Lecture 4: Magnetostatics – Part I

- **Magnetic Forces and Torques**
  - Force on a Current-Carrying Conductor
  - Torque on a Current-Carrying Loop

- **The Biot-Savart Law**
  - Field due to Surface/Volume Current Distribution
  - Field of a Magnetic Dipole

- **Magnetic Force between Two Parallel Conductors**

- **Maxwell’s Magnetostatic Equations**
  - Gauss’s Law for Magnetism
  - Ampère’s Law
Magnetic Forces and Torques

• Concepts - Equations
  – The steady currents produce magnetic field while the steady charges produce static electric fields
  – The magnetic flux density $\mathbf{B}$ and the magnetic field intensity $\mathbf{H}$ are related by
  – For most dielectric and metals (excluding ferromagnetic materials), we have $\mu = \mu_0$.

\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{H} = \mathbf{J}. \]

\[ \mathbf{B} = \mu \mathbf{H} \quad \text{or} \quad \mu_0 \mathbf{H}. \]
Magnetic Forces and Torques

- Concepts – Magnetic Force
  - The magnetic force $\mathbf{F}_m$ acting on a particle of charge $q$ can be cast in the form of $\mathbf{F}_m = qu\times\mathbf{B}$ or $F_m = quB\sin\theta$. 

![Diagram](a) ![Diagram](b)
Magnetic Forces and Torques

• Concepts – Lorentz Force
  – The Lorentz force is defined as
    \[ F = F_e + F_m = qE + qu \times B = q(E + u \times B). \]
  - Whereas the electric force is always in the direction of the electric field, the magnetic force is always perpendicular to the magnetic field.
  - Whereas the electric force acts on a charged particle whether or not it is moving, the magnetic force acts on it only when it is in motion.
  - Whereas the electric force expends energy in displacing a charged particle, the magnetic force does no work when a particle is displaced \[ dW = F_m \cdot d\vec{l} = (F_m \cdot u) dt = 0. \]
Magnetic Forces and Torques

• **Force on a Current-Carrying Conductor**
  - Consider a wire containing a free-electron charge density $\rho_{ve} = -N_e e$, where $N_e$ is the number of moving electrons per unit volume. Then the total amount of moving charge contained in an elementary volume of the wire is $dQ = \rho_{ve} Adl = -N_e e Adl$.
  - Therefore, the corresponding magnetic force acting on the $dQ$ in a field $\mathbf{B}$ is $d\mathbf{F}_m = dQ \mathbf{u}_e \times \mathbf{B} = -N_e e A (dl \mathbf{u}_e) \times \mathbf{B} = N_e e A u_e dl \times \mathbf{B}$, where $\mathbf{u}_e$ is the drift velocity of the electrons and $dl \mathbf{u}_e = - u_e dl$.
  - The current flowing through a cross-sectional area $A$ due to electrons with density is $I = \rho_{ve} (-u_e) A = (-N_e e) (-u_e) A = N_e e A u_e$, therefore we re-write the force as $d\mathbf{F}_m = I dl \times \mathbf{B}$ (N).
  - For a closed circuit of contour $C$ carrying a current $I$, the total magnetic force is

\[
\mathbf{F}_m = I \oint_C dl \times \mathbf{B}, \quad \text{(N)}.
\]
Magnetic Forces and Torques

- **Force on a Current-Carrying Conductor**
  - Closed Circuit in a Uniform $\mathbf{B}$ Field
    \[
    \mathbf{F}_m = I \int_C d\mathbf{l} \times \mathbf{B} = I \left( \int_C d\mathbf{l} \right) \times \mathbf{B} = 0.
    \]
    The total magnetic force on any closed current loop in a uniform magnetic field is zero.
  - Curved Wire in a Uniform $\mathbf{B}$ Field
    \[
    \mathbf{F}_m = I \int_{a}^{b} d\mathbf{l} \times \mathbf{B} = I \left( \int_{a}^{b} d\mathbf{l} \right) \times \mathbf{B} = I\mathbf{l} \times \mathbf{B}.
    \]
    where $\mathbf{l}$ is the vector directed from $a$ to $b$, and irrespective of the path between $a$ and $b$. 
**Example 1**

**Exercise 4.4**  A horizontal wire with a mass per unit length of 0.2 kg/m carries a current of 4 A in the +x-direction. If the wire is placed in a uniform magnetic flux density $\mathbf{B}$, what should the direction and minimum magnitude of $\mathbf{B}$ be in order to magnetically lift the wire vertically upward? (Hint: The acceleration due to gravity is $g = -\hat{z}9.8 \text{ m/s}^2$.)

**Solution:** For a length $l$,

\[
\mathbf{F}_g = -\hat{z}0.2l \times 9.8 = -\hat{z}1.96l \quad \text{ (N)}
\]

\[
\mathbf{F}_m = \hat{z}ll \times \mathbf{B}
\]

For $\mathbf{F}_m + \mathbf{F}_g = 0$, $\mathbf{F}_m$ has to be along $+\hat{z}$, which means that $\mathbf{B}$ has to be along $+\hat{y}$. Hence,

\[
1.96l = llB
\]

\[
B = \frac{1.96}{l} = 0.49 \text{ (T)}, \text{ and}
\]

\[
\mathbf{B} = \hat{y}0.49 \text{ (T)}.
\]
Magnetic Forces and Torques

- **Torque on a Current-Carrying Loop**
  
  - **Concepts**
    - The *torque* $T$ is defined as the cross product of *distance vector* $\mathbf{d}$ and the *magnetic force* $\mathbf{F}$, i.e., $T = \mathbf{d} \times \mathbf{F}$ (N.m), or
      $$T = \hat{z} |\mathbf{d}| |\mathbf{F}| \sin \theta = \hat{z} r F \sin \theta.$$
    - The *unit* for *torque* $T$ is the same as that for *work* or *energy*, but torque is a vector and makes an angle between $\mathbf{d}$ and $\mathbf{F}$, as shown.
    - *Right-hand rule*: When the thumb of the right hand is pointed along the direction of the torque, the four fingers indicate the direction that the torque is trying to rotate the body.
Magnetic Forces and Torques

- **Torque on a Current-Carrying Loop**
  - **Magnetic Field in the Plane of the Loop**
    - A loop with a current \( I \) lies in the \( x-y \) plane and is pivoted about the axis shown.
    - Therefore, only arms 1 and 3 of the loop are subjected to forces \( \mathbf{F}_1 \) and \( \mathbf{F}_3 \) given by
      \[
      \mathbf{F}_1 = I \left ( -\hat{y}b \right ) \times \left ( \mathbf{x}B_0 \right ) = \hat{z}IB_0, \\
      \mathbf{F}_3 = I \left ( \hat{y}b \right ) \times \left ( \mathbf{x}B_0 \right ) = -\hat{z}IB_0.
      \]
    - The torque is thus given by
      \[
      \mathbf{T} = \mathbf{d}_1 \times \mathbf{F}_1 + \mathbf{d}_3 \times \mathbf{F}_3 \\
      = \left ( -\hat{x} \frac{a}{2} \right ) \times \hat{z}IB_0 + \left ( \hat{x} \frac{a}{2} \right ) \times \left ( -\hat{z}IB_0 \right ) \\
      = \hat{y}IabB_0 = \hat{y}IAB_0.
      \]
Magnetic Forces and Torques

- Torque on a Current-Carrying Loop

- Magnetic Field Perpendicular to the Axis of a Rectangular Loop
  - There is an angle $\theta$ produced where $T = IAB_0 \sin \theta$.
  - The torques $F_2$ and $F_4$ are equal in magnitude but opposite in direction, thus their net torque contributions will be zero.
  - If the loop consists of $N$ turns, then the torque is $T = NIAB_0 \sin \theta$.
  - Therefore, the torque can be written in vector form as follows:
    \[ T = m \times B \] (N.m) where $m = \hat{n} NI A$ (A.m²)
Magnetic Forces and Torques

**Example 2**

**Exercise 4.5** A square coil of 200 turns and 0.5-m-long sides is in a region with a uniform magnetic flux density of 0.2 T. If the maximum magnetic torque exerted on the coil is $4 \times 10^{-2}$ (N·m), what is the current flowing in the coil?

**Solution:**

$$T_{\text{max}} = NIAB_0$$

$$I = \frac{T_{\text{max}}}{NAB_0} = \frac{4 \times 10^{-2}}{200 \times (0.5)^2 \times 0.2} = 4 \text{ (mA)}.$$
The Biot-Savart Law

- Hans Oersted established that currents induce magnetic fields that form closed loops around the wires. Jean Biot and Felix Savart arrived at an expression that results the magnetic field $\mathbf{H}$ at any point in space to the current $I$ that generates $\mathbf{H}$.

- The Biot-Savart Law states that the differential magnetic field $d\mathbf{H}$ generated by a steady current $I$ flowing through a different length $dl$ is given by

$$d\mathbf{H} = \frac{I}{4\pi} \frac{dl \times \hat{R}}{R^2}, \quad (\text{A/m})$$

or

$$\mathbf{H} = \frac{I}{4\pi} \int \frac{dl \times \hat{R}}{R^2}$$
The Biot-Savart Law

- Field due to Surface/Volume Current Distribution
  - The Biot-Savart Law can be also extended and expressed in terms of distributed sources \( I d \mathbf{l} = J_s ds = J \, dv \):
    - Volume current density \( J \), measured in \((A/m^2)\), or
    - Surface current density \( J_s \), measured in \((A/m)\).

\[
\mathbf{H} = \frac{1}{4\pi} \int_\mathbf{S} J_s \times \hat{\mathbf{R}} \, ds, \quad \text{for a surface current}
\]

\[
\mathbf{H} = \frac{1}{4\pi} \int_\mathbf{V} J \times \hat{\mathbf{R}} \, dv, \quad \text{for a volume current}
\]
The Biot-Savart Law

• Home Work

- **Question 1:** A linear conductor of length $l$ and carrying $I$ is placed along the $z$-axis as shown in the figure. Prove that the magnetic field is

$$H = \hat{\phi} \frac{I}{4\pi r} (\cos \theta_1 - \cos \theta_2), \quad \text{(finite line).}$$

- **Question 2:** Also deduce the field expression due to an infinitely long wire such that $l >> r$, as

$$H = \hat{\phi} \frac{I}{2\pi r}, \quad \text{(infinite line).}$$
The Biot-Savart Law

• **Example 3**

**Question:** A pie-shaped loop of radius \( a \) carries a current \( I \), as shown in the figure. Determine the magnetic field at the apex \( O \).

**Solution:** For a straight segments \( OA \) and \( OC \), the magnetic field at \( O \) is identically zero. This is because, for all points along these segment, \( d\mathbf{l} \) is parallel or anti-parallel to the vector \( \mathbf{R} \), and hence \( d\mathbf{l} \times \mathbf{R} = 0 \). For segment \( AC \), \( d\mathbf{l} \) is perpendicular to \( \mathbf{R} \), so we have

\[
\mathbf{H} = \frac{I}{4\pi} \int_{A}^{C} \frac{d\mathbf{l} \times \mathbf{R}}{R^2} = \frac{I}{4\pi} \int_{0}^{\phi} \frac{\hat{z} d\phi}{a^2} = \hat{z} \frac{I}{4\pi} \int_{0}^{\phi} \frac{a d\phi}{a^2} = \hat{z} \frac{I \phi}{4\pi a}.
\]
The Biot-Savart Law

• **Example 4**

  **Question:** A circular loop of radius $a$ carries a current $I$, as shown. Determine the magnetic field $\mathbf{H}$ at a point on the axis of the loop.

  **Solution:** Consider an elementary element $d\mathbf{l}$. Its magnetic field is

  \[ d\mathbf{H} = \hat{z}dH_z = \hat{z}dH \cos \theta = \hat{z} \frac{I \cos \theta}{4\pi(a^2 + z^2)} dl. \]

  \[ \mathbf{H} = \hat{z} \frac{I}{4\pi(a^2 + z^2)} \cos \theta \int dl = \hat{z} \frac{I}{4\pi(a^2 + z^2)} a \frac{a}{\sqrt{a^2 + z^2}} \left( \frac{2\pi}{a} \right) \]

  \[ \mathbf{H} = \hat{z} \frac{Ia^2}{2(a^2 + z^2)^{3/2}} = \hat{z} \left\{ \begin{array}{l} \frac{I}{2a'}, \\
 2z \end{array} \right. \]

  \[ \frac{Ia^2}{2|z|^3} = \frac{I\pi a^2}{2\pi|z|^3} = \frac{m}{2\pi|z|^3}, \quad |z| >> a. \]
The Biot-Savart Law

- Field of a Magnetic Dipole

A magnetic dipole moment is \( m = IS = I(\pi a^2) \). So the magnetic field due to a dipole is

\[
H = \hat{z} \frac{m}{2\pi |z|^3}, \text{ at } |z| >> a.
\]
The Biot-Savart Law

Example 5

Exercise 4.6  A semiinfinite linear conductor extends between \( z = 0 \) and \( z = \infty \) along the \( z \)-axis. If the current \( I \) in the conductor flows along the positive \( z \)-direction, find \( \mathbf{H} \) at a point in the \( x-y \) plane at a radial distance \( r \) from the conductor.

Solution: From the homework, we have the following formula

\[
\mathbf{H} = \hat{\phi} \frac{I}{4\pi r} (\cos \theta_1 - \cos \theta_2), \quad \text{(finite line)}.
\]

For a conductor extending from \( z = 0 \) to \( z = \infty \), so that \( \theta_1 = \pi/2 \) and \( \theta_2 = \pi \). Hence, we have

\[
\mathbf{H} = \hat{\phi} \frac{I}{4\pi r} (\cos \theta_1 - \cos \theta_2)
\]

\[
= \hat{\phi} \frac{I}{4\pi r} (0 - (-1)) = \hat{\phi} \frac{I}{4\pi r}.
\]
Magnetic Force between Two Parallel Conductors

- **Magnetic Force**
  - The magnetic field at the point of $I_2$ due to current $I_1$ is $B_1 = -\hat{x} \frac{\mu_0 I_1}{2\pi d}$.
  - The force $F_2$ exerted on a length $l$ of wire $I_2$ due to its presence in field $B_1$ may be obtained by

$$F_2 = I_2\hat{z} \times B_1 = I_2\hat{z} \times (-\hat{x}) \frac{\mu_0 I_1}{2\pi d} = (-\hat{y}) \frac{\mu_0 I_1 I_2 l}{2\pi d}$$

- Force per unit length is thus $F'_2 = (-\hat{y}) \frac{\mu_0 I_1 I_2}{2\pi d} = -F'_1$. 

Maxwell’s Magnetostatic Equations

- **Gauss’s Law for Magnetism**
  
  **- Gauss’s Law for \( E \)-Field**
  
  - The net electric flux through a closed surface surrounding a charge is *not* zero.

  \[
  \nabla \cdot \mathbf{D} = \rho_s \iff \oint_S \mathbf{D} \cdot d\mathbf{s} = Q. 
  \]

  **- Gauss’s Law for \( H \)-Field**
  
  - The net magnetic flux through a closed surface surrounding one of the poles of a magnet is *zero*.

  \[
  \nabla \cdot \mathbf{B} = 0 \iff \oint_S \mathbf{B} \cdot d\mathbf{s} = 0. 
  \]
Maxwell’s Magnetostatic Equations

- **Ampère’s Circuit Law**
  - **Differential Form**
    \[ \nabla \times \mathbf{H} = \mathbf{J}. \]
  - **Stoke’s Theorem**
    \[ \iint_S (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \oint_C \mathbf{H} \cdot d\mathbf{l}. \]
  - **Integral Form**
    \[ \oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{s} = I. \]
Questions & Answers

- What is the fundamental difference between *electric* field lines and *magnetic* field lines?
- If the line integral of $\mathbf{H}$ over a closed contour is zero, does it follow that $\mathbf{H} = 0$ everywhere on the contour? If not, what then does it imply?
- Compare the utility of applying the *Biot-Savart law* versus *Ampere’s law* for computing the magnetic field due to current-carrying conductors.
Maxwell’s Magnetostatic Equations

- **Example 6: Magnetic Field of a Long Wire**
  - **Ampère’s Law**
    \[ \oint_{C} \mathbf{H} \cdot d\mathbf{l} = H \oint_{C} dl = H(2\pi r) = I_{encl}, \text{ so } H = \frac{I_{encl}}{2\pi r}. \]
  - **Enclosed Current**
    \[ I_{encl} = \sigma_\mathcal{S} = \begin{cases} \left( \frac{I}{\pi a^2} \right)(\pi r^2), & r \leq a; \\ I, & r \geq a. \end{cases} \]
Maxwell’s Magnetostatic Equations

- **Example 7: \(H\)-Field inside a Toroidal Coil**

  **Question:** For a toroid with \(N\) turns carrying a current \(I\), determine the magnetic field \(H\) in each of the following 3 regions: (i) \(r < a\), (ii) \(a < r < b\), and (iii) \(r > b\).

  **Solution:**
  (i) From symmetry, we see that the field in the region \(r < a\) should be zero according to the Ampere’s law because there is no current enclosed in the contour. (iii) Similarly, when a contour is set at \(r > b\), the net current is also zero and thus the magnetic field is also zero. (ii) For the region inside the core, we have
  \[
  \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_0^{2\pi} \left(-\hat{\phi}H\right) \cdot (\hat{\phi}r d\phi) = -2\pi r H = -NI.
  \]
  \[
  H = -\hat{\phi}H = -\hat{\phi} \frac{NI}{2\pi r}.
  \]
Maxwell’s Magnetostatic Equations

- **Example 8: A Thin Current Sheet**

  **Question:** The x-y plane in the figure below contains an infinite current sheet with surface current density $J_s = (J_s, 0, 0)$. Find the magnetic field $H$.

  **Solution:** From symmetry and also the right-hand rule, we see that the magnetic field must be expressed as,

  $$H = \begin{cases} -\hat{y}H, & \text{for } z > 0; \\ \hat{y}H, & \text{for } z < 0. \end{cases}$$

  Applying the Ampère’s law, we have,

  $$\oint H \cdot dl = Hl + 0 + Hl + 0 = 2Hl = J_s l$$

  Further, we have the final expression:

  $$H = \begin{cases} -\hat{y} \frac{J_s}{2}, & \text{for } z > 0; \\ \hat{y} \frac{J_s}{2}, & \text{for } z < 0. \end{cases}$$

  in contrast with $E = \begin{cases} +\hat{z} \frac{\sigma_s}{2\varepsilon_0}, & \text{for } z > 0; \\ -\hat{z} \frac{\sigma_s}{2\varepsilon_0}, & \text{for } z < 0. \end{cases}$