Lecture 2: Electrostatics – Part I

- Maxwell’s Equations
- Charge and Current Distributions
  - Charge Density
  - Current Density
- Coulomb’s Law
  - Multiple Point Charges
  - Charge Distribution
- Gauss’s Law
- Electric Scalar Potential
  - Electric Potential as a Function of E-Field
  - Electric Potential due to Point Charge
  - Electric Potential due to Continuous Distributions
  - E-Field as a Function of Electrical Potential
  - Poisson’s Equation
Lecture 3: Electrostatics – Part II

- **Electrical Properties of Materials**
- **Conductors**
  - Resistance
  - Joule’s Law
- **Dielectrics**
- **Electric Boundary Conditions**
  - Dielectric-Conductor Boundary
  - Conductor-Conductor Boundary
- **Capacitance**
- **Electrostatic Potential Energy**
- **Image Theory**

(* Optional contents for knowledge enrichment)
Maxwell’s Equations

- Maxwell’s Equations

![Maxwell's Equations](image)

### Maxwell equations in time- & spectral-domains

<table>
<thead>
<tr>
<th>Time-domain</th>
<th>Relation</th>
<th>Spectral-domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M} )</td>
<td>( A = \text{Re} \left[ Ae^{j\omega t} \right] ) in time/spectral-domains.</td>
<td>( \nabla \times \mathbf{E} = -j\omega \mathbf{B} - \mathbf{M} )</td>
</tr>
<tr>
<td>( \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} )</td>
<td>( \nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J} )</td>
<td></td>
</tr>
<tr>
<td>( \nabla \cdot \mathbf{D} = \rho(t) )</td>
<td>( \nabla \cdot \mathbf{D} = \rho )</td>
<td></td>
</tr>
<tr>
<td>( \nabla \cdot \mathbf{B} = m(t) )</td>
<td>( \nabla \cdot \mathbf{B} = m )</td>
<td></td>
</tr>
<tr>
<td>( \nabla \cdot \mathbf{J} = -\frac{\partial \rho(t)}{\partial t} )</td>
<td>( \nabla \cdot \mathbf{J} = -j\omega \rho )</td>
<td></td>
</tr>
<tr>
<td>( \nabla \cdot \mathbf{M} = -\frac{\partial m(t)}{\partial t} )</td>
<td>( \nabla \cdot \mathbf{M} = -j\omega m )</td>
<td></td>
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</tbody>
</table>
### Maxwell’s Equations

- **Physical Quantities**

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{E}$, ($\vec{E}$)</td>
<td>Electric field strength</td>
<td>Volts/m</td>
</tr>
<tr>
<td>$\mathcal{H}$, ($\vec{H}$)</td>
<td>Magnetic field strength</td>
<td>Amperes/m</td>
</tr>
<tr>
<td>$\mathcal{D}$, ($\vec{D}$)</td>
<td>Electric current</td>
<td>Coulombs/m²</td>
</tr>
<tr>
<td>$\mathcal{J}$, ($\vec{J}$)</td>
<td>Electric current density</td>
<td>Amperes/m²</td>
</tr>
<tr>
<td>$\mathcal{B}$, ($\vec{B}$)</td>
<td>Magnetic flux density</td>
<td>Webers/m²</td>
</tr>
<tr>
<td>$\mathcal{M}$, ($\vec{M}$)</td>
<td>Impressed magnetic current density</td>
<td>Volts/m²</td>
</tr>
<tr>
<td>$\rho(t)$, $\rho$</td>
<td>Electric charge density</td>
<td>Coulombs/m³</td>
</tr>
<tr>
<td>$m(t)$, $m$</td>
<td>Impressed magnetic charge density</td>
<td>Webers/m³</td>
</tr>
</tbody>
</table>
Maxwell’s Equations

• Physical Significance - I

- The 1st equation: the differential form of Faraday’s law of induction,
- The 2nd equation: a generalization of Ampère’s circuital law (also referred to as the law of Biot-Savart),
- The 3rd equation: the differential form of Gauss’ law,
Maxwell’s Equations

- Physical Significance - II

- The 4th equation: the magnetic lines of flux form a system of closed loops and nowhere terminate on “magnetic” charge,
- The 5th equation: the equation of conservation of electric charge,
- The 6th equation: the equation of conservation of magnetic charge.
Maxwell’s Equations

• Some Notes

- The introducing of the magnetic current distribution does not make any speculations about the existence of magnetic monopoles.
- Physically, there does not exist an magnetic source. The magnetic source, first introduced by Heaviside in 1885, is considered as an equivalent source adopted for convenience.
Maxwell’s Equations

• Independence of Equations

The above 6 equations are not all independent.

From $\nabla \times \mathcal{E} = -\frac{\partial \mathcal{B}}{\partial t} - \mathcal{M}$, and $\nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} + \mathcal{J}$,

we have $\nabla \cdot \frac{\partial \mathcal{B}}{\partial t} + \nabla \cdot \mathcal{M} = 0$, and

$\nabla \cdot \frac{\partial \mathcal{D}}{\partial t} + \nabla \cdot \mathcal{J} = 0$, because $\nabla \cdot (\nabla \times \mathbf{A}) = 0$.

Using the 3rd and 4th equations

$\nabla \cdot \mathcal{D} = \rho(t)$, \hspace{0.2cm} $\nabla \cdot \mathcal{B} = m(t)$, we readily obtain

the last two equations after the substitution, i.e.,

$\nabla \cdot \mathcal{J} = -\frac{\partial \rho(t)}{\partial t}$ and $\nabla \cdot \mathcal{M} = -\frac{\partial m(t)}{\partial t}$. 
Maxwell’s Equations

- Special Cases of Maxwell’s Equations

<table>
<thead>
<tr>
<th>Electric source</th>
<th>Magnetic source</th>
<th>Static fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \overrightarrow{M} = 0, m = 0 )</td>
<td>( \overrightarrow{J} = 0, \rho = 0 )</td>
<td>( \partial / (\partial t) = 0 )</td>
</tr>
<tr>
<td>( \nabla \times \overrightarrow{E} = -j\omega \overrightarrow{B} )</td>
<td>( \nabla \times \overrightarrow{E} = -j\omega \overrightarrow{B} - \overrightarrow{M} )</td>
<td>( \nabla \times \overrightarrow{E} = -\overrightarrow{M} )</td>
</tr>
<tr>
<td>( \nabla \times \overrightarrow{H} = j\omega \overrightarrow{D} + \overrightarrow{J} )</td>
<td>( \nabla \times \overrightarrow{H} = j\omega \overrightarrow{D} )</td>
<td>( \nabla \times \overrightarrow{H} = \overrightarrow{J} )</td>
</tr>
<tr>
<td>( \nabla \cdot \overrightarrow{D} = \rho )</td>
<td>( \nabla \cdot \overrightarrow{D} = 0 )</td>
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<tr>
<td>( \nabla \cdot \overrightarrow{B} = 0 )</td>
<td>( \nabla \cdot \overrightarrow{B} = m )</td>
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</tr>
<tr>
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<td>( \nabla \cdot \overrightarrow{J} = 0 )</td>
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<td>( \nabla \cdot \overrightarrow{M} = 0 )</td>
</tr>
</tbody>
</table>
Maxwell’s Equations

- Maxwell’s Equations
  \[
  \begin{align*}
  \nabla \cdot \mathbf{D} &= \rho_v \\
  \nabla \cdot \mathbf{B} &= 0 \\
  \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
  \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}
  \end{align*}
  \]

  Static case happens when all charges are permanently fixed in space, or, if they move, they do so at a steady rate so that \(\rho_v\) and \(\mathbf{J}\) are constant in time.

  The electric and magnetic fields are no longer interconnected in the static case.

- Electrostatics
  \[
  \begin{align*}
  \nabla \cdot \mathbf{D} &= \rho_v \\
  \nabla \times \mathbf{E} &= 0.
  \end{align*}
  \]

- Magnetostatics
  \[
  \begin{align*}
  \nabla \cdot \mathbf{B} &= 0 \\
  \nabla \times \mathbf{H} &= \mathbf{J}.
  \end{align*}
  \]

- Constitutive Relations
  \[
  \begin{align*}
  \mathbf{D} &= \varepsilon \mathbf{E} \\
  \mathbf{B} &= \mu \mathbf{H}.
  \end{align*}
  \]
• Charge and Current

  – In electromagnetics, we encounter various forms of electric charge distributions;

  – If the charges are in motion, they constitute current distributions;

  – Charge may be distributed over a volume of space, across a surface, or along a line.
### Charge and Current Distributions

#### Charge Density

- **Volume Charge Density**
  \[
  \rho_v = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv} \quad \text{(C/m}^3\text{)}
  \]

- **Surface Charge Density**
  \[
  \rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad \text{(C/m}^2\text{)}
  \]

- **Line Charge Density**
  \[
  \rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)}
  \]

\[
Q = \iiint \rho_v dv \quad \text{or} \quad \iint \rho_s ds \quad \text{or} \quad \int \rho_l dl.
\]
Charge and Current Distributions

Example 1

Exercise 3.1  A square plate in the $x$–$y$ plane is situated in the space defined by $-3 \text{ m} \leq x \leq 3 \text{ m}$ and $-3 \text{ m} \leq y \leq 3 \text{ m}$. Find the total charge on the plate if the surface charge density is given by $\rho_s = 2y^2 \text{ (C/m}^2\text{)}$.

Solution:

\[
\rho_s = 2y^2
\]

\[
Q = \int_S \rho_s \, ds
\]

\[
= \int_{-3}^{3} \int_{-3}^{3} 2y^2 \, dx \, dy
\]

\[
= \frac{2y^3x}{3} \bigg|_{-3}^{3} \bigg|_{-3}^{3} = 216 \mu\text{C} = 0.216 \text{ (mC)}.
\]
Example 2

Exercise 3.2 A spherical shell centered at the origin extends between $R = 2 \text{ cm}$ and $R = 3 \text{ cm}$. If the volume charge density is given by $\rho_v = 6R \times 10^{-4} \text{ (C/m}^3\text{)}$, find the total charge contained in the shell.

Solution:

\[
\rho_v = 6R \times 10^{-4}
\]

\[
Q = \int \rho_v \, dV
\]

\[
= \int_{R=2 \text{ cm}}^{R=3 \text{ cm}} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} 6R \times 10^{-4} \cdot R^2 \sin \theta \, dR \, d\theta \, d\phi
\]

\[
= \frac{6R^4}{4} \times 10^{-4} \left[ \frac{3 \text{ cm}}{2 \text{ cm}} \right] \times 2 \times 2\pi
\]

\[
= 6\pi \times 10^{-4} \left[ (3 \times 10^{-2})^4 - (2 \times 10^{-2})^4 \right] = 1.22 \text{ (nC)}.
\]
Charge and Current Distributions

- **Current Density**

\[ \Delta q' = \rho_v u \Delta s' \Delta t \]

\[ \Delta q = \rho_v u \varepsilon \Delta s \Delta t \]

\[ \Delta q = \rho_v u \Delta s \Delta t \cos \theta \]

\[ \Delta I = \frac{\Delta q}{\Delta t} = \rho_v u \cdot \Delta s = J \cdot \Delta s \]
Charge and Current Distributions

(a) Orbiting electron  (b) Spinning electron

Current Density
- Convection Current Density and Electric Current
- Difference Between Convection and Conduction Currents

Conduction current obeys Ohms Law
- move of some of outermost electrons
- atom, but atoms of the conducting medium do not move (atomic nuclei–still and electron shells of atom ––moving, in a metal wire).

Convection
- flow of charges or electrons (charged cloud, cathode–ray tubes of TVs, etc)
Coulomb’s Law

- **Coulomb’s Law**
  - An isolated charge \( q \) induces an electric field \( \mathbf{E} \) at every point in space, and at any specific point \( P \), \( \mathbf{E} \) is given by

\[
\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2}, \quad \text{(V/m)}
\]

where \( \hat{\mathbf{R}} \) is a unit vector point from \( q \) to \( P \), \( R \) is the distance between them, and \( \varepsilon \) is the electric permittivity of the medium containing \( P \) point.
Coulomb’s Law

Coulomb’s Law

- In the presence of an electric field \( \mathbf{E} \) at a given point in space, which may be due to a single charge or a distribution of many charges, the force acting on a test charge \( q' \), when the charge is placed at that point, is given by

\[
F = q' \mathbf{E}, \quad (N); \quad D = \varepsilon \mathbf{E} = \varepsilon_0 \varepsilon_r \mathbf{E};
\]

where \( F \) measured in Newtons (N) and \( q' \) in Coulombs (C), the unit of \( \mathbf{E} \) is N/C shown to be volt per meter (V/m) later on.
Coulomb’s Law

- **Coulomb’s Law** ❌ Skip this slide
Coulomb’s Law

- **Multiple Point Charges**
  - **Point Charge Q1**
    \[ E_1 = \frac{q_1 (R - R_1)}{4\pi \varepsilon |R - R_1|^3}, \text{ (V/m)} \]
  - **Point Charge Q2**
    \[ E_2 = \frac{q_2 (R - R_2)}{4\pi \varepsilon |R - R_2|^3}, \text{ (V/m)} \]
  - **Superposition Principle**
    \[ E = E_1 + E_2 = \frac{q_1 (R - R_1)}{4\pi \varepsilon |R - R_1|^3} + \frac{q_2 (R - R_2)}{4\pi \varepsilon |R - R_2|^3}, \text{ (V/m)} \]
  - **Multiple Point Charges**
    \[ E = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i (R - R_i)}{|R - R_i|^3}, \text{ (V/m)} \]
Coulomb’s Law

Example 3

Exercise 3.3  Four charges of 10 μC each are located in free space at (−3, 0, 0), (3, 0, 0), (0, −3, 0), and (0, 3, 0) in a Cartesian coordinate system. Find the force on a 40-μC charge located at (0, 0, 4). All distances are in meters.

Solution:

\[ \mathbf{R}_1 = -\hat{x}3 \]
\[ \mathbf{R}_2 = \hat{x}3 \]
\[ \mathbf{R}_3 = -\hat{y}3 \]
\[ \mathbf{R}_4 = \hat{y}3 \]
\[ \mathbf{R} = \hat{z}4 \]

\[ \mathbf{F}_1 = \frac{QQ_1}{4\pi\varepsilon_0} \frac{\mathbf{R} - \mathbf{R}_1}{|\mathbf{R} - \mathbf{R}_1|^3} = \frac{QQ_1}{4\pi\varepsilon_0} \frac{\hat{z}4 + \hat{x}3}{125} = \frac{QQ_1}{500\pi\varepsilon_0} (\hat{z}4 + \hat{x}3) \]

\[ \mathbf{F}_2 = \frac{QQ_2}{4\pi\varepsilon_0} \frac{\mathbf{R} - \mathbf{R}_2}{|\mathbf{R} - \mathbf{R}_2|^3} = \frac{QQ_2}{4\pi\varepsilon_0} \frac{\hat{z}4 - \hat{x}3}{125} = \frac{QQ_2}{500\pi\varepsilon_0} (\hat{z}4 - \hat{x}3) \]

\[ \mathbf{F}_3 = \frac{QQ_3}{4\pi\varepsilon_0} \frac{\mathbf{R} - \mathbf{R}_3}{|\mathbf{R} - \mathbf{R}_3|^3} = \frac{QQ_3}{4\pi\varepsilon_0} \frac{\hat{z}4 + \hat{y}3}{125} = \frac{QQ_3}{500\pi\varepsilon_0} (\hat{z}4 + \hat{y}3) \]

\[ \mathbf{F}_4 = \frac{QQ_4}{4\pi\varepsilon_0} \frac{\mathbf{R} - \mathbf{R}_4}{|\mathbf{R} - \mathbf{R}_4|^3} = \frac{QQ_4}{4\pi\varepsilon_0} \frac{\hat{z}4 - \hat{y}3}{125} = \frac{QQ_4}{500\pi\varepsilon_0} (\hat{z}4 - \hat{y}3) \]

\[ \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 \]

\[ = \frac{400 \times 10^{-12}}{500\pi\varepsilon_0} (\hat{z}16) = \frac{64 \times 10^{-12}}{5\pi \times 8.85 \times 10^{-12}} = \hat{z}0.46 \quad \text{(N)} \]
**Coulomb’s Law**

- **Examples 4 and 5**

  **Exercise 3.4** Two identical charges are located on the x-axis at \( x = 3 \) and \( x = 7 \). At what point in space is the net electric field zero?

  **Solution:** Since both charges are on the x-axis, the point at which the fields due to the two charges can cancel has to lie on the x-axis also. Intuitively, since the two charges are identical, that point is midway between them at \((5,0,0)\).

  **Exercise 3.5** In a hydrogen atom the electron and proton are separated by an average distance of \(5.3 \times 10^{-11}\) m. Find the magnitude of the electrical force \(F_e\) between the two particles, and compare it with the gravitational force \(F_g\) between them.

  **Solution:**

  \[
  F_e = \frac{q_e q_p}{4\pi \varepsilon_0 R^2} = \frac{(1.6 \times 10^{-19})^2}{4\pi \times 8.85 \times 10^{-12}(5.3 \times 10^{-11})^2} = 8.2 \times 10^{-8}\ N.
  \]

  \[
  F_g = \frac{Gm_e m_p}{R^2} = \frac{6.67 \times 10^{-11} \times 9.11 \times 10^{-31} \times 1.67 \times 10^{-27}}{(5.3 \times 10^{-11})^2} = 3.6 \times 10^{-47}\ N.
  \]
Coulomb’s Law

- **Charge Distribution**
  - Differential Charge $dq = \rho_v \, dv$
    \[
    dE = \hat{R}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{R}' \frac{\rho_v \, dv'}{4\pi\varepsilon R'^2}, \quad (V/m).
    \]
  - Electric Field
    \[
    E = \frac{1}{4\pi\varepsilon} \iiint_{V'} \frac{\rho_v \, dv'}{R'^2} \quad (V/m).
    \]
  - Surface and Line Distributions
    \[
    E = \frac{1}{4\pi\varepsilon} \iint_{S'} \frac{\rho_s \, ds'}{R'^2} \quad \text{or} \quad \frac{1}{4\pi\varepsilon} \int_{l'} \frac{\rho_l \, dl'}{R'^2}, \quad (V/m).
    \]
Coulomb's Law

• Shielding Effects in the Presence of a Conductor
  – Outer charges or currents **DO NOT** effect the inner electric or magnetic fields
  – The inner charges or currents will produce the induced charges or currents and thus will further produce the outer electric and magnetic fields
Coulomb’s Law

- Shielding Effects in the Presence of a Conductor
Example 6

Questions: A ring of charge of radius $b$ is characterised by a uniform line charge density of positive polarity $\rho_l$. With the ring in free space and positioned in the $x$-$y$ plane as shown in the figure, determine the electric field intensity $\mathbf{E}$ at a point $P(0, 0, h)$ along the axis of the ring at a distance $h$ from its center.

Solution: Consider the electric field generated by a differential segment of the ring, e.g., segment 1 located at $(b, \phi, 0)$ in the figure. The segment has length $dl = b d\phi$ and contains charge $dq = \rho_l dl = \rho_l b d\phi$. The distance vector $\mathbf{R}_1'$ from segment 1 to point $P(0, 0, h)$ is

$$R_1' = |\mathbf{R}_1'| = \sqrt{b^2 + h^2}, \quad \hat{\mathbf{R}}_1' = \frac{\mathbf{R}_1'}{|\mathbf{R}_1'|} = \frac{-\hat{x} - \hat{z}h}{\sqrt{b^2 + h^2}}.$$
Coulomb’s Law

• Example 6 - continued
  - **Solution:** The electric field intensity \( E \) at the point \( P(0, 0, h) \) due to the charge of segment is
  \[
dE_1 = \frac{1}{4\pi\varepsilon_0} \frac{\rho_l dl}{R_1^2} = \frac{\rho_l b}{4\pi\varepsilon_0} \frac{-\hat{r} b + \hat{z} h}{(b^2 + h^2)^{3/2}} d\phi.
  \]
  - The \( z \)-directional contribution due to two opposite segments is thus given by
  \[
dE = dE_1 + dE_2 = \hat{z} \frac{\rho_l b h}{2\pi\varepsilon_0 (b^2 + h^2)^{3/2}} d\phi.
  \]
  - The total electric field is thus integrated as
  \[
  E = \hat{z} \frac{\rho_l b h}{2\varepsilon_0 (b^2 + h^2)^{3/2}} \int_0^{\pi} d\phi
  \]
  \[
  = \hat{z} \frac{\rho_l b h}{2\varepsilon_0 (b^2 + h^2)^{3/2}} = \hat{z} \frac{Q h}{4\pi\varepsilon_0 (b^2 + h^2)^{3/2}},
  \]
  where \( Q = 2\pi b \rho_l \).
### Coulomb’s Law

#### Example 7

- **Question:** Find the electric field at a point \( P(0, 0, h) \) in free space at a height \( h \) on the \( z \)-axis due to a circular disk of charge in the \( x-y \) plane with uniform charge density \( \rho_s \), as shown in the figure, and then evaluate \( \mathbf{E} \) for the infinite-sheet case by letting \( a \to \infty \).

- **Solution:** A ring of radius \( r \) and width \( dr \) has an area \( ds = 2\pi r \, dr \) and contains charge \( dq = \rho_s \, ds = 2\pi \rho_s \, r \, dr \). Using the expression in the last example, and replacing \( b \) with \( r \), we have

\[
d\mathbf{E} = \hat{z} \frac{h}{4\pi \varepsilon_0 \left( r^2 + h^2 \right)^{3/2}} \left( 2\pi \rho_s \, r \, dr \right)
\]

\[
\mathbf{E} = \hat{z} \frac{\rho_s}{2\varepsilon_0} \left[ \frac{r \, dr}{\left( r^2 + h^2 \right)^{3/2}} \right] \frac{a}{a \to \infty} \to \pm \hat{z} \frac{\rho_s}{2\varepsilon_0}.
\]
Coulomb’s Law

Example 8

Exercise 3.6  An infinite sheet of charge with uniform surface charge density \( \rho_s \) is located at \( z = 0 \) (x–y plane), and another infinite sheet with density \(-\rho_s\) is located at \( z = 2 \text{ m} \), both in free space. Determine \( \mathbf{E} \) in all regions.

Solution:  Per Eq. (3.25), for the sheet at \( z = 0 \),

\[
\mathbf{E}_1 = \begin{cases} 
\hat{z} \frac{\rho_s}{2\varepsilon_0}, & \text{for } z > 0, \\
-\hat{z} \frac{\rho_s}{2\varepsilon_0}, & \text{for } z < 0.
\end{cases}
\]

Similarly, for the sheet at \( z = 2 \text{ m} \) with charge density \(-\rho_s\),

\[
\mathbf{E}_2 = \begin{cases} 
-\hat{z} \frac{\rho_s}{2\varepsilon_0}, & \text{for } z > 2 \text{ m}, \\
\hat{z} \frac{\rho_s}{2\varepsilon_0}, & \text{for } z < 2 \text{ m}.
\end{cases}
\]

Hence,

\[
\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \begin{cases} 
0, & \text{for } z < 0, \\
\hat{z} \frac{\rho_s}{\varepsilon_0}, & \text{for } 0 < z < 2 \text{ m}, \\
0, & \text{for } z > 2 \text{ m}.
\end{cases}
\]
Gauss’s Law

- **Gauss’s Law**
  - **Differential Form**
    \[ \nabla \cdot \mathbf{D} = \rho_v \n\]
  - **Total Charge**
    \[ \iiint_{v} \nabla \cdot \mathbf{D} \, dv = \iiint_{v} \rho_v \, dv = Q \n\]
  - **Divergence Theorem**
    \[ \iiint_{v} \nabla \cdot \mathbf{D} \, dv = \oiint_{S} \mathbf{D} \cdot ds \n\]
  - **Integral Form**
    \[ \oiint_{S} \mathbf{D} \cdot ds = Q \n\]

Gauss’s law states that the outward flux of \( D \) through a surface is proportional to the enclosed charge \( Q \).
Gauss’s Law

- Electric Field due to a Point Charge
  - Radially Outward \( \mathbf{D} \)
    \[
    \mathbf{D}(\mathbf{R}) = \hat{\mathbf{R}} \mathbf{D}_R
    \]
  - Gauss’s Law
    \[
    \oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S \hat{\mathbf{R}} \mathbf{D}_R \cdot d\mathbf{s} = \int_S \mathbf{D}_R \cdot d\mathbf{s} = D_R \left( 4\pi R^2 \right) = q.
    \]
  - Electric field – Coulomb’s Law
    \[
    \mathbf{E}(\mathbf{R}) = \frac{\mathbf{D}(\mathbf{R})}{\varepsilon} = \hat{\mathbf{R}} \frac{q}{4\pi \varepsilon R^2}, \quad \text{(V/m)}.
    \]

- Gauss’s Law is easier to apply than Coulomb’s Law, but limited to symmetrical charge distributions
Gauss’s Law

- **Example 9**

  **Exercise 3.7**  Two infinite lines of charge, each carrying a charge density \( \rho_l \), are parallel to the \( z \)-axis and located at \( x = 1 \) and \( x = -1 \). Determine \( \mathbf{E} \) at an arbitrary point in free space along the \( y \)-axis.

  **Solution:**

  The distance between either line of charge and a point at \( y \) on the \( y \)-axis is \( r = (1 + y^2)^{1/2} \).

  For line 1,

  \[
  \hat{r}_1 = \frac{r_1}{r} = \frac{-\hat{x} + \hat{y} y}{(1 + y^2)^{1/2}}.
  \]

  For line 2,

  \[
  \hat{r}_2 = \frac{r_2}{r} = \frac{\hat{x} + \hat{y} y}{(1 + y^2)^{1/2}}.
  \]

  Using Eq. (3.33),

  \[
  \mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \frac{\hat{r}_1 \rho_l}{2\pi \varepsilon_0 r} + \frac{\hat{r}_2 \rho_l}{2\pi \varepsilon_0 r} = \frac{(-\hat{x} + \hat{y} y)\rho_l}{2\pi \varepsilon_0 (1 + y^2)} + \frac{(\hat{x} + \hat{y} y)\rho_l}{2\pi \varepsilon_0 (1 + y^2)} = \frac{\hat{y} \rho_l y}{\pi \varepsilon_0 (y^2 + 1)}.
  \]
Gauss’s Law

- **Home Work**
  - **Question 1:** Prove that the electric field due to an infinite wire of charge density $\rho_l$ is given by
    \[ E = \hat{r} \frac{\rho_l}{2\pi \varepsilon_0 r}, \quad \text{(infinite line).} \]
  
  - **Questions 2:** Derive the electric field anywhere along the $z$-axis resulted from a regular wire triangle lying in the $x$-$y$ plane, whose centre is located at coordinate origin, side dimension of $a$ and total charge of $Q$. 
Example 10

Exercise 3.9  A spherical volume of radius $a$ contains a uniform volume charge density $\rho_v$. Use Gauss’s law to determine $\mathbf{D}$ for (a) $R \leq a$ and (b) $R \geq a$.

Solution:

For $R \leq a$,

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \oint_S \mathbf{D}_r \ ds = D_r (4\pi R^2)$$

$Q$ within a sphere of radius $R$ is

$$Q = \frac{4}{3} \pi R^3 \rho_v$$

Hence,

$$4\pi R^2 D_R = \frac{4}{3} \pi R^3 \rho_v$$

$$D_r = \frac{\rho_v R}{3}$$

$$\mathbf{D} = \hat{\mathbf{R}} D_r = \hat{\mathbf{R}} \frac{\rho_v R}{3}, \quad R \leq a.$$
Example 10 (Continued)

Exercise 3.9  A spherical volume of radius \( a \) contains a uniform volume charge density \( \rho_v \). Use Gauss’s law to determine \( \mathbf{D} \) for (a) \( R \leq a \) and (b) \( R \geq a \).

Solution:

For \( R \geq a \), total charge in sphere is

\[
Q = \frac{4}{3} \pi a^3 \rho_v,
\]

\[
4\pi R^2 D_R = \frac{4}{3} \pi a^3 \rho_v,
\]

\[
\mathbf{D} = \hat{\mathbf{R}} D_R = \hat{\mathbf{R}} \frac{\rho_v a^3}{3R^2}, \quad R \geq a.
\]
Electric Scalar Potential

- Electric Potential as a Function of $E$-Field

Work done in moving a charge $q$ a distance $dy$ against the electric field $\mathbf{E}$ is $dW = qE dy$. 
Electric Scalar Potential

• Concept

– The voltage $V$ between two points in the circuit represents the amount of work, or potential energy, required to move a unit charge between the two points.

– In fact, the term “voltage” is a short form of “voltage potential” and is the same as electric potential.

– An electric field between two points gives rise to the voltage difference between circuits, such as across a resistor or a capacitor.
**Electric Scalar Potential**

- **Electric Potential as a Function of E-Field**
  - Consider a simple case of a positive charge $q$ in a uniform electric field $\mathbf{E} = (0, -E)$, parallel to the $-y$-direction.
  - The presence of the field $\mathbf{E}$ exerts a force $\mathbf{F}_e = q\mathbf{E}$ on the charge in the negative $y$-direction (against the force $\mathbf{F}_e$).
  - If we attempt to move the charge along the positive $y$-direction (against the force $\mathbf{F}_e$) at a constant speed, we need an external force $\mathbf{F}_{ext}$ which should have the same magnitude but opposite direction: $\mathbf{F}_{ext} = - \mathbf{F}_e = -q\mathbf{E}$.
  - The work done in Joules by moving any object a vector differential distance $d\mathbf{l}$ under the force $\mathbf{F}_{ext}$ is $dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l}$, or specifically $dW = -q(0, -E)(0, dy) = qEdy$. 
  
  Work done in moving a charge $q$ a distance $dy$ against the electric field $\mathbf{E}$ is $dW = qEdy$. 

Electric Scalar Potential

- **Electric Potential as a Function of $E$-Field**
  - Differential electric potential
    \[ dV = \frac{dW}{q} = -E \cdot dl \]
  - Potential difference
    \[ V_{21} = V_2 - V_1 = \int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} E \cdot dl \]
  - Kirchhoff’s voltage law
    \[ \oint_C E \cdot dl = 0 \quad (\text{Electrostatics}) \]
    which is conservative or irrotational field
  - General form
    \[ V = -\int_{\infty}^{P} E \cdot dl, \quad (V) \]

Potential difference between $P_2$ and $P_1$ is the same irrespective of the path used for calculating the line integral of the electric field between them.
Electric Scalar Potential

- Electric Potential due to a Point Charge
  - Electric field
    \[ E = \hat{R} \frac{q}{4\pi \varepsilon R^2} \]
  - Potential due to a point charge
    \[ V = -\int_{\infty}^{R} \left( \hat{R} \frac{q}{4\pi \varepsilon R^2} \right) \cdot \hat{R} dR = \frac{q}{4\pi \varepsilon R} = \frac{q}{4\pi \varepsilon |R - R_1|} \]
    if the charge is located at \( R_1 \).
  - Potential due to \( N \) discrete point charges
    \[ V = \frac{1}{4\pi \varepsilon} \sum_{i=1}^{N} \frac{q_i}{|R - R_i|} \quad (V) \]
Electric Scalar Potential

- Electric Potential due to Continuous Distributions

\[ V(R) = \frac{1}{4\pi\varepsilon} \left\{ \iiint_{v'} \frac{\rho_v}{R'} dv', \quad \text{volume;} \right. \]
\[ \left. \iint_{s'} \frac{\rho_s}{R'} ds', \quad \text{surface;} \right. \]
\[ \int_{l'} \frac{\rho_l}{R'} dl', \quad \text{line.} \]

- \textit{E}-Field as a Function of Electrical Potential

\[ dV = -E \cdot d\mathbf{l} = \nabla V \cdot d\mathbf{l} \]
\[ \mathbf{E} = -\nabla V \]

\[ V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l}, \quad (V) \]
Electric Scalar Potential

- **Example 11**

**Exercise 3.10** Determine the electric potential at the origin in free space due to four charges of 20 μC each located at the corners of a square in the x–y plane and whose center is at the origin. The square has sides of 2 m each.

**Solution:** For four identical charges all equidistant from the origin:

\[
V = \frac{4Q}{4\pi \varepsilon_0 R}, \quad R = \sqrt{2} \quad (\text{m})
\]

\[
= \frac{4 \times 20 \times 10^{-6}}{4\pi \varepsilon_0 \sqrt{2}} = \frac{\sqrt{2} \times 10^{-5}}{\pi \varepsilon_0} \quad (\text{V}).
\]
Electric Scalar Potential

• **Poisson’s Equation**

  – **Gauss’s Law and Electric Potential**
    \[ \nabla \cdot (\varepsilon \mathbf{E}) = \rho_v \text{ and } \mathbf{E} = -\nabla V \text{ so we have } \nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon} \nabla \]

  – **Poisson’s Equation**
    \[ \nabla^2 V = \nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \]

    **satisfied by**
    \[ V = \frac{1}{4\pi\varepsilon} \int \int \int \frac{\rho_v}{R'} \, dv' \]

  – **Laplace’s Equation**
    \[ \nabla^2 V = 0 \]
Electric Scalar Potential

- **Example 12**

**Exercise 3.11**  A spherical shell of radius $R$ has a uniform surface charge density $\rho_s$. Determine the electric potential at the center of the shell.

**Solution:** Application of (3.48b):

\[
V(R) = \frac{1}{4\pi \varepsilon} \int_{S'} \frac{\rho_s}{R'} ds'
\]

\[
= \frac{1}{4\pi \varepsilon R} \cdot \rho_s (4\pi R^2)
\]

\[
= \frac{\rho_s R}{\varepsilon}.
\]