Optimization-free superoscillatory lens using phase and amplitude masks

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To observe microscale objects, people always pursue superresolution imaging by decreasing the focused spot [1], tailoring the evanescent wave [2, 3], utilizing the nonlinear effect [4], exploiting the digital-image-processing technique [5, 6] and developing novel equipments [7, 8]. The newly demonstrated optical microscopy based on superoscillatory focusing provides another route to superresolution imaging [9]. This superoscillatory optical microscopy with the resolution of $\lambda/6$ has gained much attention because its focused spot can be infinitesimally sharp according to the superoscillation theory, which opens up a promising conceptual avenue to imaging arbitrarily small objects. Nevertheless, the superoscillatory spot with smaller feature suffers from its higher sidelobe, which, to some extent, imposes great challenges in the further application in high-resolution imaging resolution. Since the superoscillatory spot is inevitably accompanied by its high sidelobe [10, 11], one cannot eliminate the sidelobe if the superoscillation arises. Hence, it is nontrivial and imperative to push the high sidelobe far enough apart from the center, so as to produce realistic applications. However, this requires the elaborate manipulation over superoscillation via complicated lens design. The reported methods of constructing a superoscillatory pattern in an optical lens mainly rely on optimizing algorithms [9, 12] for FZP. Hence, the underlying physics, relating every feature of the physical lens structure and their contribution on the imaging plane, is not revealed yet, which in turn limits the flexible and controlled design of the superoscillation imaging in not only FZP but also binary-phase masks [13].

It is well known that the superoscillation in optics is one kind of destructive interference of light with different frequencies at some points at small intervals by matching the amplitude of every frequency [14]. This implies that one can control the optical superoscillation by choosing a suitable amplitude and frequency of light for the destructive interference at the prescribed position, which is a prototype inverse problem. We find that this inverse problem in some realistic optical devices, e.g., a zone plate (amplitude mask) or a binary-phase lens system (phase mask), can be described by a nonlinear matrix equation. Solving that can produce a customized superoscillatory pattern or control the superoscillation optionally. In contrast to using optimization for designing multiple rings as the only way, the unveiled fundamental physics behind the matrix enables us to analytically design a superfocusing central spot and push the high sidelobe away from the center for several wavelengths. In addition, we also attempt to propose a superoscillatory criterion in optical focusing, $r_S = 0.38/f_{max}$ ($f_{max}$ is the maximum spatial frequency), which determines whether the superoscillatory focusing occurs or not.

In contrast to the nanohole array [15], the zone plate with the amplitude modulation of 0 or 1 is an easy method to focus light into a superoscillatory spot. Optimization turns out to be the only method reported so far that can optimize the central radius and width of every belt in a
In order to evaluate the focusing properties of a single belt, we use the root-mean-square error (RMSE, whose definition is available in Supplementary Materials) between its diffracting intensity at the target plane and its corresponding zero-order Bessel function of \(|J_0(kr \sin \theta)|^2\) with the same \(\sin \theta\) of a single belt. The light from a belt has the different intensity profile at the target plane when the geometry of the transparent belt in Fig. 1a changes. Only the light passing through the belt with its geometry located in the colored region of Fig. 1b has a better focusing pattern with small RMSE, which can be approximated as a zero-order Bessel function of the first kind as shown in Fig. 1c, at the target plane. However, the intensity profile for the case of A in Fig. 1d might destroy the total intensity of the superoscillatory focusing for a subwavelength spot due to its poor focusing property at \(r = 0\) and the incomplete destructive interference at its first valley. The optimizing algorithm behaves poorly in rejecting the case of A by itself. In addition, even if all the belts in a zone plate have the geometry located in the colored region, it is still an arduous task for the optimizing method to realize the prescribed intensity (i.e., complete destructive interference with zero intensity) at the customized radial (r) position in the total intensity of the zone plate. To achieve the customized intensity pattern, we here suggest a mathematical method by solving a nonlinear matrix equation, without any optimizing technique involved, to design a superoscillatory mask.

Although some attempts based on the inverse of the matrix have been made to construct a superoscillatory pattern and diffraction-free beam [14, 16, 17], this method is only constrained to the case that the unknown amplitude-modulation coefficients are independent of the spatial frequency. For the zone plate, the amplitude-modulation coefficient from every spatial frequency has a tight relationship, which can be approximated as

\[
U_n(r) = \frac{1}{2\pi} \int_{R_n - \Delta r/2}^{R_n + \Delta r/2} \int_0^{2\pi} u(r, \phi) \frac{\partial}{\partial z} \left[ \frac{\exp(ikR)}{R} \right] \rho d\rho d\phi,
\]

where \(R^2 = z^2 + r^2 + \rho^2 - 2\rho \cos(\theta - \phi)\), the complex amplitude \(u(\rho, \phi)\) of the incident beam is taken as unity for the uniform illumination here. The electric field at the target plane beyond the evanescent region is

\[
U_n(r) = \frac{1}{2\pi} \int_{R_n - \Delta r/2}^{R_n + \Delta r/2} \int_0^{2\pi} u(r, \phi) \frac{\partial}{\partial z} \left[ \frac{\exp(ikR)}{R} \right] \rho d\rho d\phi,
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amplitude $A_n(r) = U_n(r)/C_n$. Figure 1b shows the root-mean-square error (RMSE) between $|A_n(r)|^2$ and its corresponding zero-order Bessel function of $|J_0(kr\sin\alpha)|^2$ with the same $\sin\alpha = R_n/(R_n^2 + z^2)^{1/2}$). In Fig. 1c, one can see that $|C_n|$ has a strong dependence on the width $\Delta r$ and the spatial frequency designated as $\sin\alpha$. Then, the total electric field of light passing through a zone plate containing $N$ belts can be expressed as

$$U(r) = \sum_{n=1}^{N} C_n A_n(r).$$

To realize the intensity $F = [f_1, f_2, \ldots, f_M]^T$ at the position $r = [r_1, r_2, \ldots, r_M]^T$ in the target plane, we can describe this problem as

$$SC = F,$$

where $S$ is an $M \times N$ matrix with its matrix element $S_{mn} = A_n(r_m)$ according to Eq. (2) and $C = [C_1, C_2, \ldots, C_N]^T$, where the sign $T$ means the transpose of matrix. The solution of Eq. (3) exists if $M \leq N$. Here, we just consider the case $M = N$ for which Eq. (3) has the only solution. Because the $S_{mn}$ and $C_n$ are dependent on the unknown $R_n$ (or $\sin\alpha$) when the width $\Delta r$ and $z$ are fixed, it is a nonlinear problem to solve the matrix equation for $R_n$. Although, in general, Eq. (3) has no analytical solution like the cases in [14, 17], its numerical solution can be easily obtained by using the well-developed Newton’s theory, which has been widely used to deal with the nonlinear problem in many areas [19, 20]. Newton’s theory for nonlinear problems solves Eq. (3) on the basis of the exact solution of its subproblem [20], which makes it a powerful tool to efficiently approach the exact solution without any search-based optimizing algorithm. The method described in Eq. (3) provides a very useful way to design a superoscillatory zone plate despite the fact that the solution is numerically approximated.

To verify the validity of our method, we show a constructed superoscillatory spot with size of about 0.57 $r_2$ (r2 is the Rayleigh limitation) and its sidelobe is about 1.8â€‰λ away from the center by using a zone plate, shown in Fig. 2a, which is designed by our method. In order to realize the goal of pushing away the sidelobe, we pad the zero intensity at the locations between the sidelobe and the center to suppress the high sidelobe near the center. The customized position $r$ with zero intensity must be carefully chosen to reject the generation of any high intensity between the high sidelobe and the center when solving Eq. (3). Therefore, we choose $F = [1, 0, 0, 0]^T$ at $r = [0, 0.33\lambda, 1.29\lambda, 1.73\lambda]^T$ for achieving a superoscillatory spot with the size of 0.57 $r_2$ (r2) and its sidelobe about $2\lambda$ away from the center in Fig. 2b. In the customized $F$ and $r$, $f_1 = 1$, $f_2 = 0$ and $r_1 = 0.33\lambda$ are used to define the superoscillatory spot and the rest is responsible for suppressing the sidelobe between the main spot and the high sidelobe. According to the result in Fig. 1b, we assume that the width $\Delta r$ of every belt has the same size of 0.3â€‰λ and the target plane is located in $z = 20\lambda$, in the simulation for removing the case of A in Fig. 1d. To obtain the unknown $R_n$ of every belt, we solve its inverse problem described in Eq. (3) by using the trust-region dogleg Newton theory that is introduced in the Supplementary Materials [20]. The solved $R_n$ is shown in the inset of Fig. 2b and their corresponding $\sin\alpha_n = [0.1387, 0.2576, 0.5643, 0.6638, 0.9548]$. Conventionally, in order to obtain a supersmall focused spot, one always prefers to focus the light of high spatial frequency with large amplitude, which leads to a small size spot dominating at the target plane, and interfere the light from different spatial frequencies constructively, which enhances the focused spot. However, in superoscillatory focusing, we here show an abnormal phenomenon that the maximum amplitude ($|C_n|$) is located at the frequency with the intermediate value. This counterintuitive requirement for obtaining a small spot by superoscillation mainly depends on the fact that the superoscillation always oscillates with very small amplitude that can be considered as almost destructive interference [14]. The destructive interference in the superoscillation is also reflected by the phase of $C_n$ that is shown in Fig. 2c. The phase difference between two neighboring belts in the designed zone plate is nearly $\pi$, which implies that the destructive interference is essentially required for realizing the superoscillation pattern in Fig. 2b. Thus, we can claim that the
amplitude-modulated coefficient $C_n$ has the alternating sign of $(-1)^n$ with its modulus small for low and high spatial frequency and large for the intermediate frequency, which is further confirmed by the case of focusing the light with rigorous single spatial frequencies (see Supplementary Materials). Nevertheless, this conclusion predicts that the zone plate is not ideal to realize a superoscillatory spot in Fig. 2b. Although the belt in the zone plate shows the excellent focusing property in a long range of $R_n$, shown in Fig. 2e, the phase of $C_n$, that is the case of $r = 0$ in Fig. 2d, varies from 0 to $2\pi$ quasi-periodically with the increase of $R_n$. As a result, much effort must be made to obtain the phase difference of $\pi$ for the alternating sign of $C_n$. Therefore, the zone plate may not be the best candidate to achieve a superoscillatory spot with its high sidelobe away, although we can use it to realize the superoscillatory spot in Fig. 2b.

Considering the difficulty of phase matching from a zone plate, we suggest another optical system containing a binary phase and a high numerical-aperture (NA) lens in Fig. 3a to realize the superoscillatory subwavelength focusing. The binary element with the phase 0 or $\pi$ located in the entrance pupil of the focusing lens provides the phase difference of $\pi$ for the generation of superoscillations in focusing [13, 21]. In the uniform illumination of an unpolarized beam, the electric field at the focal plane can be approximated by the Debye theory [22, 23]

$$U(r) = \frac{2\pi i}{\lambda} \int_0^\pi P(\theta) J_0(\kappa r \sin \theta) \sin \theta d\theta$$

where $P(\theta)$ is the apodization function that equals $p(\theta)\cos(\theta)^{1/2}$ for the lens obeying the sine condition [23, 24], $p(\theta)$ is the entrance pupil function that is $(-1)^n$ for the uniform illumination with the modulation of binary phase. The relationship between $R_n$ and $\theta_n (n = 0, 1, 2, \ldots, N)$ with $\theta_0 = 0, \theta_N = \alpha$ is $R_n f = \sin \theta_n$ for the sine lens used here, where $f$ is the focal length of focusing lens. We define the amplitude modulation coefficient $C_n = (-1)^n f_{in}(0)$ and $A_n (r) = U_n(r)/U_n(0)$. Similarly, the inverse problem of constructing the superoscillatory focusing lens [23] can also be expressed by Eq. (3) with the unknown variable $R_n$ (or $\sin \theta_n$). The phase $\theta_n (n = 0, 1, 2, \ldots, N)$ of every belt is increasing so that the amplitude modulation $|C_n|$ shows the monotonically increasing tendency from the low spatial frequency to the high spatial frequency. However, for a sine lens, the corresponding angle width $\Delta \theta_n (\approx \theta_n - \theta_{n-1})$ of every belt is increasing so that the amplitude modulation $|C_n|$ shows the monotonically increasing tendency from the low spatial frequency to the high spatial frequency. We can enlarge the distance further by padding more zero-intensity positions between the high sidelobe and the center. Figure 3c shows the structure of the designed binary phase by our method. The width $\Delta \theta_n (\approx \theta_n - \theta_{n-1})$ of belts in the binary phase tends to be diminishing at the outmost belts that are relative to the high spatial frequency. Considering the difficulty of phase matching from a zone plate, we suggest another optical system containing a binary phase and a high numerical-aperture (NA) lens in Fig. 3a to realize the superoscillatory subwavelength focusing. The binary element with the phase 0 or $\pi$ located in the entrance pupil of the focusing lens provides the phase difference of $\pi$ for the generation of superoscillations in focusing [13, 21]. In the uniform illumination of an unpolarized beam, the electric field at the focal plane can be approximated by the Debye theory [22, 23].
solve the inverse problem of superoscillation by using the optical system in Fig. 3a.

Next, we discuss the method that distinguishes a superoscillatory spot in optical focusing. Although the superoscillatory spot has been widely investigated in optical focusing and imaging [9, 12, 15, 25], none provides a clear demonstration as to how small a spot has to be so that it can be considered as a superoscillatory spot. To our knowledge, the Rayleigh criterion \( (r_R = 0.61/\lambda/NA) \) is mostly used to judge a superoscillatory spot in optical focusing [26]. However, it is a very rough method because there is no definition of superoscillation involved. In optics, a relevant and natural definition of superoscillation has been proposed by measuring the changing rate of the phase of a band-limited function in a local region [27, 28]. In particular, for the case of the 1-dimensional (or axisymmetric) band-limited function, i.e. the zone plate and a binary-phase-based lens, Berry and Dennis proposed a practical method for measuring the local wave number, \( k(r) = \text{Im}\{\partial F(r)/\partial r\} \), where \( F(r) \) is the band-limited function [28]. Therefore, the definition of local wave number by Berry and Dennis is preferred in optical focusing. However, when we use Berry and Dennis’s suggestion to evaluate the local wave number of a superoscillatory band-limited function in Fig. 4a, the calculated wave number in Fig. 4b is larger than the wave number of its maximum Fourier component only when the band-limited function has zero intensity. This means that, though the band-limited function indeed oscillates faster in the whole region \( x \in [-0.8\lambda, 0.8\lambda] \) than its maximum Fourier component, Berry and Dennis’s suggestion only predicts the superoscillation at the zero-intensity position. It is worth pointing out that Berry and Dennis’s suggestion gives the wave number at a certain position but not in a region where Fig. 4b shows the large wave number only at the zero-intensity position. Therefore, in optical focusing, it is better to define the superoscillatory spot by measuring the phase-changing rate in a certain region.

In optical focusing, we constrain the definition of a superoscillatory spot on three conditions: 1) The optical system is axisymmetric so that a circular spot could be generated. 2) The superoscillatory spot must oscillate faster in a certain region of the target plane than its maximum Fourier frequency component. 3) “A certain region” is located at \( r \leq r_S \), where \( r_S \) is the first zero-intensity position of the electric field at the target plane by focusing the light only from the maximum Fourier frequency component. The reason for choosing the region \( r \leq r_S \) is to exclude the case shown by the black curve of Fig. 4c, which has the fast superoscillation at \( r > r_S \) while its spot size is very large. In this region \( r \leq r_S \), the maximum Fourier frequency component only oscillates for one time without changing its phase, which is shown by the blue curve in Fig. 4c. If a spot oscillates faster in \( r \leq r_S \), this will lead to the generation of the intensity valley, where the high local wave number is located [28]. Thus, we can define a superoscillatory spot in optical focusing when its local wave number that is larger than the wave number of the maximum Fourier frequency is located in the region \( r \leq r_S \). In that case, a spot with its zero intensity located in \( r \leq r_S \) is superoscillatory, which is shown by red curves in Figs. 4a and c. This means that a superoscillatory spot has a smaller size than that \( r_{S} \) by only focusing its maximum spatial frequency, which implies that \( r_S \) can be taken as the superoscillatory criterion. When light with a single spatial frequency of \( \sin \alpha/\lambda \) (\( \alpha \) is the angle between the optical axis and the maximum convergent ray) is focused, its electric field at the target plane is proportional to the zero-order Bessel function \( J_0(2\pi r/\sin \alpha/\lambda) \) of the first kind, which gives \( r_S = 0.38/\lambda/\sin \alpha \). The superoscillatory criterion \( r_S \) has a similar shape to the Rayleigh criterion \( r_R \). Figure 4d shows the spot size in different NA that is usually in terms of \( \sin \alpha/\lambda/\sin \alpha \). For a given NA, the spot in all the cyan and dark-blue areas below the Rayleigh criterion (black curve) can be called the superresolution spot and the spot in the dark-blue area below the superoscillation criterion (white curve) is the superoscillation spot, which means that the superoscillation spot is one subaggregate of the superresolution spot. The finely distinguished roadmap in Fig. 4d provides an instructive guide that the cyan area between the Rayleigh and superoscillation criterion is the best choice when one pursues a superresolution focusing spot without high sidelobe beyond the evanescent range.
importantly, $r_3$ implies a limitation of 0.38$\lambda$, for the application of subwavelength spot without high sidelobe.

In summary, we have demonstrated a physical design roadmap of the superoscillatory focusing by using a zone plate or a binary-phase-based lens, with significantly enlarged field of view. The described inverse problem of superoscillation in terms of a nonlinear matrix equation enables construction of a customized superoscillatory pattern possible to be implemented without the traditional optimizing technique involved in the reported superoscillatory lens. This paves the way to a new scheme in further improving the resolution of the optical far-field imaging, and narrowing width of longitudinally polarized needle light for advanced data-storage performance [13]. In achieving a supersmall spot beyond the evanescent region, our result shows a counterintuitive phenomenon that the large spatial frequency with low intensity and destructive interference must be involved. Furthermore, the superoscillatory criterion proposed here gives us the direct insight into the spot pattern beyond the Rayleigh limitation, which sets a theoretical limitation of 0.38$\lambda$ for the spot size in some applications that demand the narrow spot and low sidelobe simultaneously, i.e. optical lithography [29], high-intensity optical machining [30] and high-contrast superresolution imaging [31–33].

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