Smith Chart

Smith chart is a graphical plot of the normalized resistance and reactance functions in the complex reflection-coefficient plane. It is a graph showing both the normalized impedance and the reflection coefficient.

Smith chart is convenient for transmission line and circuit calculations. It is also a useful tool in impedance matching circuit design.
Recall that:

\[ \Gamma(\ell) = \frac{Z(\ell) - Z_0}{Z(\ell) + Z_0} \]

\[ Z(\ell) = Z_0 \frac{1 + \Gamma(\ell)}{1 - \Gamma(\ell)} \]

Now normalize the impedance \(Z(\ell)\) by \(Z_0\).

\[ z = \frac{Z(\ell)}{Z_0} \]
In terms of the normalized impedance $z$ (drop the $\ell$ dependence), we can write:

$$\Gamma = \frac{z - 1}{z + 1} = \Gamma_{re} + j\Gamma_{im}$$

$$z = r + jx = \frac{1 + \Gamma}{1 - \Gamma} = \frac{(1 + \Gamma_{re}) + j\Gamma_{im}}{(1 - \Gamma_{re}) - j\Gamma_{im}}$$

From the last equation, we have

$$r = \frac{1 - \Gamma_{re}^2 - \Gamma_{im}^2}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$

$$x = \frac{2\Gamma_{im}}{(1 - \Gamma_{re})^2 + \Gamma_{im}^2}$$
The last two equations of $r$ and $x$ define two families of circles in the complex plane of reflection coefficient $\Gamma$.

The Smith char is the superposition of these two families of circles together in the complex plane of reflection coefficient $\Gamma$. 
The Smith Chart
A point in the Smith chart gives the values of the normalized impedance \( z \) and the complex reflection coefficient \( \Gamma \) at the same point on a transmission line.
When the angle of $\Gamma$ is zero, $\Gamma$ is real and $\Gamma = |\Gamma|$. Then,

$$\frac{1 + \Gamma}{1 - \Gamma} = \frac{1 + |\Gamma|}{1 - |\Gamma|} = S$$

But when the angle of $\Gamma$ is zero,

$$\frac{1 + \Gamma}{1 - \Gamma} = z = r$$

Thus, the value of $S$ is same as $r$ when the angle of $\Gamma$ is zero and can be read out directly from the Smith chart by noting the $r$ value ($S = r$).

Since all points on the dotted black circle have the same $|\Gamma|$, they must also have the same $S$. This circle is known as the constant VSWR circle.
For example,

All points on this circle have a $S = r = 3$
### Example 1
Plot the following impedances on to the Smith chart.

<table>
<thead>
<tr>
<th>$Z$ (Ω)</th>
<th>$z$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z_1 = 100 + j50$</td>
<td>$z_1 = 2 + j$</td>
<td>$\Gamma_1 = 0.45 \angle 27^\circ$</td>
</tr>
<tr>
<td>$Z_2 = 75 - j100$</td>
<td>$z_2 = 1.5 - j2$</td>
<td>$\Gamma_2 = 0.65 \angle -38^\circ$</td>
</tr>
<tr>
<td>$Z_3 = j200$</td>
<td>$z_3 = j4$</td>
<td>$\Gamma_3 = 1\angle 28^\circ$</td>
</tr>
<tr>
<td>$Z_4 = 150$</td>
<td>$z_4 = 3$</td>
<td>$\Gamma_4 = 0.5 \angle 0^\circ$</td>
</tr>
<tr>
<td>$Z_5 = \infty$</td>
<td>$z_5 = \infty$</td>
<td>$\Gamma_5 = 1 \angle 0^\circ$</td>
</tr>
<tr>
<td>$Z_6 = 0$</td>
<td>$z_6 = 0$</td>
<td>$\Gamma_6 = 1 \angle 180^\circ$</td>
</tr>
<tr>
<td>$Z_7 = 50$</td>
<td>$z_7 = 1$</td>
<td>$\Gamma_7 = 0$</td>
</tr>
<tr>
<td>$Z_8 = 184 - j900$</td>
<td>$z_8 = 3.68 - j18$</td>
<td>$\Gamma_8 = 0.97 \angle -6^\circ$</td>
</tr>
</tbody>
</table>
Solutions
Recall that on a transmission line:

\[ \Gamma(\ell) = \Gamma = \Gamma_L e^{-j2k\ell} \]

\[ |\Gamma| = |\Gamma_L| \]
Hence, $\Gamma$ can be obtained from $\Gamma_L$ by moving clockwise along a constant circle on the Smith chart with a radius $|\Gamma_L|$ through an angle $-2k\ell$ which is equivalent to $\ell/\lambda$ wavelengths measured towards the generator on the periphery of the Smith chart.

This circle is also known as the constant VSWR circle. All the points on this circle has the same $S$ and same $|\Gamma|$.
1.2 Reading on Smith chart

Several scales around the outside of the Smith chart are used to determine the distance along the line. Some Smith charts have a number of scales at the bottom of the chart for measuring the reflection coefficient magnitude and others.
Two scales on the periphery (in wavelengths):
- Wavelengths towards generator (WTG scale), clockwise sense
- Wavelengths towards load (WTL scale), anticlock sense

Note also that a complete turn around the Smith chart corresponds to a total length of $\lambda/2$. Because:

$$\Gamma(\ell_2) = \Gamma(\ell_1)e^{-j2k(\ell_2-\ell_1)}$$

When $\ell_2 - \ell_1 = 0.5\lambda$, the phase turned from $\Gamma(\ell_1)$ to $\Gamma(\ell_2)$ is:

$$2k(\ell_2 - \ell_1) = 2\frac{2\pi}{\lambda} \times 0.5\lambda = 2\pi = 360^\circ$$
Impedance ↔ admittance

Any point reflected through the centre point converts an impedance to an admittance and vice versa.

Top Half: inductive reactance, $X_L = \omega L$
  or
  capacitive susceptance, $B_C = \omega C$

Bottom Half: capacitive reactance, $X_C = 1/\omega C$
  or
  inductive susceptance, $B_L = 1/\omega L$

\[
\begin{align*}
z & = 1.8 + j2 \\
y & = 0.25 - j0.28 \\
& = 1/(1.8 + j2)
\end{align*}
\]
Example 2
Use Smith chart to find the input impedance $Z_{in}$ looking at the input of a transmission line.

(a) Actual circuit
(b) Normalized circuit
Example 2 (cont’d):

\[ \Gamma_L = 0.7 \angle 45^\circ \]

\[ = 0.7e^{j\frac{\pi}{4}} \]
Example 2 (cont’d):

\[ \Gamma_{in} = \Gamma_L e^{-j2kz'} \]

\[ = 0.7 e^{j\frac{\pi}{4} - j2 \times \frac{2\pi}{\lambda} \times 0.3\lambda} \]

\[ = 0.7 e^{-j0.95\pi} \]

\[ z_{in} = \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} = 0.1775 - j0.0762 \text{ (at point c)} \]

\[ Z_{in} = z_{in}Z_0 = 8.8765 - j3.8118 \ \Omega \]

See animation “Transmission Line Impedance Calculation”
Voltage Maxima and Minima in Smith Chart

Voltage maxima occur when the angle of the reflection coefficient \( \Gamma(\ell) \) \( \theta = -2n\pi \) (\( n = 0, 1, 2, \ldots \)). This corresponds to the right-most point in the Smith chart. Voltage minima occur when the angle of the reflection coefficient \( \Gamma(\ell) \) \( \theta = -2(n+1)\pi \) (\( n = 0, 1, 2, \ldots \)). This corresponds to the left-most point in the Smith chart. See an example shown on next page.
$$z_L = 0$$
2 Impedance Matching

Meaning of impedance matching
Impedance matching is to eliminate the reflected voltage or current on a transmission line.

Reasons for impedance matching:
1. Maximize power transfer to the load
2. The input impedance remains constant at the value $Z_0$. Therefore, the input impedance is independent of the length of transmission line.
3. VSWR = 1. Therefore there are no voltage peaks on the transmission line.

Two matching techniques:
1. Quarter-wave transformer
2. Single-stub matching network
2.1 Quarter-wave transformer

For a transmission line of length \( d = \lambda / 4 \), characteristic impedance = \( Z_s \), and terminated in an impedance \( R_L \),

\[
Z_i = Z(\ell = \lambda / 4) = Z_s \frac{R_L + jZ_s \tan(\pi/2)}{Z_s + jR_L \tan(\pi/2)} = \frac{Z_s^2}{R_L}
\]

When \( Z_s \) is real, we can change \( Z_s \) to achieve a desired \( Z_i \).
Example 3
A signal generator has an internal impedance of 50 $\Omega$. It needs to feed equal power through a lossless 50 $\Omega$ transmission line to two separate resistive loads of 64 $\Omega$ and 25 $\Omega$ at a frequency of 10 MHz. Quarter-wave transformers are used to match the loads to the 50 $\Omega$ line, as shown below.
(a) Determine the required characteristic impedances and the physical lengths of the quarter-wavelength lines assuming the phase velocities of the waves traveling on the them is 0.5$c$. (b) Find the standing-wave ratios on the matching line sections.
Solutions

(a) As the two quarter-wave transformers are connected in parallel to the 50-Ω line, if equal powers are required to the two loads, the input impedances of the two branches looking at the junction from the 50-Ω line must be equal to 100 Ω so that when they add together in parallel, the total impedance is 50 Ω.

\[ R_{in1} = 100 \, \Omega \]

\[ R_{in2} = 100 \, \Omega \]

\[ R_0 = 50 \, (\Omega) \]

\[ R_{L1} = 64 \, (\Omega) \]

\[ R_{L2} = 25 \, (\Omega) \]
Solutions (cont’d):

Therefore, \( Z_{in1} = R_{in1} = 100 \, \Omega \)
\( Z_{in2} = R_{in2} = 100 \, \Omega \)

The characteristic impedances \( R'_{01} \) and \( R'_{02} \) can be found by:
\[
R'_{01} = \sqrt{R_{in1}R_L} = \sqrt{100 \times 64} = 80 \, \Omega \\
R'_{02} = \sqrt{R_{in2}R_L} = \sqrt{100 \times 25} = 50 \, \Omega 
\]

\[
u_p = \frac{1}{\sqrt{\mu \varepsilon}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0 \varepsilon_r}} \times \frac{c}{2} \Rightarrow \varepsilon_r = 4
\]

\[
\lambda = \text{wavelength along the transformers} = \frac{\lambda_0}{\sqrt{\varepsilon_r}} = \frac{30 \, \text{m}}{2} = 15 \, \text{m}
\]

Physical length of the transformers = \( \lambda / 4 = 3.75 \, \text{m} \)
Solutions (cont’d):
(b) Under matched conditions, there are no standing waves on the main transmission line, i.e. $S = 1$. The standing wave ratios on the two matching line sections are as follows:

Matching section No. 1:

$$\Gamma_{L1} = \frac{R_{L1} - R'_{01}}{R_{L1} + R'_{01}} = \frac{64 - 80}{64 + 80} = -0.11$$

$$S_1 = \frac{1 + |\Gamma_{L1}|}{1 - |\Gamma_{L1}|} = \frac{1 + 0.11}{1 - 0.11} = 1.25$$

Matching section No. 2:

$$\Gamma_{L2} = \frac{R_{L2} - R'_{02}}{R_{L2} + R'_{02}} = \frac{25 - 50}{25 + 50} = -0.33$$

$$S_2 = \frac{1 + |\Gamma_{L2}|}{1 - |\Gamma_{L2}|} = \frac{1 + 0.33}{1 - 0.33} = 1.99$$
2.2 Single-stub matching network

What is a stub?
A stub is a short section of transmission line (shorted or open at one end) whose input impedance can be changed by varying its length.

- B-B' = matching point
- d = matching position
- ℓ = length of the matching stud
- \( y_B \) = normalized admittance of line at B-B'
- \( y_s \) = normalized admittance of the stub
- \( y_L \) = normalized admittance of the load
When a transmission line is matched at the matching point,

\[ y_i = y_B + y_s = 1 \quad \text{or} \quad Y_i = Y_B + Y_s = Y_0 \]

After matched, there is no reflection on the line to the left of B-B’. But there are reflections on the line to the right of B-B’ and on the stub.
Use of admittance

For parallel stub matching, the stub is connected in parallel with the transmission line. Hence it will be more convenient to use admittance rather than impedance. In Smith chart, when a impedance $z$ is known, the corresponding admittance, $y = 1/z$, can be obtained by a reflection through the centre of the Smith chart. The admittance $y$ is represented on the same Smith chart but its position is different from that of $z$. When every impedance point on the Smith chart is reflected in this way, we transform the impedance Smith chart to an admittance Smith chart in which every point now represents a normalized admittance. For parallel-stub matching, we work in the admittance Smith chart.
**Reminder…**

On plotting into the Smith chart, all values have to be normalized by the characteristic impedance $Z_0$ (or the characteristic admittance $Y_0 = 1/ Z_0$) first. Normalized values are usually represented by small letters while un-normalized values by CAPITAL LETTERS. For example:

<table>
<thead>
<tr>
<th>Normalized quantity</th>
<th>Un-normalized quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_L = Z_L/ Z_0$</td>
<td>$Z_L$</td>
</tr>
<tr>
<td>$y_L = Y_L/ Y_0$</td>
<td>$Y_L$</td>
</tr>
<tr>
<td>$z_{in} = Z_{in}/ Z_0$</td>
<td>$Z_{in}$</td>
</tr>
<tr>
<td>$y_{in} = Y_L/ Y_0$</td>
<td>$Y_{in}$</td>
</tr>
</tbody>
</table>
**Method to determine d and ℓ**

Choose \(d\) such that:

\[
y_B = \frac{1}{z(\ell = d)} = \frac{Z_0 + jZ_L \tan(kd)}{Z_L + jZ_0 \tan(kd)} = 1 + jb_B
\]

Choose \(\ell\) such that:

\[
y_s = \frac{1}{j \tan(k\ell)} = -j \cot(k\ell) = -jb_B
\]

So that:

\[
y_i = y_B + y_s = 1
\]
Steps in single-stub matching (using normalized $z$ and $y$):

1. Convert the load impedance $z_L$ to an equivalent admittance $y_L = 1/z_L$.

2. Use a line of length $d$ and a characteristic impedance $Z_0$ (characteristic admittance $Y_0 = 1/Z_0$) to transform $y_L$ to $y_B = 1 + jb_B$ at B-B'.

3. Connect a parallel stub of length $\ell$ and characteristic impedance $Z_0$ at B-B' with an input admittance $y_s = -jb_B = -j\cot(2\pi\ell/\lambda)$.

4. Then, the total admittance at B-B' is:

$$ y_i = y_B + y_s = 1 + jb_B - jb_B = 1 \Rightarrow \text{matched} $$
The detailed matching steps in the Smith chart will be explained by using the example shown below.

Example 4
A 50 Ω lossless transmission line is connected to a load impedance \( Z_L = 35 - j47.5 \Omega \). Find the position \( d \) and length \( l \) of a short-circuit stub required to match the load at a frequency of 200 MHz. Assume that the transmission line is a coaxial line filled with a dielectric material for which \( \varepsilon_r = 9 \).

Solutions
Given \( Z_0 = R_0 = 50 \Omega \) and \( Z_L = 35 - j47.5 \Omega \). \( \Rightarrow z_L = Z_L / Z_0 = 0.7 - j 0.95 \).

- Enter \( z_L \) at point \( P_1 \).
- Draw a \(|\Gamma|\) -circle centred at \( O \) with radius \( \overline{OP_1} \).
- Draw straight line from \( P_1 \) through \( O \) to point \( P_2' \) on the perimeter, intersecting the \(|\Gamma|\) -circle at \( P_2 \), which represents \( y_L \). Note 0.109 at \( P_2' \) on the “wavelengths toward generator” scale.
• Note the two points of intersection of the $|\Gamma|$-circle with the $g = 1$ circle:
  
  o At $P_3$: $y_{B1} = 1 + j1.2 = 1 + jb_{B1}$
  
  o At $P_4$: $y_{B2} = 1 - j1.2 = 1 + jb_{B2}$

• Solutions for the position of the stub:
  
  o For $P_3$ (from $P_2'$ to $P_3'$): $d_1 = (0.168 - 0.109)\lambda = 0.059\lambda$
  
  o For $P_4$ (from $P_2'$ to $P_4'$): $d_2 = (0.332 - 0.109)\lambda = 0.223\lambda$

• Solutions for the length of the short-circuited stub to provide $y_s = -jb_B$:
  
  o For $P_3$, $y_s = -jb_{B1} = -j1.2$ (from $P_{sc}$ to $P_3''$): $l_1 = (0.361 - 0.250)\lambda = 0.111\lambda$
  
  o For $P_4$, $y_s = -jb_{B2} = j1.2$ (from $P_{sc}$ to $P_4''$): $l_2 = (0.139 + 0.250)\lambda = 0.389\lambda$

To compute the physical lengths of the transmission line sections, we need to calculate the wavelength on the transmission line. Therefore

$$\lambda = \frac{u_p}{f} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{\varepsilon_r}} \approx 0.5 \text{ m}.$$ 

Thus:

<table>
<thead>
<tr>
<th>$d_1$ $= 0.059\lambda = 29.5 \text{ mm}$</th>
<th>$l_1 = 0.111\lambda = 55.5 \text{ mm}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d_2 = 0.223\lambda = 111.5 \text{ mm}$</td>
<td>$l_2 = 0.389\lambda = 194.5 \text{ mm}$</td>
</tr>
</tbody>
</table>
Either of these two sets of solutions would match the load. In fact, there is a whole range of possible solutions. For example, when calculating $d_1$, instead of going straight from $P_2'$ to $P_3'$, we could have started at $P_2'$, rotated clockwise around the Smith chart $n$ times (representing an additional length of $n\lambda/2$) and continued on to $P_3'$, yielding $d_1 = 0.059\lambda + n\lambda/2$, $n = 0, 1, 2, \ldots$. The same argument applies for $d_2$, $l_1$ and $l_2$.

**Reasons for Choosing a Shorter-Line Solution**

Sometimes, out of the two basic sets of solutions, $(d_1, l_1)$ and $(d_2, l_2)$, it is advantageous to choose a shorter combination of $d$ and $l$. There are at least two reasons for this:

1. Shorter lines can always reduce the inevitable loss along the lines, though this is small.
2. Shorter lines result in a smaller Q-factor for the resonant circuit consisting of the stub and the transmission line to the right of the matching point. A smaller Q-factor produces a wider matching bandwidth, i.e., the matching condition being less sensitive to frequency change.

**Drawings on Smith chart shown on next page**
Solutions on Smith chart for Example 4

See animation
“Parallel Stub Matching – Short”