d\rho}\right|_{\rho=\rho_f}, \tag{70}$$

$$J_j = J_n(\lambda_{jf}\rho_f), \tag{71}$$

$$\partial J_j = \left.\frac{d[J_n(\lambda_{jf}\rho)]}{d\rho}\right|_{\rho=\rho_f}. \tag{72}$$

The terms τ_{2jf}, τ_{3jf}, τ_{4jf}, τ_{5jf}, and τ_{7jf} are the weighting factors associated with the scattering coefficients A_{lj}^{fs} and B_{lj}^{fs} where $l=M$, Q, U, or V. They have the same forms as those in Eqs. (2) and (50) with the only change that each term relating to wave numbers (e.g., λ) will have a subscript of jf (e.g., λ_{jf}) and each term relating to material parameters (e.g., ϵ_z) will have a subscript of f (e.g., ϵ_{zf}) where $j=1$, 2 and f represents the fth region. The following matrices are also used in the formulation:

$$[\Upsilon_{lj,fs}] = \begin{bmatrix} A_{lj}^{fs} & B_{lj}^{fs} \\ C_{lj}^{fs} & D_{lj}^{fs} \end{bmatrix}, \tag{73}$$

$$[U_f] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \tag{74}$$

$$[D_f] = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \tag{75}$$

Defining the following transmission T matrix

$$[T_{ljf}] = [F_{lj,(f+1)f}]^{-1} \cdot [F_{lj,ff}], \tag{76}$$

where $[F_{lj,(f+1)f}]^{-1}$ is the inverse matrix of $[F_{lj,(f+1)f}]$, we rewrite the linear equation into

$$[\Upsilon_{lj,(f+1)s}] = [T_{ljf}] \cdot \{[\Upsilon_{lj,fs}] + \delta_f^s[D_f]\} - \delta_{f+1}^s[U_{(f+1)}]. \tag{77}$$

To shorten the expression, we also define

$$[T_{lj}^K]_{2\times 2} = [T_{lj,N-1}][T_{lj,N-2}]\cdots[T_{lj,K+1}][T_{lj,K}] = \begin{bmatrix} T_{lj,11}^K & T_{lj,12}^K \\ T_{lj,21}^K & T_{lj,22}^K \end{bmatrix}. \tag{78}$$

It should be noted that the coefficient matrices of the first and the last regions have the following specific forms

$$[\Upsilon_{lj,1s}] = \begin{bmatrix} A_{lj}^{1s} & B_{lj}^{1s} \\ 0 & 0 \end{bmatrix} \tag{79}$$

$$[\Upsilon_{lj,Ns}] = \begin{bmatrix} 0 & 0 \\ C_{lj}^{Ns} & D_{lj}^{Ns} \end{bmatrix}. \tag{80}$$

Then we utilize the previously obtained recursive formula to obtain all the coefficients of A_{lj}^{fs}, B_{lj}^{fs}, C_{lj}^{fs}, and D_{lj}^{fs}.

B. Specific applications: Three cases

To gain insight into the specific mathematical expressions of the physical quantities such as the transmission and reflection coefficient matrices, the following three cases are considered subsequently to demonstrate how these coefficients are determined by using the recursive algorithm when the source point is located in the first, the intermediate, and the last regions.

1. Source in an intermediate layer

$$[\Upsilon_{lj,1s}] = \begin{bmatrix} A_{lj}^{1s} & B_{lj}^{1s} \\ 0 & 0 \end{bmatrix}, \tag{81}$$

$$[\Upsilon_{lj,ms}] = \begin{bmatrix} A_{lj}^{ms} & B_{lj}^{ms} \\ C_{lj}^{ms} & D_{lj}^{ms} \end{bmatrix} \tag{82}$$

$$[\Upsilon_{lj,Ns}] = \begin{bmatrix} 0 & 0 \\ C_{lj}^{Ns} & D_{lj}^{Ns} \end{bmatrix}. \tag{83}$$

From Eq. (78), the recurrence equation becomes

$$[\Upsilon_{lj,fs}] = [T_{lj,f-1}]\cdots[T_{lj,s}]\{[T_{lj,s-1}]\cdots[T_{lj,1}][\Upsilon_{lj,1s}]\ u(f-s-1)[D_s] - u(f-s)[U_s]\}, \tag{84}$$

where $u(x-x_0)$ denotes the unit step function. When $f=N$, the coefficients for the first region are given by:

$$A_{lj}^{1s} = \frac{T_{lj,11}^{(s)}}{T_{lj,11}^{(1)}}, \quad (85)$$

$$B_{lj}^{1s} = -\frac{T_{lj,12}^{(s)}}{T_{lj,11}^{(1)}}. \quad (86)$$

For the last region, the coefficients are given by

$$C_{lj}^{Ns} = T_{lj,21}^{(1)} A_{lj}^{1s} - T_{lj,21}^{(s)}, \quad D_{lj}^{Ns} = T_{lj,21}^{(1)} B_{lj}^{1s} + T_{lj,22}^{(s)}. \quad (87)$$

Substituting Eqs. (85) to (87) into Eq. (84), the remaining coefficients can be obtained for the dyadic Green's functions. If the source is located in the first or last region (i.e., $s=1$ or N), the formulation of coefficients can be tremendously simplified.

2. Source in the first region

When the current source is located in the first region (i.e., $s=1$), the terms containing $(1-\delta_s^1)$ in Eq. (57) vanishes. The coefficient matrices in (73) and (79) will be further reduced to:

$$[\Upsilon_{lj,11}] = \begin{bmatrix} 0 & B_{lj}^{11} \\ 0 & 0 \end{bmatrix}, \quad (88)$$

$$[\Upsilon_{lj,m1}] = \begin{bmatrix} 0 & B_{lj}^{m1} \\ 0 & D_{lj}^{m1} \end{bmatrix}, \quad (89)$$

$$[\Upsilon_{lj,N1}] = \begin{bmatrix} 0 & 0 \\ 0 & D_{lj}^{N1} \end{bmatrix}, \quad (90)$$

where $m=2,3,\ldots,N-1$. It can be seen that only four coefficients for the first region and the last region, but eight coefficients for each of the remaining regions or layers, need to be solved for. By following Eq. (77), the recurrence relation for coeficients in the fth layer becomes

$$[\Upsilon_{lj,f1}] = [T_{lj,f-1}] \cdots [T_{lj,1}]\{[\Upsilon_{lj,11}] + [D_1]\}. \quad (91)$$

When $f=N$ in Eq. (91), a matrix equation satisfied by the coefficient matrices in Eqs. (88)–(90) can be obtained. The coefficients for the first region where the source is located (i.e., $s=1$) is given by:

$$B_{lj}^{11} = -\frac{T_{lj,12}^{(1)}}{T_{lj,11}^{(1)}}. \quad (92)$$

The coefficients for the last region can be derived in terms of the coefficients for the first region given by:

$$D_{lj}^{N1} = T_{lj,21}^{(1)} B_{lj}^{11} + T_{lj,22}^{(1)}. \quad (93)$$

The coefficients for the intermediate layers can be then obtained by substituting the coefficients in Eqs. (92) and (93) to Eq. (91). Thus, all the coefficients can be obtained by these procedures.

3. Source in the last region

When the current source is located in the first region (i.e., $s=N$), the coefficients are:

$$[\Upsilon_{lj,1N}] = \begin{bmatrix} A_{lj}^{1N} & 0 \\ 0 & 0 \end{bmatrix}, \quad (94)$$

$$[\Upsilon_{lj,mN}] = \begin{bmatrix} A_{lj}^{mN} & 0 \\ C_{lj}^{mN} & 0 \end{bmatrix}, \quad (95)$$

$$[\Upsilon_{lj,NN}] = \begin{bmatrix} 0 & 0 \\ C_{lj}^{NN} & 0 \end{bmatrix}. \quad (96)$$

From the recurrence equation in Eq. (77), similarly we have

$$[\Upsilon_{lj,fN}] = [T_{lj,f-1}] \cdots [T_{lj,1}][\Upsilon_{lj,1N}] - u(f-N)[U_N]. \quad (97)$$

By letting $s=N$, the coefficient for the first region is

$$A_{lj}^{1N} = \frac{1}{T_{lj,11}^{(1)}}. \quad (98)$$

And for the last region, it is found that

$$C_{lj}^{NN} = T_{lj,21}^{(1)} A_{lj}^{1N}. \quad (99)$$

Similarly, the rest of the coefficients can be obtained by substituting Eq. (98) and Eq. (99) into Eq. (97).

So far, for gyrotropic Faraday chiral media in layered structures, we have obtained a complete set of solutions to the DGFs in terms of the cylindrical vector wave functions and their scattering coefficients in terms of compact matrices. Reduction can be made for formulating the dyadic Green's functions in less complex media, e.g, an anisotropic medium where $\xi_c=0$, a bi-isotropic medium where $g=w=0$, a gyroelectric medium where $w=0$ and $\mu=\mu_z$, a chiroferrite medium where $p=0$ and $\epsilon=\epsilon_z$, or an isotropic medium where $\xi_c=g=w=0$, $\epsilon_z=\epsilon$, and $\mu=\mu_z$.

IV. NEGATIVE REFRACTION AND BACKWARD EIGENWAVES INSIDE GYROTROPIC FARADAY CHIRAL MEDIA

The permittivity and permeability considered here are defined in Eq. (1). The generalized media studied in this paper can be reduced to: (i) chiroplasma consisting of chiral objects embedded in a magnetically biased plasma, or (ii) a chiroferrite by immersing chiral objects into magnetically biased ferrite. Assume that wave is given in form by $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$, and it propagates along the z axis inside the gyrotropic Faraday chiral media. There are two approaches for obtaining the eigenmodes and wave numbers: (i) starting from Eq. (5) by setting $\boldsymbol{J}$ equal to zero and then obtaining the nontrivial solutions of wave numbers, and (ii) starting from Eq. (1) directly, listing all the relations of the field components and then solving the final equation consisting of wave numbers and parameters only. Here we choose the second method, which is less cumbersome and gives more insight to physical properties of the electromagnetic waves inside the media.

Consider Eq. (1) and

$$\nabla \times \boldsymbol{E} = i\omega \boldsymbol{B}, \quad (100)$$

TABLE I. Helicity and polarization states of k_{p-} and k_{a-} in three cases, assuming $-\alpha_t<\sigma<\alpha_t$ and $\xi_c>0$.

	[b]					
	$g<-\epsilon$		$-\epsilon<g<\epsilon$		$g>\epsilon$	
	HEL	POL	HEL	POL	HEL	POL
k_{p-}	$\ominus$	LCP	$\ominus$	LCP	$\ominus$[a]	RCP[a]
k_{a-}	$\oplus$[a]	RCP[a]	$\oplus$	LCP	$\oplus$	LCP

[a]Backward wave regimes.

$$\nabla \times \boldsymbol{H} = - i\omega \boldsymbol{D}. \tag{101}$$

Since the electromagnetic fields, $\boldsymbol{E}$ and $\boldsymbol{H}$, have only transverse components and the parameters are in gyrotropic form, D_z and B_z vanish. We will finally arrive at two equations where lengthy intermediate procedures have been suppressed

$$\begin{bmatrix} k_z(H_y + i\xi_c E_y) \\ - k_z(H_x + i\xi_c E_x) \end{bmatrix} = \omega \begin{bmatrix} \epsilon E_x - igE_y \\ \epsilon E_y + igE_x \end{bmatrix}, \tag{102}$$

$$\begin{bmatrix} H_x \\ H_y \end{bmatrix} = i\xi_c \begin{bmatrix} E_x \\ E_y \end{bmatrix} + \frac{k_z}{\omega}\begin{bmatrix} - \alpha_t E_y - \alpha_a E_x \\ - \alpha_a E_y + \alpha_t E_x \end{bmatrix}. \tag{103}$$

Thus one set of equations can be obtained, and it contains only transverse electric components

$$\left(\frac{k_z}{\omega}\alpha_t - \frac{\omega}{k_z}\epsilon\right)E_y = \left(2i\xi_c + i\frac{\omega}{k_z}g - \frac{k_z}{\omega}\alpha_a\right)E_x, \tag{104}$$

$$\left(2i\xi_c + i\frac{\omega}{k_z}g - \frac{k_z}{\omega}\alpha_a\right)E_y = \left(- \frac{k_z}{\omega}\alpha_t + \frac{\omega}{k_z}\epsilon\right)E_x. \tag{105}$$

Hence, we can obtain four wave numbers for the eigenwaves propagating along the z axis as follows:

$$k_{p\pm} = \omega\frac{\pm \xi_c + \sqrt{\xi_c^2 + (\alpha_t \pm \sigma)(\epsilon \pm g)}}{\alpha_t \pm \sigma}, \tag{106}$$

$$k_{a\pm} = \omega\frac{\mp \xi_c - \sqrt{\xi_c^2 + (\alpha_t \mp \sigma)(\epsilon \mp g)}}{\alpha_t \mp \sigma}, \tag{107}$$

where p and a denote the parallel and antiparallel directions of the real part of Poynting's vector and $\sigma=i\alpha_a=w/(\mu^2-w^2)$, while plus and minus signs refer to as the right- and left-circularly polarized (RCP and LCP) forwarding waves, respectively. k_{p-} and k_{a-} are of particular interest since they will represent the properties of backward waves under specific cases as shown in Table I.

The realization of negative refraction is of particular interest. Taking into account Eq. (4) and respective polarization states, we can finally obtain two refraction indices for those backward eigenwaves:

$$n_{\pm} = \frac{c_0}{(\alpha_t \pm \sigma)}[\sqrt{\xi_c^2 + (\alpha_t \pm \sigma)(\epsilon \pm g)} - \xi_c], \tag{108}$$

where plus and minus signs refer to as k_{a-} and k_{p-}, respectively. It can be seen that n_+ will be negative when $g<-\epsilon$ and when n_- will possess a minus sign when $g>\epsilon$ (which means that a backward wave can propagate in such a medium) if the conditions in Table I are maintained. It also shows that even when the chirality ξ is small, a negative index of refraction may be easily achieved. By considering the Poynting's vectors and the Maxwell equations, we will finally arrive at two impedances η_1 (for k_{a-}) and η_2 (for k_{p-}), which are given as follows:

$$\eta_1 = \frac{1}{\sqrt{\xi_c^2 + (\alpha_t - \sigma)(\epsilon + g)}} = \frac{1}{\sqrt{\xi_c^2 + \dfrac{\epsilon + g}{\mu + w}}}, \tag{109}$$

$$\eta_2 = \frac{1}{\sqrt{\xi_c^2 + (\alpha_t + \sigma)(\epsilon - g)}} = \frac{1}{\sqrt{\xi_c^2 + \dfrac{\epsilon - g}{\mu - w}}}. \tag{110}$$

Hence, when a Faraday chiral medium is employed to fabricate a perfect lens, impedance matching should be carefully carried out at the air-material interfaces. Within certain frequency bands, there are two backward waves. Therefore, each time only the one whose impedance is matched can be transmitted into the slab and double focusing effect can take place. Most of the other backward waves would be reflected due to the mismatch at the interface. Note that $g>\epsilon$ can be realized with some modern technology in the future based on the theory of off-diagonal parameter amplification in gyrotropic media.[37] Hence, negative refractive media can be realized even when all the parameters in material tensors are positive.

It is possible to achieve a nearly zero-index material

$$n_- \approx 0$$

by requring $\epsilon - g \approx 0$ while the material still possesses a positive wave impedance $1/\xi_c$. Such a material has wide potential applications in airborne radome design, and high-directivity antenna design since the phase of the propagating waves will keep unchanged inside this material. A zero-index or nearly zero-index medium provides potentials in quantum devices because the discrete quantized field will be greatly enhanced. The electric field strength with respect to n_k photons of k mode can be expressed

$$E_k = \sqrt{\frac{(n_k + 1/2)\hbar\omega}{\dfrac{n}{\mu}\dfrac{d(n\omega)}{d\omega}}V}, \tag{111}$$

where $\hbar$ is the Dirac constant and V stands for the volume of the medium.

For a single photon, we have the critical field strength

$$E_c = \sqrt{\frac{3\hbar\omega}{2\dfrac{n}{\mu}\dfrac{d(n\omega)}{d\omega}}V}. \tag{112}$$

If the field strength has the same order of magnitude of E_c or less than E_c, the field can be viewed as a quantized one or a fluctuation of quantum vacuum. It is obvious that the critical field strength becomes very large when the refractive index is almost zero. Hence, the quantum vacuum fluctuation field becomes strong.

In summary, a gyrotropic Faraday chiral medium provides us a very exciting opportunity to realize negative refraction, backward wave propagation, and related quantum effects.

V. CONCLUSIONS

In this paper, we studied some important electromagnetic properties of gyrotropic Faraday chiral media due to their significant potentials for metamaterials and interactions between electromagnetic waves and materials. Green dyadics, backward wave propagation, and negative refractive index associated with gyrotropic Faraday media have been well investigated in this paper. The unbounded DGF for a gyrotropic Faraday chiral medium is formulated by use of the Ohm-Rayleigh method and cylindrical vector wave function expansion. Scattering DGFs for cylindrically stratified structures (where each layer is an arbitrarily characterized gyrotropic Faraday chiral medium) are constructed and a recursive algorithm is proposed to obtain those scattering coefficients based on the boundary conditions matched at each interface. Those formulations can be reduced in form to the counterparts for anisotropic, chiroplasma, chiral, and isotropic media. In what follows, the eigenmodes and wave numbers in a gyrotropic Faraday chiral medium are studied. Finally, the properties of backward propagating eigenmodes and realization of negative refraction are well explored and discussed.

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[1] J. A. Kong and D. K. Cheng, IEEE Trans. Microwave Theory Tech. **19**, 99 (1968).
[2] R. D. Graglia, M. S. Sarto, and P. L. E. Uslenghi, IEEE Trans. Microwave Theory Tech. **44**, 1470 (1996).
[3] J. B. Pendry, A. J. Holden, D. J. Robbins, and W. J. Stewart, IEEE Trans. Microwave Theory Tech. **47**, 2075 (1999).
[4] X. Gong, W. H. She, E. E. Hoppenjans, Z. N. Wing, R. G. Geyer, J. W. Halloran, and W. J. Chappell, IEEE Trans. Microwave Theory Tech. **53**, 3638 (2005).
[5] V. G. Veselago, Sov. Phys. Usp. **10**, 509 (1968).
[6] H. Chen, L. Ran, J. Huangfu, X. Zhang, K. Chen, T. M. Grzegorczyk, and J. Au Kong, Phys. Rev. E **70**, 057605 (2004).
[7] A. Grbic and G. V. Eleftheriades, Phys. Rev. Lett. **92**, 117403 (2004).
[8] N. Katsarakis, T. Koschny, M. Kafesaki, E. N. Economou, Ekmel Ozbay, and C. M. Soukoulis, Phys. Rev. B **70**, 201101(R) (2004).
[9] N.-C. Panoiu and R. M. Osgood, Jr., Phys. Rev. E **68**, 016611 (2003).
[10] V. Kuzmiak and A. A. Maradudin, Phys. Rev. B **66**, 045116 (2002).
[11] J. Pendry and S. A. Ramakrishna, J. Phys.: Condens. Matter **15**, 6345 (2003).
[12] Zhen Ye, Phys. Rev. B **67**, 193106 (2003).
[13] S. A. Ramakrishna and J. B. Pendry, Phys. Rev. B **69**, 115115 (2004).
[14] J. B. Pendry, Phys. Rev. Lett. **85**, 3966 (2000).
[15] J. D. Baena, L. Jelinek, R. Marqués, and F. Medina, Phys. Rev. B **72**, 075116 (2005).
[16] R. Marqués, F. Medina, and R. Rafii-El-Idrissi, Phys. Rev. B **65**, 144440 (2002).
[17] J. D. Baena, R. Marqués, F. Medina, and J. Martel, Phys. Rev. B **69**, 014402 (2004).
[18] L. Liu, C. Caloz, and T. Itoh, Electron. Lett. **38**, 1414 (2002).
[19] N. Engheta, IEEE Microw. Wirel. Compon. Lett. **1**, 10 (2002).
[20] R. A. Shelby, D. R. Smith, and S. Schultz, Science **292**, 77 (2001).
[21] A. Ishimaru, S. W. Lee, Y. Kuga, and V. Jandhyala, IEEE Trans. Antennas Propag. **51**, 2550 (2003).
[22] X. Chen, B. I. Wu, J. A. Kong, and T. M. Grzegorczyk, Phys. Rev. E **71**, 046610 (2005).
[23] G. V. Eleftheriades, A. K. Iyer, and P. C. Kremer, IEEE Trans. Microwave Theory Tech. **50**, 2702 (2002).
[24] A. Lai, T. Itoh, and C. Caloz, IEEE Microw. Mag. **5**, 34 (2004).
[25] S. Tretyakov, I. Nefedov, A. Sihvola, S. Maslovski, and C. Simovski, J. Electromagn. Waves Appl. **17**, 695 (2003).
[26] J. B. Pendry, Science **306**, 1353 (2004).
[27] C. T. Tai, *Dyadic Green's Functions in Electromagnetic Theory*, 2nd ed. (IEEE Press, Piscataway, New Jersey, 1994).
[28] D. Cheng and W. Ren, Phys. Rev. E **54**, 2917 (1996).
[29] L. W. Li, P. S. Kooi, M. S. Leong, and T. S. Yeo, IEEE Trans. Microwave Theory Tech. **42**, 2302 (1994).
[30] L. W. Li, P. S. Kooi, M. S. Leong, and T. S. Yeo, IEEE Trans. Antennas Propag. **43**, 232 (1995).
[31] I. V. Lindell and F. Olyslager, IEEE Trans. Antennas Propag. **45**, 1563 (1997).
[32] Dajun Cheng, Phys. Rev. E **55**, 1950 (1997).
[33] L. W. Li, M. S. Leong, P. S. Kooi, and T. S. Yeo, Phys. Rev. E **59**, 3767 (1999).
[34] I. V. Lindell and W. S. Weiglhofer, IEEE Trans. Antennas Propag. **42**, 1013 (1994).
[35] J. A. Kong, *Electromagnetic Wave Theory*, 3rd ed. (John Wiley & Sons, New York, 1990).
[36] W. C. Chew, *Waves and Fields in Inhomogeneous Media* (Van Nostrand Reinhold, New York, 1990).
[37] F. Jonsson and C. Flytzanis, J. Opt. A, Pure Appl. Opt. **2**, 299 (2000).