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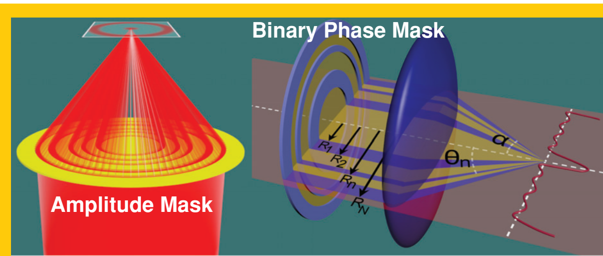
ORIGINAL PAPER

**Abstract** A superoscillatory focusing lens has been experimentally demonstrated by optimizing Fresnel zone plates (FZP), with limited physical insight as to how the lens feature contributes to the focal formation. It is therefore imperative to establish a generalized viable account for both FZP (amplitude mask) and binary optics (phase mask). Arbitrary superoscillatory spots can now be customized and realized by a realistic optical device, without using optimization. It is counterintuitively found that high spatial frequency with small amplitude and destructive interference are favorable in superfocusing of a superoscillation pattern. The inevitably high sidelobe is pushed 15λ away from the central subwavelength spot, resulting in significantly enlarged field of view for viable imaging applications. This work therefore not only reveals the explicit physical role of any given metallic/dielectric rings but also provides an alternative design roadmap of superresolution imaging. The robust method is readily applicable in superthin longitudinally polarized needle light, quantum physics and information theory.




# Optimization-free superoscillatory lens using phase and amplitude masks

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To observe microscale objects, people always pursue superresolution imaging by decreasing the focused spot [1], tailoring the evanescent wave [2, 3], utilizing the nonlinear effect [4], exploiting the digital-image-processing technique [5, 6] and developing novel equipments [7, 8]. The newly demonstrated optical microscopy based on superoscillatory focusing provides another route to superresolution imaging [9]. This superoscillatory optical microscopy with the resolution of λ/6 has gained much attention because its focused spot can be infinitesimally sharp according to the superoscillation theory, which opens up a promising conceptual avenue to imaging arbitrarily small objects. Nevertheless, the superoscillatory spot with smaller feature suffers from its higher sidelobe, which, to some extent, imposes great challenges in the further application in high-resolution imaging resolution. Since the superoscillatory spot is inevitably accompanied by its high sidelobe [10, 11], one cannot eliminate the sidelobe if the superoscillation arises. Hence, it is nontrivial and imperative to push the high sidelobe far enough apart from the center, so as to produce realistic applications. However, this requires the elaborate manipulation over superoscillation via complicated lens design. The reported methods of constructing a superoscillatory pattern in an optical lens mainly rely on optimizing algorithms [9, 12] for FZP. Hence, the underlying physics, relating every feature of the physical lens structure and their contribution on the imaging plane, is not revealed yet, which in turn limits the flexible and controlled design of the superoscillation imaging in not only FZP but also binary-phase masks [13].

It is well known that the superoscillation in optics is one kind of destructive interference of light with different frequencies at some points at small intervals by matching the amplitude of every frequency [14]. This implies that one can control the optical superoscillation by choosing a suitable amplitude and frequency of light for the destructive interference at the prescribed position, which is a prototype inverse problem. We find that this inverse problem in some realistic optical devices, e.g., a zone plate (amplitude mask) or a binary-phase lens system (phase mask), can be described by a nonlinear matrix equation. Solving that can produce a customized superoscillatory pattern or control the superoscillation optionally. In contrast to using optimization for designing multiple rings as the only way, the unveiled fundamental physics behind the matrix enables us to analytically design a superfocusing central spot and push the high sidelobe away from the center for several wavelengths. In addition, we also attempt to propose a superoscillatory criterion in optical focusing, $r_S = 0.38/f_{max}$ ($f_{max}$ is the maximum spatial frequency), which determines whether the superoscillatory focusing occurs or not.

In contrast to the nanohole array [15], the zone plate with the amplitude modulation of 0 or 1 is an easy method to focus light into a superoscillatory spot. Optimization turns out to be the only method reported so far that can optimize the central radius and width of every belt in a

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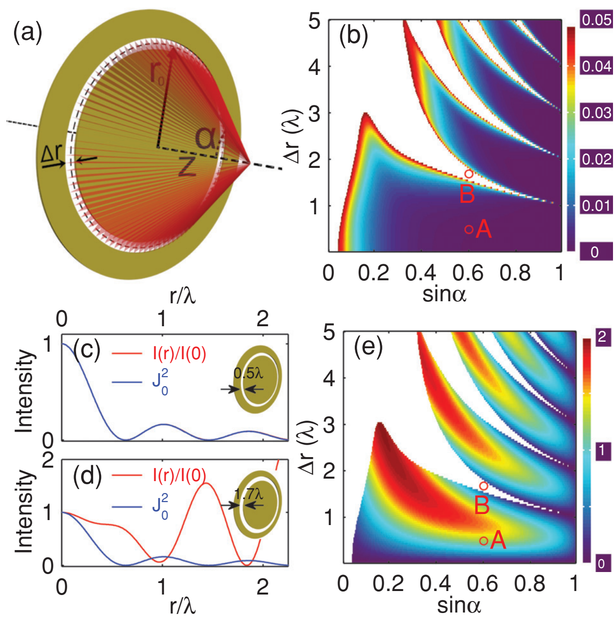


**Figure 1** The single belt's diffraction at the intermediate region (where $z = 20\lambda$) between the evanescent (near-field) and far-field region. (a) The optical system describing the diffraction of a single belt with its width $\Delta r$ and radius $r_0$. (b) The dependence of RMSE on the width $\Delta r$ and radius $r_0$ (or $\sin\alpha$). The smaller the RMSE, the better the approximation between the intensity profile at the target plane and its zero-order Bessel function $|J_0(kr\sin\alpha)|^2$ with the same $\sin\alpha$ ($= r_0/(r_0{}^2+z^2)^{1/2}$). We just show the cases with small RMSE located in the colored region. The geometry parameters of the single belt are $(\sin\alpha, \Delta r)_A = (0.6, 1.7\lambda)$ at A and $(\sin\alpha, \Delta r)_A = (0.6, 0.5\lambda)$ at B. (c and d) The 1-dimensional intensity profiles (red) of light passing through the belt with its parameters at position A (c) and position B (d) and their corresponding Bessel functions with the same $\sin\alpha$ (blue). The intensity profile in (c) shows an excellent coincidence with the Bessel function so that it is hard to distinguish them. (e) The dependence of the amplitude-modulation coefficient $|C_n|$ in Eq. (1) on the width $\Delta r$ and radius $r_0$ (or $\sin\alpha$).

zone plate [9]. Unfortunately, such an optimization-based method presents little physical information but a fitness function containing lens parameters and the designed superoscillatory spot, which is not able to provide an insightful means for controlling the superoscillatory focusing with a customized pattern in the imaging plane, e.g., the peak ratio of sideband over central spot, the distance between high sidelobe and the center. These are actually nontrivial in practical imaging industries, when one wants to use a superoscillatory lens. In this connection, the contribution of our optimization-free design principle for a superoscillatory lens is three-fold: First, the design process is fully guided by the proposed theory; Secondly, the approach applies not only to an amplitude mask but also a phase mask; Thirdly, customization and solving the disadvantage of a superoscillatory lens, i.e. higher sidelobe too close to the central spot.

Figure 1 shows the diffraction of light by a single belt with its geometry of radius $r_0$ and width $\Delta r$, as shown in Fig. 1a. In order to evaluate the focusing properties of a single belt, we use the root-mean-square error (RMSE, whose definition is available in Supplementary Materials) between its diffracting intensity at the target plane and its corresponding zero-order Bessel function of $|J_0(kr\sin\alpha_0)|^2$ with the same $\sin\alpha_0$ ($= r_0/(r_0{}^2+z^2)^{1/2}$). Figure 1b shows the relationship between RMSE and the geometry (in terms of width $\Delta r$ and radius $r_0$) of a single belt. The light from a belt has the different intensity profile at the target plane when the geometry of the transparent belt in Fig. 1a changes. Only the light passing through the belt with its geometry located in the colored region of Fig. 1b has a better focusing pattern with small RMSE, which can be approximated as a zero-order Bessel function of the first kind as shown in Fig. 1c, at the target plane. However, the intensity profile for the case of A in Fig. 1d might destroy the total intensity of the superoscillatory focusing for a subwavelength spot due to its poor focusing property at $r = 0$ and the incomplete destructive interference at its first valley. The optimizing algorithm behaves poorly in rejecting the case of A by itself. In addition, even if all the belts in a zone plate have the geometry located in the colored region, it is still an arduous task for the optimizing method to realize the prescribed intensity (i.e. complete destructive interference with zero intensity) at the customized radial ($r$) position in the total intensity of the zone plate. To achieve the customized intensity pattern, we here suggest a mathematical method by solving a nonlinear matrix equation, without any optimizing technique involved, to design a superoscillatory mask.

Although some attempts based on the inverse of the matrix have been made to construct a superoscillatory pattern and diffraction-free beam [14, 16, 17], this method is only constrained to the case that the unknown amplitude-modulation coefficients are independent of the spatial frequency. For the zone plate, the amplitude-modulation coefficient from every spatial frequency has a tight relationship, shown in Fig. 1e, with the geometry of the transparent belt in the zone plate, which makes designing of a superoscillatory zone plate very challenging. Here, we develop this method further to design a superoscillatory mask with customized pattern in a realistic optical system. For simplicity, we assume that the illuminated light of the mask is an unpolarized plane wave with uniform distribution in the paper. According to the scalar Rayleigh–Sommerfeld diffraction theory [18], for an unpolarized incident beam passing through the unobstructed belt with radius $R_n$ and width $\Delta r$ in Fig. 2a, its electric field at the target plane beyond the evanescent region is

$$U_n(r) = \frac{1}{2\pi}\int_{R_n-\Delta r/2}^{R_n+\Delta r/2}\int_0^{2\pi} u(r,\phi)\frac{\partial}{\partial z}\left[\frac{\exp(ikR)}{R}\right]\rho \mathrm{d}\rho \mathrm{d}\phi, \tag{1}$$

where $R^2 = z^2+r^2+\rho^2–2r\rho\cos(\theta–\varphi)$, the complex amplitude $u(\rho,\varphi)$ of the incident beam is taken as unity for the uniform illumination here. The electric field mainly depends on the $R_n$, $\Delta r$ and $z$. We define the amplitude-modulation coefficient $C_n = U_n(0)$ and the normalized

 

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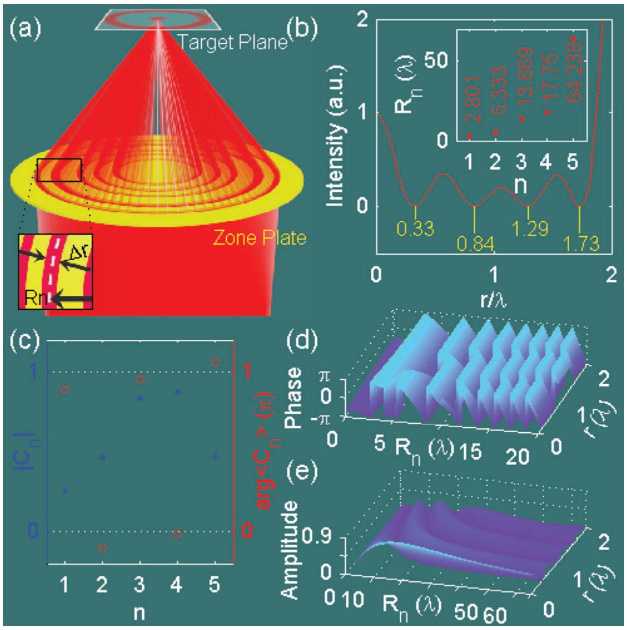


**Figure 2** Generation of superoscillatory focusing with the sidelobe away from the center by using a zone plate. (a) The sketch of focusing light beyond the evanescent region by using the zone plate. The $n$th belt in the zone plate has the radius of $R_n$ and width $\Delta d$. (b) The constructed optical superoscillatory pattern with the prescribed position $r = [0,\ 0.33\lambda,\ 0.84\lambda,\ 1.29\lambda,\ 1.73\lambda]$ and the customized intensity $\mathbf{F} = [1, 0, 0, 0, 0]$ at $\mathbf{r}$. Inset: the solved $R_n$ of every belt with fixed $\Delta r = 0.3\lambda$. (c) The modulus (dot) and phase (circle) of the amplitude-modulated coefficient $C_n$ in the solved zone plate of (b). (d and e) The phase (d) and intensity (e) profiles of a belt with its width $\Delta r = 0.3\lambda$ and the changing radius $R_n$.

amplitude $A_n(r) = U_n(r)/C_n$. Figure 1b shows the root-mean-square error (RMSE) between $|A_n(r)|^2$ and its corresponding zero-order Bessel function of $|J_0(kr\sin\alpha_n)|^2$ with the same $\sin\alpha_n$ ($= R_n/(R_n{}^2+z^2)^{1/2}$). In Fig. 1c, one can see that $|C_n|$ has a strong dependence on the width $\Delta r$ and the spatial frequency designated as $\sin\alpha/\lambda$. Then, the total electric field of light passing through a zone plate containing $N$ belts can be expressed as

$$U(r) = \sum_{n=1}^{N} C_n A_n(r). \qquad (2)$$

To realize the intensity $\mathbf{F} = [f_1, f_2, .., f_M]^T$ at the position $\mathbf{r} = [r_1, r_2, \ldots, r_M]^T$ in the target plane, we can described this problem as

$$\mathbf{SC} = \mathbf{F}, \qquad (3)$$

where $\mathbf{S}$ is an $M\times N$ matrix with its matrix element $S_{mn} = A_n(r_m)$ according to Eq. (2) and $\mathbf{C} = [C_1, C_2, \ldots, C_N]^T$, where the sign $T$ means the transpose of matrix. The solution of Eq. (3) exists if $M \le N$. Here, we just consider the case $M = N$ for which Eq. (3) has the only solution. Because the $S_{mn}$ and $C_n$ are dependent on the unknown $R_n$ (or $\sin\alpha_n$) when the width $\Delta r$ and z are fixed, it is a nonlinear problem to solve the matrix equation for $R_n$. Although, in general, Eq. (3) has no analytical solution like the cases in [14, 17], its numerical solution can be easily obtained by using the well-developed Newton's theory, which has been widely used to deal with the nonlinear problem in many areas [19,20]. Newton's theory for nonlinear problems solves Eq. (3) on the basis of the exact solution of its subproblem [20], which makes it a powerful tool to efficiently approach the exact solution without any search-based optimizing algorithm. The method described in Eq. (3) provides a very useful way to design a superoscillatory zone plate despite the fact that the solution is numerically approximated.

To verify the validity of our method, we show a constructed superoscillatory spot with size of about $0.5r_R$ ($r_R$ is the Rayleigh limitation) and its sidelobe is about $1.8\lambda$ away from the center by using a zone plate, shown in Fig. 2a, which is designed by our method. In order to realize the goal of pushing away the sidelobe, we pad the zero intensity at the locations between the sidelobe and the center to suppress the high sidelobe near the center. The customized position $\mathbf{r}$ with zero intensity must be carefully chosen to reject the generation of any high intensity between the high sidelobe and the center when solving Eq. (3). Therefore, we choose $\mathbf{F} = [1, 0, 0, 0, 0]^T$ at $\mathbf{r} = [0, 0.33\lambda, 0.84\lambda, 1.29\lambda, 1.73\lambda]^T$ for achieving a superoscillatory spot with the size of $0.5r_R$ ($0.33\lambda$) and its sidelobe about $2\lambda$ away from the center in Fig. 2b. In the customized $\mathbf{F}$ and $\mathbf{r}$, $f_1 = 1$, $f_2 = 0$ and $r_1 = 0.33\lambda$ are used to define the superoscillatory spot and the rest is responsible for suppressing the sidelobe between the main spot and the high sidelobe. According to the result in Fig. 1b, we assume that the width $\Delta r$ of every belt has the same size of $0.3\lambda$ and the target plane is located in $z = 20\lambda$ in the simulation for removing the case of A in Fig. 1d. To obtain the unknown $R_n$ of every belt, we solve its inverse problem described in Eq. (3) by using the trust-region dogleg Newton theory that is introduced in the Supplementary Materials [20]. The solved $R_n$ is shown in the inset of Fig. 2b and their corresponding $\sin\alpha_n = [0.1387, 0.2576, 0.5643, 0.6638, 0.9548]$.

Conventionally, in order to obtain a supersmall focused spot, one always prefers to focus the light of high spatial frequency with large amplitude, which leads to a small size spot dominating at the target plane, and interfere the light from different spatial frequencies constructively, which enhances the focused spot. However, in superoscillatory focusing, we here show an abnormal phenomenon that the maximum amplitude ($|C_n|$) is located at the frequency with the intermediate value. This counterintuitive requirement for obtaining a small spot by superoscillation mainly depends on the fact that the superoscillation always oscillates with very small amplitude that can be considered as almost destructive interference [14]. The destructive interference in the superoscillation is also reflected by the phase of $C_n$ that is shown in Fig. 2c. The phase difference between two neighboring belts in the designed zone plate is nearly $\pi$, which implies that the destructive interference is essentially required for realizing the superoscillation pattern in Fig. 2b. Thus, we can claim that the

 

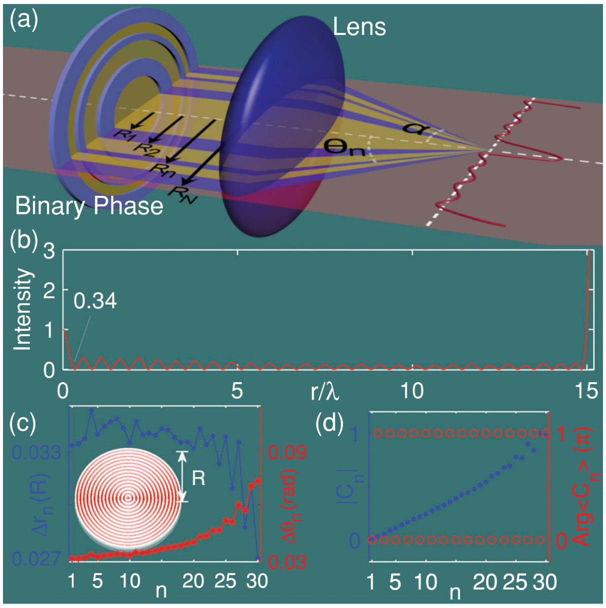


**Figure 3** Generation of superoscillatory focusing with the sidelobe away from the center by using a binary phase and a lens. (a) The sketch of focusing a binary-phase modulated beam by a lens. The binary element has the phase of 0 and $\pi$, whose boundary is the circle with radius of $R_n$ ($n = 1,2,\ldots,N$). The lens has an NA of sin$\alpha$, where $\alpha$ is the maximum convergent angle. (b) A superoscillatory spot with size of 0.34λ and its sidelobe about 15λ away from the center by solving its inverse problem. Inset: 2-dimensional intensity profiles in the range $r \leq \lambda$. The specific radii of individual dielectric grooves are given in Supplementary Materials. (c) Width $\Delta r_n$ (blue dot) of every belt and its corresponding angle width $\Delta\theta_n$ (red star) in the designed binary phase. Inset: 3-dimensional phase profile of this binary phase plate. (d) Modulus (solid circle) and phase (hollow circle) of amplitude-modulated coefficient $C_n$.

amplitude-modulated coefficient $C_n$ has the alternating sign of $(-1)^n$ with its modulus small for low and high spatial frequency and large for the intermediate frequency, which is further confirmed by the case of focusing the light with rigorous single spatial frequencies (see Supplementary Materials). Nevertheless, this conclusion predicts that the zone plate is not ideal to realize a superoscillatory spot in Fig. 2b. Although the belt in the zone plate shows the excellent focusing property in a long range of $R_n$ shown in Fig. 2e, the phase of $C_n$, that is the case of $r = 0$ in Fig. 2d, varies from 0 to $2\pi$ quasiperiodically with the increase of $R_n$. As a result, much effort must be made to obtain the phase difference of $\pi$ for the alternating sign of $C_n$. Therefore, the zone plate may not be the best candidate to achieve a superoscillatory spot with its high sidelobe away, although we can use it to realize the superoscillatory spot in Fig. 2b.

Considering the difficulty of phase matching from a zone plate, we suggest another optical system containing a binary phase and a high numerical-aperture (NA) lens in Fig. 3a to realize the superoscillatory subwavelength focusing. The binary element with the phase 0 or $\pi$ located in the entrance pupil of the focusing lens provides the phase difference of $\pi$ for the generation of superoscillation in focusing [13, 21]. In the uniform illumination of an unpolarized beam, the electric field at the focal plane can be approximated by the Debye theory [22, 23]

$$\begin{aligned} U(r) &= \frac{2\pi i}{\lambda}\int_0^{\alpha} P(\theta) J_0(kr\sin\theta)\sin\theta \mathrm{d}\theta \\ &= \sum_{n=1}^{N}(-1)^n \frac{2\pi i}{\lambda}\int_{\theta_{n-1}}^{\theta_n}\sqrt{\cos\theta}\, J_0(kr\sin\theta)\sin\theta \mathrm{d}\theta \\ &= \sum_{n=1}^{N}(-1)^n U_n(r), \end{aligned} \tag{4}$$

where $P(\theta)$ is the apodization function that equals $p(\theta)\cdot\cos(\theta)^{1/2}$ for the lens obeying the sine condition [23, 24], $p(\theta)$ is the entrance pupil function that is $(-1)^n$ for the uniform illumination with the modulation of binary phase. The relationship between $R_n$ and $\theta_n$ ($n = 0, 1, 2, \ldots, N$ with $\theta_0 = 0$, $\theta_N = \alpha$) is $R_n/f = \sin\theta_n$ for the sine lens used here, where $f$ is the focal length of focusing lens. We define the amplitude modulation coefficient $C_n = (-1)^n U_n(0)$ and $A_n(r) = U_n(r)/U_n(0)$. Similarly, the inverse problem of constructing the superoscillation using the optical system in Fig. 3a can also be expressed by Eq. (3) with the unknown variable $R_n$ (or sin$\theta_n$). The amplitude modulation coefficient $C_n$ with the alternating sign of $(-1)^n$ makes it easier to solve the inverse problem for generating the superoscillatory focusing. Figure 3b shows a constructed superoscillatory spot with a size of about $0.5r_R$ (0.34λ) and the high sidelobe about 15λ away from the center by using a 0.95 NA lens (see Supplementary Materials for the radius parameters).

This superoscillatory spot is obtained by padding 29 zero-intensity positions between the main spot and the sidelobe when solving its inverse problem with $N = 30$ variables. Compared with the result in Fig. 2b by using a zone plate, the spot in Fig. 3b almost keeps the same size, while the distance between its high sidelobe and center is nearly 10 times that in Fig. 2b, which mainly benefits from the binary phase (with a phase difference of $\pi$) for destructive interference. We can enlarge the distance further by padding more zero-intensity positions between the high sidelobe and the center. Figure 3c shows the structure of the designed binary phase by our method. The width $\Delta r_n$ ($= R_n - R_{n-1}$) of belts in the binary phase tends to be diminishing at the outmost belts that are relative to the high spatial frequency. However, for a sine lens, the corresponding angle width $\Delta\theta_n$ ($= \theta_n - \theta_{n-1}$) of every belt is increasing so that the amplitude modulation $|C_n|$ shows the monotonically increasing tendency from the low spatial frequency to the high in Fig. 3d, which is different from the case in Fig. 2c. This is mainly attributed to the fact that every belt of binary phase corresponds to the spectrum (from $\sin\theta_{n-1}/\lambda$ to $\sin\theta_n/\lambda$) of spatial frequency not a quasisingle spatial frequency that occurs in zone plates. Through this example, we have shown that the method suggested here is valid to solve the inverse problem of superoscillation by using the optical system in Fig. 3a.

Next, we discuss the method that distinguishes a superoscillatory spot in optical focusing. Although the superoscillatory spot has been widely investigated in optical focusing and imaging [9, 12, 15, 25], none provides a clear demonstration as to how small a spot has to be so that it can be considered as a superoscillatory spot. To our knowledge, the Rayleigh criterion ($r_R = 0.61\lambda/\mathrm{NA}$) is mostly used to judge a superoscillatory spot in optical focusing [26]. However, it is a very rough method because there is no definition of superoscillation involved. In optics, a relevant and natural definition of superoscillation has been proposed by measuring the changing rate of the phase of a band-limited function in a local region [27, 28]. In particular, for the case of the 1-dimensional (or axisymmetric) band-limited function, i.e. the zone plate and a binary-phase-based lens, Berry and Dennis proposed a practical method for measuring the local wave number, $k(r) = \mathrm{Im}\{\partial_r[\ln F(r)]\}$, where $F(r)$ is the band-limited function [28]. Therefore, the definition of local wave number by Berry and Dennis is preferred in optical focusing. However, when we use Berry and Dennis's suggestion to evaluate the local wave number of a superoscillatory band-limited function in Fig. 4a, the calculated wave number in Fig. 4b is larger than the wave number of its maximum Fourier component only when the band-limited function has zero intensity. This means that, though the band-limited function indeed oscillates faster in the whole region $x \in [-0.8\lambda, 0.8\lambda]$ than its maximum Fourier component, Berry and Dennis's suggestion only predicts the superoscillation at the zero-intensity position. It is worth pointing out that Berry and Dennis's suggestion gives the wave number at a certain position but not in a region where Fig. 4b shows the large wave number only at the zero-intensity position. Therefore, in optical focusing, it is better to define the superoscillatory spot by measuring the phase-changing rate in a certain region.

In optical focusing, we constrain the definition of a superoscillatory spot on three conditions: 1) The optical system is axisymmetric so that a circular spot could be generated. 2) The superoscillatory spot must oscillate faster in a certain region of the target plane than its maximum Fourier frequency component. 3) "A certain region" is located at $r \leq r_S$, where $r_S$ is the first zero-intensity position of the electric field at the target plane by focusing the light only from the maximum Fourier frequency component. The reason for choosing the region $r \leq r_S$ is to exclude the case shown by the black curve of Fig. 4c, which has the fast superoscillation at $r \geq r_S$ while its spot size is very large. In this region $r \leq r_S$, the maximum Fourier frequency component only oscillates for one time without changing its phase, which is shown by the blue curve in Fig. 4c. If a spot oscillates faster in $r \leq r_S$, this will lead to the generation of the intensity valley, where the high local wave number is located [28]. Thus, we can define a superoscillatory spot in optical focusing as: a spot is superoscillatory when its local wave number that is larger than the wave number of the maximum Fourier frequency is located in the region $r \leq r_S$. In that case, a spot with its zero intensity

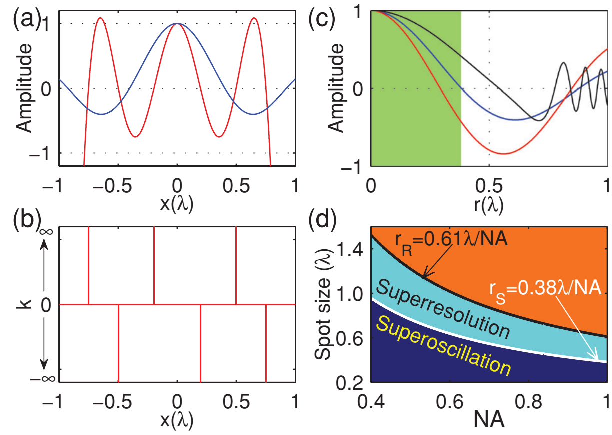


**Figure 4** Superoscillatory criterion in optical focusing. (a) The amplitude profiles of a superoscillatory band-limited function with its zero-intensity position at $x = \pm 0.2\lambda$ (red) and its maximum spatial frequency component (blue). The band-limited function is the electric field at the focal plane by using the binary-phase-based 0.95NA lens in Fig. 3a with the solved $\sin\theta_n = [0, 0.3435, 0.6523, 0.8744, 0.95]$. This superoscillatory band-limited function obviously oscillates faster in the region $x \in [-0.8\lambda, 0.8\lambda]$ than its maximum spatial frequency. (b) The local wave number of the band-limited function in (a) by using Berry's suggestion. (c) The amplitude profiles of various cases: the first zero-intensity position located in the colored region (red) and outside color region (black). The blue curve shows the amplitude of the maximum spatial frequency. (d) The spot size in different NA, which equals the sine ($\sin\alpha$) of the angle between the optical axis and the maximum convergent ray in free space. The two curves, which are the Rayleigh (black) and superoscillation (white) criterions, divide the focusing spot into three parts: subresolved (orange), superresolution (cyan) and superoscillation (dark blue).

located in $r \leq r_S$ is superoscillatory, which is shown by red curves in Figs. 4a and c. This means that a superoscillatory spot has a smaller size than that ($r_S$) by only focusing its maximum spatial frequency, which implies that $r_S$ can be taken as the superoscillatory criterion. When light with a single spatial frequency of $\sin\alpha/\lambda$ ($\alpha$ is the angle between the optical axis and the maximum convergent ray) is focused, its electric field at the target plane is proportional to the zero-order Bessel function $J_0(2\pi r\sin\alpha/\lambda)$ of the first kind, which gives $r_S = 0.38\lambda/\sin\alpha$. The superoscillatory criterion $r_S$ has a similar shape to the Rayleigh criterion $r_R$. Figure 4d shows the spot size in different NA that is usually in terms of $\sin\alpha$ in free space. For a given NA, the spot in all the cyan and dark-blue areas below the Rayleigh criterion (black curve) can be called the superresolution spot and the spot in the dark-blue area below the superoscillation criterion (white curve) is the superoscillation spot, which means that the superoscillation spot is one subaggregate of the superresolution spot. The finely distinguished roadmap in Fig. 4d provides an instructive guide that the cyan area between the Rayleigh and superoscillation criterion is the best choice when one pursues a superresolution focusing spot without high sidelobe beyond the evanescent range. More importantly, $r_S$ implies a limitation of 0.38λ for the application of subwavelength spot without high sidelobe.

In summary, we have demonstrated a physical design roadmap of the superoscillatory focusing by using a zone plate or a binary-phase-based lens, with significantly enlarged field of view. The described inverse problem of superoscillation in terms of a nonlinear matrix equation enables construction of a customized superoscillatory pattern possible to be implemented without the traditional optimizing technique involved in the reported superoscillatory lens. This paves the way to a new scheme in further improving the resolution of the optical far-field imaging, and narrowing width of longitudinally polarized needle light for advanced data-storage performance [13]. In achieving a supersmall spot beyond the evanescent region, our result shows a counterintuitive phenomenon that the large spatial frequency with low intensity and destructive interference must be involved. Furthermore, the superoscillatory criterion proposed here gives us the direct insight into the spot pattern beyond the Rayleigh limitation, which sets a theoretical limitation of 0.38λ for the spot size in some applications that demand the narrow spot and low sidelobe simultaneously, i.e. optical lithography [29], high-intensity optical machining [30] and high-contrast superresolution imaging [31–33].

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## References

[1] R. Dorn, S. Quabis, and G. Leuchs, Phys. Rev. Lett. **91**, 233901 (2003).
[2] N. Fang, H. Lee, C. Sun, and X. Zhang, Science **308**, 534–537 (2005).
[3] Z. Liu, H. Lee, Y. Xiong, C. Sun, and X. Zhang, Science **315**, 1686 (2007).
[4] W. Denk, J. H. Strickler, and W. W. Webb, Science **248**, 73–76 (1990).
[5] E. Betzig, G. Patterson, R. Sougrat, O. Lindwasser, S. Olenych, J. Bonifacino, M. Davidson, J. Lippincott-Schwartz, and H. Hess, Science **313**, 1642–1645 (2006).
[6] B. Huang, W. Wang, M. Bates, and X. Zhuang, Science **319**, 810–813 (2008).
[7] E. A. Ash and G. Nicholls, Nature **237**, 510–513 (1972).
[8] G. Binnig, H. Rohrer, C. Gerber, and E. Weibel, Appl. Phys. Lett. **40**, 178–180 (1982).
[9] E. T. F. Rogers, J. Lindberg, T. Roy, S. Savo, J. E. Chad, M. R. Dennis, and N. I. Zheludev, Nature Mater. **11**, 432–435 (2012).
[10] M. V. Berry and S. Popescu, J. Phys. A: Math. Theor. **39**, 6965–6977 (2006).
[11] P. J. S. G. Ferreira and A. Kempf, IEEE Trans. Signal Process. **54**, 3732–3740 (2006).
[12] F. M. Huang and N. Zheludev, Nano Lett. **9**, 1249–1254 (2009).
[13] H. F. Wang, L. P. Shi, B. Luk'yanchuk, C. Sheppard, and C. T. Chong, Nature Photon. **2**, 501–505 (2008).
[14] M.V. Berry, J. Phys. A: Math. Theor. **46**, 205203 (2013).
[15] F. M. Huang, N. Zheludev, Y. Chen, and F. Javier Garcia de Abajo, Appl. Phys. Lett. **90**, 091119 (2007).
[16] G. Toraldo di Francia, Nuovo Cimento **9**, 426 (1952).
[17] K. G. Makris and D. Psaltis, Opt. Lett. **36**, 4335 (2011).
[18] B. Born, and E. Wolf, Principles of Optics, 6th edn (Pergamon, Oxford, 1993).
[19] J. E. Dennis, and R. B. Schnabel, Numerical methods for unconstrained optimization and nonlinear equations (Prentice-Hall, Inc., New Jersey, 1983).
[20] J. Nocedal and S. J. Wright, Numerical Optimization, 2nd edn (Springer Series in Operations Research, Springer Verlag, New York, 2006).
[21] K. Huang, P. Shi, X. Kang, X. Zhang, and Y. Li, Opt. Lett. **35** 965–967 (2010).
[22] P. Debye, Ann. Phys. **335**, 57–136 (1909).
[23] Min Gu, Advanced Optical Imaging Theory, 75 (Springer-Verlag, New York, 1999).
[24] Q. Zhan, Adv. Opt. Photon. **1** (1), 1–57 (2009).
[25] F. M. Huang, T. S. Kao, V. A. Fedotov, Y. Chen, and N. I. Zheludev, Nano Lett. **8**, 2469–2472 (2008).
[26] E. Greenfield, R. Schley, I. Hurwitz, J. Nemirovsky, K.G. Makris, and M. Segev, Opt. Exp. **21**, 13425–13435 (2013).
[27] M. R. Dennis, A. C. Hamilton, and J. Courtial, Opt. Lett. **33**, 2976 (2008).
[28] M. V. Berry and M. R. Dennis, J. Phys. A: Math. Theor. **42**, 022003 (2009).
[29] H. Liu, B. Wang, L. Ke, J. Deng, C. C. Chum, S. L. Teo, L. Shen, S. A. Maier and J. Teng, Nano Lett. **12**, 1549–1554 (2012).
[30] M. Duocastella and C. B. Arnold, Laser Photon. Rev. **6**, 607–5621 (2012).
[31] H. Liu, B. Wang, L. Ke, J. Deng, C. C. Chum, S. L. Teo, L. Shen, S. A. Maier and J. Teng, Adv. Func. Mater. **22**, 3777–3783 (2012).
[32] C. Lu and R. H. Lipson, Laser Photon. Rev. **4**, 568–580 (2010).
[33] H. Wang, C. J. R. Sheppard, K. Ravi, S. T. Ho, and G. Vienne, Laser Photon. Rev. **6**, 354–392 (2012).

 