Image Charge Theory

Consider a charge $Q$ being placed in region 1 at a position $(0,0,a)$. Region 1 and region 2 meet at a boundary on the $x$-$y$ plane. In region 2, the electric field produced by $Q$ polarizes the molecules in this region, resulting in a negative bounded surface charge density on the boundary. Hence the total electric field $E_n^+$ just above the boundary $(0,0,0^+)$ can be calculated by the principle of superposition: that produced by $Q$ (i.e., $E_{n1}$) and that produced by the negative bounded charge density on the boundary $E_s$. That is (we consider the field magnitudes only),

$$E_n^+ = E_{n1} + E_s$$

(1)

where

$$E_{n1} = \frac{Q}{4\pi\varepsilon_1 a^2}$$

(2)

On the other hand, the total electric field $E_n^-$ just below the boundary $(0,0,0^-)$ can be calculated as that produced by $Q$ (i.e., $E_{n1}$) plus that produced by the negative charge density on the boundary $E_s$. That is,

$$E_n^- = E_{n1} - E_s$$

(3)

The negative sign now with $E_s$ is because the surface charge density is negative (see the magnified view in the figure above) and below the boundary, the field of $Q$ and the field of the negative charge density are opposite in direction.
Now using the boundary condition (no net free charges on the boundary):

\[ D_{n1} = D_{n2} \]  \hspace{1cm} (4)

we have

\[ \varepsilon_1 E_0^+ = \varepsilon_2 E_0^- \]

\[ \varepsilon_1 (E_{n1} + E_s) = \varepsilon_2 (E_{n1} - E_s) \]  \hspace{1cm} (5)

So we have

\[ E_s = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} E_{n1} \]  \hspace{1cm} (6)

Using (2) for \( E_{n1} \),

\[ E_s = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} E_{n1} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \frac{Q}{4\pi\varepsilon_0 a^2} = \frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} \frac{Q}{4\pi\varepsilon_0 a^2} \]  \hspace{1cm} (7)

From the above expression in (7), there are two cases need to be considered:

(I) When \( \varepsilon_2 > \varepsilon_1 \), we can consider \( E_s \) to be produced by an equivalent negative charge of

\[ Q_1 = -\frac{\varepsilon_2 - \varepsilon_1}{\varepsilon_2 + \varepsilon_1} Q \]  \hspace{1cm} (8)

in a medium with \( \varepsilon_1 \) and placed at a distance \( a \) below the boundary (i.e., -\( a \) in order to produce the downward pointing field just above the boundary). As this charge produces an electric field that (i) satisfies the boundary condition together with the electric field produced by charge \( Q \) in region 1, and (ii) in the region 1, there is only charge \( Q_1 \), then by the uniqueness theorem of the electromagnetic field, the electric field anywhere in region 1 can be calculated by the electric field produced by \( Q \) and \( Q_1 \) together. Note that \( Q_1 \) can also be placed at a distance \( a \) above the boundary and positive in sign instead of negative, but this placement introduces an additional charge (\( Q_1 \)) in region 1 and violates the uniqueness theorem. Hence this arrangement cannot be used.

Now in region 2, the electric fields produced by \( Q \) and \( Q_1 \) (now placed at a distance \( a \) above the boundary) satisfy the boundary condition. That is,
\[ E_n^r = E_{n1} - E_s \]
\[ = \frac{Q}{4\pi\epsilon_1 a^2} + \left( \frac{-\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \right) \frac{Q}{4\pi\epsilon_2 a^2} \left( 1 - \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \right) \]
\[ = \frac{Q}{4\pi\epsilon_1 a^2} \left( \frac{2\epsilon_1}{\epsilon_2 + \epsilon_1} \right) \]
\[ = \frac{1}{4\pi\epsilon_2 a^2} \left( \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} Q \right) \]
\[ = \frac{1}{4\pi\epsilon_2 a^2} Q_2 \] (9)

where

\[ Q_2 = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1} Q \] (10)

can be considered to be an equivalent charge placed at a distance \( a \) above the boundary (but in a medium with \( \epsilon_2 \) as shown in the last line of (9)) for producing the electric field to satisfy the boundary condition on the side of region 2. Then by the uniqueness theorem, the field everywhere in region 2 can be considered to be produced by \( Q_2 \).

The equivalent charges \( Q_1 \) and \( Q_2 \) in (8) and (10), respectively, are called the **image charges**.

(II) When \( \epsilon_2 < \epsilon_1 \), we can consider \( E_s \) to be produced by an equivalent **positive charge** of

\[ Q_3 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} Q \] (11)

in a medium with \( \epsilon_1 \) and placed at a distance \( a \) below the boundary so as to produce an upward pointing field just above the boundary. Using the same reasoning as in the first case (case I), the electric field anywhere in region 1 can be calculated by the electric field produced by \( Q \) and \( Q_3 \) together.

Now for region 2, the electric field just below the boundary is (with \( Q_3 \) placed at a distance \( a \) above the ground plane in order to satisfy the uniqueness theorem in region 2):
\[ E_n = E_{a1} - E_s \]

\[
= \frac{Q}{4\pi \varepsilon_0 a^2} + \left( \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right) \frac{Q}{4\pi \varepsilon_0 a^2} \left( 1 + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right) \\
= \frac{Q}{4\pi \varepsilon_0 a^2} \left( \frac{2\varepsilon_1}{\varepsilon_1 + \varepsilon_2} \right) \\
= \frac{1}{4\pi \varepsilon_0 a^2} \left( \frac{2\varepsilon_2}{\varepsilon_1 + \varepsilon_2} \right) Q \\
= \frac{1}{4\pi \varepsilon_0 a^2} Q_4
\]  

(12)

where

\[ Q_4 = \frac{2\varepsilon_2}{\varepsilon_2 + \varepsilon_1} Q \]

(13)

can be considered to be an equivalent charge placed at a distance \( a \) above the boundary for producing the electric field to satisfy the boundary condition on the side of region 2. Then by the uniqueness theorem, the field everywhere in region 2 can be considered to be produced by \( Q_4 \).

Similarly, the equivalent charges \( Q_3 \) and \( Q_4 \) in (11) and (13), respectively, are called the image charges for this case.