A Population Based Approach to Model the Lifetime and Energy Distribution in Battery Constrained Wireless Sensor Networks

K. Ramachandran and B. Sikdar *Senior Member, IEEE*

***Abstract*—The residual power levels of the nodes in a wireless sensor network determine its important performance metrics like the network lifetime, coverage, and connectivity. In this paper, we present a general framework to model the availability of power at sensor nodes as a function of time, based on models for population dynamics in biological studies. Models are developed for sensors with and without battery recharging and expressions are derived for the network lifetime as well as the distribution and moments of random variables describing the number of sensors with different levels of residual energy as a function of time. The model is also extended to the case where new sensors are periodically added to the network to substitute older sensors that have expended their energy. Finally, the effect of the packet arrival rates and a sensor's geographical location are modeled. Simulation results to verify the accuracy of the proposed models are presented.**

***Index Terms*—Network lifetime, sensor networks, modeling**

I. Introduction

A major constraint in the design and deployment of sensor networks is their limited battery capacity. The finite battery limits the lifetime of the network, and may also cause the network to become disconnected or lose coverage over time. To be able to provide guarantees on the performance of a sensor network and develop schemes to maximize the network lifetime, it is important to be able to characterize the available battery power at the sensors. In this paper, we present a general methodology for modeling the lifetime and available battery power of sensor nodes. The model is motivated by the methods used to study population dynamics by researchers in the area of biology.

Existing research has primarily concentrated on developing algorithms, be it either distributed or centralized, to optimize network longevity metrics. Works along the lines of actually building network models for energy consumption such as [4] fail to capture the interplay between a node's spatial location and it's energy consumption. In [5] an optimization model to evaluate the maximum network lifetime is proposed, taking into account the network topology and data aggregation scheme. A Markov chain model for calculating the energy dissipation in sensors networks is presented in [6] while accounting for the transitions between active and sleep modes. A model for the network lifetime in a general form that is independent of the underlying network is proposed in [7]. The node density and the lifetime upper bound which ensures that a certain portion of network area is covered is studied in [8]. The effect of increasing the number of nodes on the network lifetime is examined in [9]. However, the existing literature fails to provide a unified framework for modeling the energy consumption and residual battery levels of sensor networks that simultaneously is capable of accounting for network and device related factors such as battery recharging, the traffic patterns, and the geographical location of the nodes. This paper tries to address these issues.

A number of deterministic as well as stochastic factors affect the battery power consumption at a sensor. These include the sensor application and the resulting traffic model, deployment scenario, the choice of communication and networking protocols etc. In our current work, we develop an unifying framework to characterize the lifetime and residual energy distribution of such energy constrained networks, and obtain insights into their working. In particular, we use techniques similar to population models for biological systems to develop our framework. Our model allows the computation of the distribution of the network lifetime and its moments, as well as the distribution of the available power at the nodes in the network. The proposed framework is general enough to accommodate scenarios with and without battery recharging, in addition to scenarios where new nodes are periodically added to the network. Our model also allows the inclusion of network related parameters in the energy calculations. We consider both *spatial* scenarios where a node's power consumption is governed by it's position in space as well as *non-spatial* scenarios where the node's location and power consumption model are independent entities.

The rest of the paper is organized as follows. Our model for the scenarios where the sensors are incapable of recharging their batteries is presented in Section II while Section III extends to model for sensors with rechargeable batteries. Section IV models scenario where new nodes are periodically added to the network and Section V quantifies the impact of network parameters on the models. Section VI presents methodologies for using the proposed models for system design. Section VII presents our simulation results and Section VIII presents the concluding remarks.

II. Sensor Networks without Battery Recharging

In this section, we develop the formulation of the analytical framework to study the network lifetime and the distribution of

K. Ramachandran and B. Sikdar are with the Department of Electrical, Computer, and Systems Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180 USA

Manuscript received March 17, 2009; revised October 19, 2009.

This work supported in part by NSF grant 0347623.

residual power in sensor networks. At any time, we categorize each sensor in terms of its residual battery level. The change in the state or residual battery level of a sensor depends on its instantaneous power consumption which in turn is dependent on several factors, such as the spatial location, routing protocols deployed, communication pattern etc.. Categorization of sensors by their power levels thus facilitates the genesis of a model that depends on the characterization of the power consumption probabilities.

To model the lifetime of energy constrained networks, we propose a generalization of Leslie's population matrix [1], which is used to study populations structured by age. The "age" of a node in our model corresponds to the amount of the battery power consumed, with one unit of power expended per packet transmitted, and the "age" of any node lies in one of the $m+1$ possible intervals; $0, 1, \cdots, m$. In other words, we assume that each sensor has enough energy to transmit m packets and the nodes in the network are structured based on this value. Our model makes the following assumptions:

1) The power is mainly expended to transmit packets.
2) The network lifetime can be discretized into "cycles", wherein each cycle spans a communication round among the nodes. This also implies that the nodes are synchronized.
3) The probability that a node receives i packets (its own as well as those it forwards), $i = 0, 1, \cdots, m$, to transmit is same in all cycles and we denote this probability by p_i.

Note that the sleep-wake cycles used by many sensor networks to conserve energy is incorporated in our model by choosing p_0 (the probability that no energy is consumed in a slot) appropriately. Thus in these scenarios, p_0 includes the fraction of cycles where the node is in the sleep state in addition to the fraction of cycles in which the node is awake but does not have any packets to transmit. Further, power is also consumed by sensor nodes in order to sense the environment for any phenomenon that the application running on the network is interested in. The first assumption implies that the energy expended in sensing the environment is not incorporated into the model. This energy being independent of the node's geographic location, impacts all nodes in the network uniformly, and hence is omitted. Additionally, the power consumption on communications dominates that for running the onboard circuitry [2]. Thus modeling the network lifetime based on the power spent on communications serves as a good approximation. Cases where the energy consumption of the circuitry and sensing devices is non-negligible can also be accommodated in our model. In this case, the circuit and sensing power is first normalized in terms of the power required to transmit a packet and let this normalized power be ω. Then a new set of power consumption probabilities $\hat{p}_i$ is used in the model formulation below instead of p_i, with $\hat{p}_i = 0$ for $0 \leq i < \omega$ and $\hat{p}_i = p_{i-\omega}$ for $i \geq \omega$.

Let $n(t)$ be a $(m{+}1)$-dimensional vector whose i-th element, $n_i(t)$, denotes the number of nodes in age group i at time t, i.e. $n_i(t)$ denotes the number of nodes which have used up i units of the total battery capacity of m at time t. Note that the time t is discretized and is measured in units of cycles. Unlike biological population models where in each time step the age of each individual increases by 1, our model allows for arbitrary power consumption or increase in age in each time step. The number of nodes at each energy level at an arbitrary time step is given by

$$\begin{aligned}
n_0(t{+}1) &= p_0 n_0(t) \\
n_1(t{+}1) &= p_0 n_1(t) + p_1 n_0(t) \\
&\vdots \\
n_{m\text{-}1}(t{+}1) &= p_0 n_{m-1}(t) + p_1 n_{m-2}(t) + \cdots + p_{m-1} n_0(t) \\
n_m(t{+}1) &= n_m(t) + \sum_{i=1}^{m} p_i n_{m-1}(t) + \sum_{i=2}^{m} p_i n_{m-2}(t) \\
&\quad + \cdots + \sum_{i=m-1}^{m} p_i n_1(t) + p_m n_0(t)
\end{aligned} \tag{1}$$

The rationale behind the above formulation can be justified as follows. A node with full power at time t (class n_0) will retain it's entire battery reserve only if it receives no packets to transmit for the duration of the cycle. The probability of this event is p_0, and since each node has the same probability distribution p_i, the expected number of nodes who receive zero packets is $p_0 n_0(t)$, which in turn is the count of nodes with full battery power at time $t+1$. Similarly the number of nodes in class n_1 at time $t+1$ is the sum of nodes in class n_1 who transmit zero packets, and the nodes in class n_0 that spend one unit of energy at time t. For evaluating the number of nodes in class m, note that a sensor in class n_i, $i = 0, \cdots, m-1$ will expend all its energy if it transmits more than $m-i$ packets in a cycle and the probability of this event is given by $\sum_{k=(m-i)}^{m} p_i$, $i = 0, \cdots, m$. Also, in the scenarios where the batteries are not capable of recharging or replenishing expended energy, a sensor that had no battery power during the cycle starting at t will continue to remain powered down at $t+1$ and hence the equation for $n_m(t+1)$. Note that the power spent on idle listening and receiving are almost the same (p. 75 of [3]). Thus whether a packet is received or not, almost the same energy is spent, if the sensor is not transmitting. Thus the receiving power is not included in the model. The receiving/idle listening power can be easily incorporated by shifting the index i of the p_i's appropriately.

The above formulation can also be expressed in a vector-matrix form. To this end, we first define the $(m{+}1) \times (m{+}1)$-dimensional "projection" matrix A as

$$A = \begin{bmatrix} p_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ p_1 & p_0 & 0 & 0 & \dots & 0 & 0 \\ p_2 & p_1 & p_0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ p_m & \sum_{m-1}^{m} p_i & \sum_{m-2}^{m} p_i & \dots & \dots & \sum_{1}^{m} p_i & 1 \end{bmatrix} \tag{2}$$

The model then can be expressed as the vector difference equation

$$n(t+1) = An(t) \tag{3}$$

This formulation is equivalent to a discrete time Markov chain as the number of nodes at a particular energy level is dependent only on the number at the previous cycle. The solution of this difference equation is easily obtained, using a recursive definition, as:

$$n(t+1) = A^{t+1} n(0) \tag{4}$$

where $n(0)$ is the initial distribution of nodes among the various energy levels. In practical situations, it is reasonable to assume that at time $t = 0$, all the nodes are fully powered, i.e. $n_i(0) = 0 \quad \forall i > 0$ and $n_0(0) = N$. What now remains is determining the probabilities for the energy consumption during a cycle and this is done in Section V.

A. Network Lifetime

When the batteries at sensor nodes do not have the capability to recharge, the network lifetime is an important quantity of interest. In this section we characterize the expected network lifetime and the probability of the availability of nodes with non-empty battery as a function of time. To this end, we use techniques that have been developed in [11] for calculating the extinction dynamics in biological populations.

We start by modeling the impact of the initial battery states on the network lifetime. From Eqn. (4) the dynamics of the energy model in the interval 0 to t can be represented as a product of t projection matrices A. In existing literature on population dynamics [10] it has been shown that asymptotically

$$n(t) \approx R(0,t)\langle v_0, n(0)\rangle u_0 \tag{5}$$

where $R(0,t)$ is a scalar representing the growth of the matrix product, v_0 and u_0 are the dominant left and right eigenvectors of the matrix product, normalized such that $\langle v_0, u_0\rangle = 1$ and the notation $\langle c, d\rangle$ is used to represent the scalar or dot product of vectors c and d. Consider the non-normalized dominant left eigenvector v of the matrix A. The impact of the initial battery states on the longevity of the network is then given by

$$V_0 = \langle v, n(0)\rangle \tag{6}$$

The rate at which the number of sensors without any remaining energy increases in the network is dependent on the dominant eigenvalue of the matrix A. In population studies, the size of the species under consideration varies with time. In contrast, the number of sensors in the network stays constant (in the absence of new nodes being added). Now state m in the model in Eqns. (2) and (3) corresponds to the state where a sensor has no remaining battery power. This is an absorbing state since the batteries do not have any recharging capability. Then we may consider the model

$$\hat{n}(t+1) = \hat{A}\hat{n}(t) \tag{7}$$

where $\hat{n}(t)$ is a m-dimensional vector corresponding to the number of sensors at time t in states 0 to $m-1$ of the original model in Eqns. (2) and (3) and $\hat{A}$ is a $m \times m$ matrix obtained from the matrix A by eliminating its $(m+1)$-th row and column. This modified model can now be used to evaluate the network lifetime by treating the model in Eqn. (7) as a population model and computing the extinction time of the "species" $\hat{n}$ modeled by the "population" projection matrix $\hat{A}$. In [12] it has been shown that the infinitesimal long-run growth (or decay) rate of the population μ and its infinitesimal variance σ^2 are given by

$$\mu \approx \ln \lambda_0 - \frac{\sigma^2}{2} \tag{8}$$

$$\sigma^2 \approx \frac{1}{\lambda_0^2}\delta^T C \delta \tag{9}$$

where λ_0 is the dominant eigenvalue of the projection matrix $\hat{A}$ and δ is a column vector of the sensitivity coefficients $\frac{\partial \lambda_0}{\partial \hat{a}_{i,j}}$ with $\hat{a}_{i,j}$ being the (i,j)-th element of $\hat{A}$. The transpose of δ is denoted by δ^T and the sensitivity coefficients are given by $\frac{\partial \lambda_0}{\partial \hat{a}_{i,j}} = v_0^i u_0^j$ where v_0^i and u_0^j are the i-th and j-th elements of the normalized left and right eigenvectors of $\hat{A}$. The normalization is done such that $\sum_i u_0^i = 1$ and $\langle v_0, u_0\rangle = 1$. Finally, C is the variance-covariance matrix of the elements in $\hat{A}$. Let x represent the natural logarithm of the total population $\sum_i \hat{n}_i$ representing the number of sensors in states 0 to $m-1$ and let $x_0 = \ln V_0$ be its adjusted initial value at time $t = 0$. Let $\varrho \triangleq \varrho(x,t|x_0)$ be the probability density function of the log population size x at time t, given that its initial value was x_0. The function characterizing ϱ quickly approaches the solution of the diffusion equation for the Weiner process ([13] p. 151)

$$\frac{\partial \varrho}{\partial t} = -\mu \frac{\partial \varrho}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 \varrho}{\partial x^2} \tag{10}$$

with the initial condition $\varrho(x,0|x_0) = \delta(x - x_0)$ where $\delta(x - x_0)$ is the Dirac delta function at x_0. Also, since the population becomes extinct (i.e. all sensors move to state m) when the population becomes less than one, we have the boundary condition

$$\varrho(0,t|x_0) = 0. \tag{11}$$

To obtain the solution for Eqn. (10) subject to the above initial and boundary conditions, we use the known solutions for Weiner processes with absorbing barriers [14]. This requires a linear transform of the coordinates and the solution to the system in Eqns. (10) and (11) is given by (refer to the appendix for details)

$$\varrho(x,t|x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}}\left[e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} - e^{-\frac{2\mu x_0}{\sigma^2} - \frac{(x+x_0-\mu t)^2}{2\sigma^2 t}}\right] \tag{12}$$

Now, the probability that the population exists at time t, i.e. has a size of at least one at time t, is given by $\int_0^\infty \varrho(x,t|x_0)dx$ where the lower limit starts from 0 since $\varrho(x,t|x_0)$ is defined for the log of the population size (i.e. the sensors in states 0 to $m-1$). The rate of decrease of the probability that the population exists at time t then corresponds to the probability density function of the random variable denoting the population is extinct at time t. Let $g(t|x_0)$ denote the probability that the population becomes extinct in an interval t and $t+dt$. Then $g(t|x_0)$ can be obtained by taking the derivative of the total probability of the event that the population is not extinct at time t:

$$g(t|x_0) = -\frac{d}{dt}\int_0^\infty \varrho(x,t|x_0)dx \tag{13}$$

$$= \frac{x_0}{\sqrt{2\pi\sigma^2 t^3}} e^{-\frac{(x_0+\mu t)^2}{2\sigma^2 t}} \tag{14}$$

From Eqns. (12) and (14), the cumulative probability that the population is extinct before time t is then

$$G(t|x_0) = \int_0^t g(t'|x_0)dt' \tag{15}$$

$$= \Phi\left[-\frac{x_0+\mu t}{\sigma\sqrt{t}}\right] + e^{-\frac{2\mu x_0}{\sigma^2}}\left[1-\Phi\left[\frac{x_0-\mu t}{\sigma\sqrt{t}}\right]\right] \tag{16}$$

where $\Phi[a]$ is the standard normal probability integral

$$\Phi[a] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{a} e^{-\frac{z^2}{2}}dz \tag{17}$$

Note that when nodes cannot recharge their batteries, we have $\mu \le 0$ and thus $G(\infty|x_0) = 1$, i.e. the network eventually runs out of energy.

The definition of network lifetime used here corresponds to the time when all sensors run out of energy. Other definitions that have been used in literature include the time when the first sensor runs out of energy, when a certain percentage of sensors run out of energy, when the network becomes disconnected, to name a few. The lifetime for cases when it is defined as the time when the first or a certain threshold ϵ of the sensors run out of energy can be evaluated using the derivation above by taking x_0 to be the distance from the adjusted initial size ($\ln V_0$) to the threshold in the log scale [12]. The time when a network becomes disconnected depends on various additional factors such as the topology and is not considered here.

B. Moments of the Network Lifetime

From Eqn. (12), the distribution of the extinction time or the network lifetime has an inverse Gaussian distribution. The mean and variance of this distribution and the network lifetime are

$$\bar{t} = \frac{x_0}{|\mu|} \quad \text{and} \quad \bar{\sigma}^2 = \frac{x_0\sigma^2}{|\mu|^3} \tag{18}$$

Thus the expected network lifetime is equal to the logarithm of the adjusted initial population size, divided by the absolute value of the long-run growth rate. The dependence of the network lifetime on the infinitesimal variance is only through its effect on μ. Also, the network lifetime distribution is positively skewed and the third central moment is $\frac{3x_0\sigma^4}{|\mu|^5}$ and the mode of the network lifetime is less than the mean.

III. Sensors with Rechargeable Batteries

Equipping sensors nodes with rechargeable batteries is the easiest way to extend the lifetime of the network. In the typical deployment scenarios of sensor networks, these batteries would typically recharge by scavenging energy from the environment such as by converting solar power, ambient heat, and motion into electricity [15], [16]. In this section we extend our model to accommodate sensors with rechargeable batteries. We consider an arbitrary recharge process governing the replenishing of the sensor batteries. We denote by α_i the probability that a sensor generates i units of energy in a cycle, with $i = 0, 1, \cdots m$. We assume that the recharge energy generated or harvested in a cycle becomes available for use at the end of the cycle. Also, the recharge process is assumed to be independent of the traffic at the node.

A sensor in state j at time t stays in the same state at time $t+1$ if the amount of energy it expends in time cycle t is the same as the amount of energy it generates. Since the traffic and energy generation processes are independent, this occurs with probability $\sum_{i=0}^{m} p_i\alpha_i$. Along the same lines, a sensor moves from state j to state i after a cycle, $j < i < m$, if the energy consumed in the cycle is $i-j$ units more than that generated in the cycle. The probability of this event is then $\sum_{k=0}^{m} p_{k+i-j}\alpha_k$. Similarly, the probability that a node in state j moves to state i after a cycle, $i < j < m$, is given by $\sum_{k=0}^{m} p_k\alpha_{k+j-i}$. For the boundary conditions where we consider the transition to states 0 and m, additional events need to be considered while calculating the transition probabilities. In particular, a sensor in state i at time t, $0 \le i < m$, moves to state m at time $t+1$ if at least $m-i$ more units of energy were consumed than generated in the time cycle. Similarly, a sensor in state i at time t, $0 < i \le m$, moves to state 0 at time $t+1$ if at least i more units of energy were generated than consumed in the cycle. Then, the number of nodes at each energy level at an arbitrary time step is given by

$$\begin{aligned}
n_0(t+1) &= n_0(t)\sum_{i=0}^{m}\alpha_i\sum_{j=0}^{i}p_j + n_1(t)\sum_{i=1}^{m}\alpha_i\sum_{j=0}^{i-1}p_j \\
&\quad + \cdots + n_{m-1}(t)\sum_{i=m-1}^{m}\alpha_i\sum_{j=0}^{i-m+1}p_j + n_m(t)\sum_{i=m}^{m}\alpha_i\sum_{j=0}^{i-m}p_j \\
n_1(t+1) &= n_0(t)\sum_{i=1}^{m}p_i\alpha_{i-1} + n_1(t)\sum_{i=0}^{m}p_i\alpha_i \\
&\quad + \cdots + n_{m-1}(t)\sum_{i=0}^{m}p_i\alpha_{i+m-2} + n_m(t)\sum_{i=0}^{m}p_i\alpha_{i+m-1} \\
&\vdots \\
n_{m-1}(t+1) &= n_0(t)\sum_{i=m-1}^{m}p_i\alpha_{i-m+1} + n_1(t)\sum_{i=m-2}^{m}p_i\alpha_{i-m+2} \\
&\quad + \cdots + n_{m-1}(t)\sum_{i=0}^{m}p_i\alpha_i + n_m(t)\sum_{i=0}^{m}p_i\alpha_{i+1} \\
n_m(t+1) &= n_0(t)\sum_{i=m}^{m}p_i\sum_{j=0}^{i-m}\alpha_j + n_1(t)\sum_{i=m-1}^{m}p_i\sum_{j=0}^{i-m+1}\alpha_j \\
&\quad + \cdots + n_{m-1}(t)\sum_{i=1}^{m}p_i\sum_{j=0}^{i-1}\alpha_j + n_m(t)\sum_{i=0}^{m}p_i\sum_{j=0}^{i}\alpha_j
\end{aligned}$$

The formulation above can be expressed in the form of a $(m+$

$1) \times (m+1)$-dimensional projection matrix A:

$$A = \begin{bmatrix} \sum_{i=0}^{m} \alpha_i \sum_{j=0}^{i} p_j & \sum_{i=1}^{m} \alpha_i \sum_{j=0}^{i-1} p_j & \cdots & \sum_{i=m}^{m} \alpha_i \sum_{j=0}^{i-m} p_j \\ \sum_{i=1}^{m} p_i \alpha_{i-1} & \sum_{i=0}^{m} p_i \alpha_i & \cdots & \sum_{i=0}^{m} p_i \alpha_{i+m-1} \\ \vdots & \ddots & \ddots & \vdots \\ \sum_{i=m-1}^{m} p_i \alpha_{i-m+1} & \sum_{i=m-2}^{m} p_i \alpha_{i-m+2} & \cdots & \sum_{i=0}^{m} p_i \alpha_{i+1} \\ \sum_{i=m}^{m} p_i \sum_{j=0}^{i-m} \alpha_j & \sum_{i=m-1}^{m} p_i \sum_{j=0}^{i-m+1} \alpha_j & \cdots & \sum_{i=0}^{m} p_i \sum_{j=0}^{i} \alpha_j \end{bmatrix} \quad (19)$$

The model then can be expressed as the vector difference equation

$$n(t+1) = An(t) \quad (20)$$

whose recursive solution in terms of the initial distribution of nodes can again be written as $n(t+1) = A^{t+1}n(0)$.

A. Energy Distribution

When nodes are capable of recharging their batteries, a sensor does not always stay devoid of energy. However, network properties such as connectivity and coverage are dependent on the number of nodes with non-zero energy at any given point in time. Also, the amount of available energy at the sensors determines the traffic that the network can support and this affects the application running on the sensor network. In this section we characterize the distribution of the available energy at the sensors as a function of time.

At each cycle, a sensor in any state i transits to any other state or stays in the same state according to the probabilities defined in the i-th column of A. In other words, the transition of a sensor in state i at the end of a cycle is determined according to a multinomial trial with $m+1$ possible outcomes with the probability of each outcome defined the entries in the i-th column of the matrix A. Then with $n_i(t)$ denoting the number of sensors in class i at time t, we have $n_i(t)$ multinomial trials corresponding to each sensor in class i that determines their transition at the start of time $t+1$. To characterize the vector $n(t{+}1)$, we start with a characterization of $n(t+1)$ conditioned on $n(t)$ and evaluate the probability $Pr\{n(t+1) = \theta(t+1)|n(t)\}$ where $\theta(t+1)$ is a $(m+1)$-dimensional vector of non-negative integers. Since each sensor is assumed to operate independently, we have

$$Pr\{n(t{+}1) = \theta(t{+}1)|n(t)\} = \prod_{i=0}^{m} Pr\{n_i(t{+}1) = \theta_i(t{+}1)|n(t)\} \quad (21)$$

These conditional probabilities may be computed quite readily. However unconditioning the expression to obtain the unconditional distribution is quite laborious. Thus we use a multivariate probability generating function (PGF) to characterize the number of nodes at different power levels. We define

$$\rho_t(\nu_0, \nu_1, \cdots, \nu_m) = Pr\{n(t) = \{\nu_0, \nu_1, \cdots, \nu_m\}\} \quad (22)$$

and

$$H_t(z) = \sum_{\nu_0, \nu_1, \cdots, \nu_m} \rho_t(\nu_0, \nu_1, \cdots, \nu_m) z_0^{\nu_0} z_1^{\nu_1} \cdots z_m^{\nu_m} \quad (23)$$

Now consider the conditional PGF $H_{t+1|t}(z)$. Recall that at time t, the state transition of each sensor in class i occurs as per a multinomial trial. The PGF of the resulting vector from the multinomial trials on the $n_i(t)$ members of class i at time t is given by

$$(a_{0,i}z_0{+}a_{1,i}z_1{+}\cdots{+}a_{m,i}z_m)^{n_i(t)} = \left[\sum_{k=0}^{m} a_{k,i} z_k\right]^{n_i(t)} \quad (24)$$

Now, the number of sensors in class k at time $t+1$ is the sum of the number of sensors that move to class k from each of the m other classes at the end of time t as well as the sensors of class k that do not change their state. Since we are working with the transforms of the probability mass functions, the resulting PGF is the product of the individual PGFs. Thus we have

$$H_{t+1|t}(z) = \prod_{i=0}^{m} \left[\sum_{k=0}^{m} a_{k,i} z_k\right]^{n_i(t)} \quad (25)$$

Unconditioning on t, we have

$$\begin{aligned} H_{t+1}(z) &= \sum_{n_0(t), \cdots, n_m(t)} \rho_t(n_0(t), \cdots, n_m(t)) H_{t+1|t}(z) \\ &= \sum_{n(t)} \rho_t(n(t)) \prod_{i=0}^{m} \left[\sum_{k=0}^{m} a_{k,i} z_k\right]^{n_i(t)} \quad (26) \\ &= H_t(\xi_0, \xi_1, \cdots, \xi_m) \quad (27) \end{aligned}$$

where

$$\xi_i = \sum_{k=0}^{m} a_{k,i} z_k \quad (28)$$

Then given a starting state vector $n(0)$, we can recursively build the PGF of $n(t)$ and use it to obtain the exact distributions and its confidence intervals. As an illustration, we have

$$H_0(z) = z_0^{n_0(0)} z_1^{n_1(0)} \cdots z_m^{n_m(0)} = \prod_{i=0}^{m} z_i^{n_i(0)} \quad (29)$$

and

$$H_1(z) = H_0(\xi) = \prod_{i=0}^{m} \left[\sum_{k=0}^{m} a_{k,i} z_k\right]^{n_i(0)} \quad (30)$$

and so on.

B. Approximate Distribution

In this subsection we obtain an approximation for the probability distribution function of $n(t)$ that is easier to calculate as compared to the expression in the previous subsection. Since the state transitions of the sensors are governed by independent multinomial trials, we can use the normal approximation to a multinomial distribution [17]. Then for a given $n(t)$, the random variables

$$Y_i(t{+}1) = \frac{n_i(t+1) - E(n_i(t+1)|n(t))}{[\text{var}(n_i(t+1)|n(t))]^{1/2}} \qquad i = 0, 1, \cdots, m \quad (31)$$

are each distributed approximately as the standard normal random variable $N(0,1)$ and the random vector $Y(t+1)$ is defined as

$$Y(t+1) = \frac{n(t+1) - E(n(t+1)|n(t))}{[\text{var}(n(t+1)|n(t))]^{1/2}} \tag{32}$$

where $\text{var}(n(t+1)|n(t)) = \text{diag}[\text{var}(n_i(t+1)|n(t))]$ is a diagonal matrix whose diagonal elements are the component variances. Then $Y(t+1)$ has a multi-dimensional standard normal distribution $N(0,I)$.

C. Mean and Variance of State Occupancy

Closed form expressions for the mean and variance of the state occupancy vector $n(t)$ denoting the number of sensors in each state can be obtained either by differentiating the PGF $H_t(z)$ or by the method outlined next. We first define the matrix W with elements

$$w_{i,j} = a_{i,j}(1 - a_{i,j}) \qquad\qquad 0 \le i,j \le m \tag{33}$$

where $a_{i,j}$ are the elements of the matrix A defined in Eqn. (19). We denote by $\mu(t)$ and $V(t)$ the $(m+1)$-dimensional vectors corresponding to the mean and variance of $n(t)$. We also define the matrix D_t as a diagonal matrix with

$$D_t = \text{diag}(W\mu_t) \tag{34}$$

where the diagonal elements correspond to the vector $W\mu_t$. Then, taking the expectation of Eqn. (20)

$$\mu_{t+1} = A\mu_t \tag{35}$$

$$V_{t+1} = D_t + AV_tA^T \tag{36}$$

Substituting the recurrence relations, we have

$$\mu_t = A^t n(0) \tag{37}$$

$$V_t = \sum_{k=0}^{t-1} A^k D_{t-1-k} A^{Tk} \tag{38}$$

Thus the variance of the sensor states at different cycles is a weighted sum of the one-step conditional variances of the mean process at all preceding steps.

IV. Effect of Node Additions

In some scenarios where battery recharging is not a viable option due to cost and size requirements, an alternative is to periodically add new sensors to the network to act as substitutes for sensors without any energy. In this section we extend our framework to model this scenario.

To model the periodic addition of new nodes to the network, we denote by the $(m+1)$-dimensional vector S, the number of new nodes added to the network in each cycle. Thus S_i represents the number of new nodes that belong to class i. One can imagine that in general $S_i = 0$ for $i \neq 0$ because new sensors would typically have a full battery. Then the system described by Eqns. (3) and (2) or Eqns. (20) and (19) can be rewritten as

$$n(t+1) = An(t) + S \tag{39}$$

and the recursive solution for this system is given by

$$n(t) = A^t n(0) + \sum_{i=0}^{t-1} A^i S \tag{40}$$

To evaluate the distribution of the state occupancy vector $n(t)$, we note that the system in Eqn. (39) can be written as $\tilde{n}(t+1) = \tilde{A}\tilde{n}(t)$ where the $(m+2)\times(m+2)$-dimensional matrix $\tilde{A}$ is defined using the A matrices in Eqn. (2) or Eqn. (19) as

$$\tilde{A} = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m} & S_0 \\ a_{1,0} & a_{1,1} & \cdots & a_{1,m} & S_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,0} & a_{m,1} & \cdots & a_{m,m} & S_m \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{41}$$

and $\tilde{n}(t)$ is a $(m+2)$-dimensional vector whose first $m+1$ elements denote the number of sensors in each state at time t and the $(m+2)$-th element is a constant, set equal to the total number of new sensors added to the network in a cycle. With this transformation, the expressions of Section III may be used to evaluate the distribution of $\tilde{n}(t)$ and its moments.

V. Impact of Network Parameters

The potency of the framework developed here lies in it's inherent ability to be abstracted to networks with varied node deployment as well as routing schemes. In this section we highlight and investigate the interplay between a node's geographical co-ordinates in space and it's power consumption under the aegis of shortest path routing by considering two scenarios: **(1)** a spatial model where the sensor nodes are located at the vertices of a finite grid and **(2)** a non-spatial model where nodes are randomly and homogeneously distributed such that traffic conditions at each node is statistically identical. In general, the number of packets transmitted in a cycle p_i, depends on the application, the route selecting mechanism as well as the placement of nodes. The exact methodology for deriving the p_i's depends on the network scenario and in this section we illustrate two specific cases.

A. Spatial Network

To consider the impact of a node's spatial location on its energy consumption rates and node lifetime, we consider a deployment scenario where the sensor nodes are placed at the vertices of a finite grid, as shown in Fig. (1). The co-ordinates of node i, $i = 1, \cdots, N$ in the grid (x_i, y_i) is determined as follows: $x_i = (i-1)/\sqrt{N}$ and $y_i = (i-1)\%\sqrt{N}$.

We now incorporate the contribution of a node's geographic location into the derivation of the power consumption probabilities under the assumption that the network employs *shortest path routing*. The following probabilities are assumed known: p_s, the probability that in a given cycle a sensor node (say i) has a new packet to send to another node (say j) in the grid and p_c, the probability that any two given nodes in the grid communicate. The probability that a node i has a packet to transmit during a cycle is the probability of the union of two mutually exclusive events: the event of a node initiating a

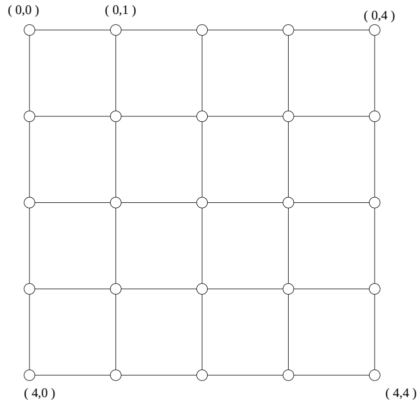


Fig. 1. An example of a grid topology for sensor networks.

communication session and the event where it receives a routing request. The probability of the latter, p_{ri}, can be obtained by using the conditional probability of it receiving a packet given two nodes in the network communicate. Mathematically, for node i

$$p_{ri} = 2\left[\sum_{\substack{j=1\\j\neq i}}^{N-1}\sum_{\substack{k=1\\k\neq i,k>j}}^{N}\begin{array}{l}Pr\{\text{session for } j\text{-}k \text{ is through } i\}\\ \times Pr\{j \text{ and } k \text{ communicate}\}\end{array}\right] \quad (42)$$

For each pair (j,k), the expression for (k,j) communicating through node i has the same numerical value since the grid is symmetric and hence the summation in Eqn (42) is multiplied by two. Now, the probability that two particular nodes say j and k communicate is: $Pr\{j\text{-}k \text{ communicate}\} = \frac{1-(1-p_s)^2}{\binom{N-1}{2}}$. In other words, the pair (j,k) can be selected from $(N-1)$ nodes (since node i is not a candidate) in $\binom{N-1}{2}$ ways and for nodes j and k to communicate, it is sufficient if either node initiates a session. The expression for $Pr\{\text{session } j\text{-}k \text{ is through } i\}$ is derived as follows. Let $(x_i, y_i), (x_j, y_j), (x_k, y_k)$ denote the co-ordinates of nodes i, j and k respectively. Defining, $\Delta x_{i,j} \triangleq |x_i - x_j|$ and $\Delta y_{i,j} \triangleq |y_i - y_j|$, we obtain: $r_{i,j} = \Delta x_{i,j} + \Delta y_{i,j}$. Similar values for $r_{j,k}$ and $r_{k,i}$ can be obtained using the previous definition. Now,

$$Pr\{\text{session } j\text{-}k \text{ is through } i\} = \begin{cases}\frac{L_{j,k,i}}{L_{j,k}} & \text{if } r_{i,k} + r_{i,j} = r_{j,k}\\ 0 & \text{otherwise}\end{cases}$$

where

$$L_{j,k,i} = \binom{r_{i,j}}{\Delta x_{i,j}}\binom{r_{i,k}}{\Delta x_{i,k}} \quad \text{and} \quad L_{j,k} = \binom{r_{j,k}}{\Delta x_{j,k}}$$

Given the probability of a sensor to initiate a session, p_s, each cycle sees an average of Np_s sessions. To obtain the energy consumption probabilities, $p_i,\ i = 0, \cdots, m$, we again condition on the node's geographic location.

$$p_i = \sum_{k=1}^{N} Pr\{i \text{ packets transmitted}|\text{node id = k}\} \times \mathcal{P}_k \quad (43)$$

Note that node k transmits $i, i > 0$ packets during a cycle if it either receives i routing packets and does not initiate a session or starts a communication session and receives $i-1$ routing requests. In our model we limit the number of communication sessions to Np_s, though theoretically the upper bound is N. The simulations validate our intuition that the expected number is a good approximation of the underlying communication process. Denoting $Pr\{\text{node id = k}\}$ by $\mathcal{P}_k$, the energy consumption probabilities can be expressed as follows:

$$p_i = \begin{cases}\left\{(1-p_s)(1-p_{rk})^{Np_s}\right\}\mathcal{P}_k & i = 0\\ \left\{(1-p_s)\binom{Np_s}{i}p_{rk}^i(1-p_{rk})^{Np_s-i} + \right.\\ \left. p_s\binom{Np_s-1}{i-1}p_{rk}^{(i-1)}(1-p_{rk})^{Np_s-i}\right\}\mathcal{P}_k & 0 < i \leq Np_s\\ 0 & \text{otherwise}\end{cases}$$

Also, the evaluation of $Pr\{\text{node id = k}\}$ has two possibilities: one where the choice of a node is equally likely among the N nodes present and the second, where the selection of the node is governed by it's location. Assuming shortest path routing, we approximate the likelihood of the node being chosen by the number of shortest paths it lies on. That is

$$Pr\{\text{node id = k}\} = \frac{\sum_{\substack{i=1\\i\neq k}}^{N-1}\sum_{\substack{j=i+1\\j\neq k}}^{N}\mathcal{I}\{r_{i,k} + r_{i,j} = r_{j,k}\}}{\sum_{k=1}^{N}\sum_{\substack{i=1\\i\neq k}}^{N-1}\sum_{\substack{j=i+1\\j\neq k}}^{N}\mathcal{I}\{r_{i,k} + r_{i,j} = r_{j,k}\}}$$

where $\mathcal{I}\{r_{i,k} + r_{i,j} = r_{j,k}\} = 1$ if $r_{i,k} + r_{i,j} = r_{j,k}$, 0 otherwise.

B. Non-spatial Homogeneous Networks

In the case of scenarios where the sensor network is homogeneous and is either assumed to span an extremely (ideally infinitely) large space or to be very densely deployed, the traffic conditions at each node can be approximated to be statistically identical. To qualitatively evaluate the node lifetimes in these scenarios, we consider a model where the number of packets transmitted by each node during a time cycle follows a Poisson distribution with mean λ, irrespective of its geographical location. The power consumption probabilities p_i in this case are given by: $p_i = \frac{e^{-\lambda}\lambda^i}{i!}$.

VI. System Design and Model Use

The modeling framework proposed in the previous sections can be used to aid in the design of system parameters. For instance, the model may be used to compute the required recharging capabilities of the sensor nodes such that the probability that their energy is not depleted is above a given threshold. In this section we present a methodology for using the proposed models to aid in the system design process.

A. Sensors with Rechargeable Batteries

In sensor networks with rechargeable batteries, the evolution of the system as described by $n(t+1) = An(t)$ in Equation (20) can be solved as a discrete time Markov chain to yield a non-trivial steady state distribution for the energy levels at each node. This is in contrast to the time-dependent state occupancy probabilities in Sections III-A and III-B. The steady-state distribution may be used to design system parameters such as the desired recharge rates or battery sizes for a desired energy distribution, including the fraction of nodes that are devoid of energy in the steady state.

The steady state solution vector ρ of the system $n(t+1) = An(t)$ is obtained by solving

$$\rho = A\rho \tag{44}$$

along with the condition $\sum \rho = 1$. Then, if the fraction of nodes that do not have energy in the steady-state is desired to be less than ϵ, $0 \le \epsilon \le 1$, we solve

$$\rho(m) \le \epsilon \tag{45}$$

for the desired parameter. If more than one parameters are required to be designed, equations for the steady state distribution of additional states and their desired values can be solved simultaneously to obtain the parameter values.

The steady state solution of the Markov chain degenerates to the trivial solution of all nodes being depleted of energy for networks without rechargeable batteries. This is also the case when a constant number of nodes are added to the network in each slot. Thus the design for a steady state distribution of node battery levels only applies to nodes with rechargeable batteries.

B. Sensors without Rechargeable Batteries

For networks without rechargeable batteries, our model may be used to design the parameters required for a desired network lifetime. The design process requires the use of the time dependent distribution of the energy levels of sensor nodes given in Eqn. (27). The PGF is inverted and the marginal probability mass function for state m, $\rho_t(m)$ is equated to the fraction ϵ of the nodes that are devoid of energy used to define the lifetime. The desired value of the lifetime is then substituted for t and the equation $\rho_t(m) \le \epsilon$ is solved for the desired parameter. For more than one parameter, the marginal probability mass functions of additional states are equated to their desired values and solved simultaneously.

A particularly important design problem is that of choosing the smallest battery capacity that leads to a desirable network lifetime. To solve this design problem, the marginal probability mass function $\rho_t(m)$ is evaluated as a function of m by inverting Eqn. (27). The expression for $\rho_t(m)$ at the desired lifetime is then equated to the ϵ used for defining the lifetime. The required battery capacity, $\hat{m}$, is then given by

$$\hat{m} = \underset{m}{\operatorname{argmax}} \{\rho(m) : \rho(m) < \epsilon\} \tag{46}$$

In cases where a closed form expression for $\rho_t(m)$ in terms of m is not available, numerical methods may be used to compute $\hat{m}$.

C. Comments on Model Usage

A requirement for using the model is the knowledge of the traffic arrival process at each node (the model may also be used to compute the allowable traffic rates for a desired lifetime). Since most sensor networks are targeted for a particular application, it is reasonable to assume that the traffic generation process for the application is known, given the rich literature on traffic modeling [18], [19]. Also, the model may be used to evaluate the impact of various strategies such as in-network processing, aggregation and compression on the network lifetime by appropriately scaling the traffic parameters. Finally, the model may be used to evaluate the impact of different traffic types and traffic mixes on the network lifetime and energy distribution.

VII. RESULTS

In this section, we evaluate the accuracy of the proposed framework by comparing the analytic results against simulations. The simulation results were generated using a custom built simulator written in C that, unlike existing simulation tools such as *ns-2*, allows the use of rechargeable batteries at nodes. To avoid repetitive results, we only present those for non-spatial networks with randomly distributed nodes (the results for spatial (grid) networks are similar). For each simulation result, ten runs of the simulation were conducted with different seeds and the average of these runs is presented in the figures. User datagram protocol (UDP) was used as the transport protocol and a code division multiple access (CDMA) based medium access control protocol with 256 codes was used to ensure that there were no collisions. The parameters used for the results have not been selected with a particular sensor platform or application in mind. This is because our goal here is to provide insights into the behavior of the network lifetime as a function of various parameters and to evaluate the accuracy of the model in diverse settings.

A. Network Lifetime

We first consider the model in Section II where batteries do not have the capability to recharge. We start with the case where lifetime is defined as the time when all nodes run out of energy and is measured in terms of the number of cycles. For this scenario, Figure 2 shows the analytic and simulation values of the expected network lifetime for a non-spatial network where 100 nodes are distributed randomly in the network and the packet arrival process at each node is modeled according to a Poisson process. The network is a 2000$\times$2000 meter square region and the transmission radius of each node is 250 meters. We consider initial battery levels of 100 (i.e. $m = 100$) or 200 ($m = 200$) and the x-axis represents the parameter λ of the Poisson process. The analytic and simulation results match closely and as expected, the network lifetime decreases as the traffic intensity increases.

Next we consider the case where the lifetime is defined as the time when a given fraction (taken as 20% of the nodes for illustrative purposes) of the nodes run out of energy. For a random network of 100 nodes, Figure 3 compares the analytic and simulation results for the network lifetime. Results are presented for initial battery levels of 100 and 200 and again there is a close match between the analysis and simulations. The analysis and simulations match closely for other network

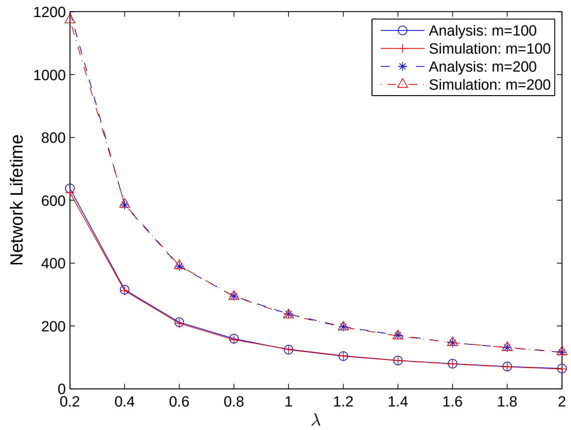


Fig. 2. Network Lifetime: Analysis versus simulation results.

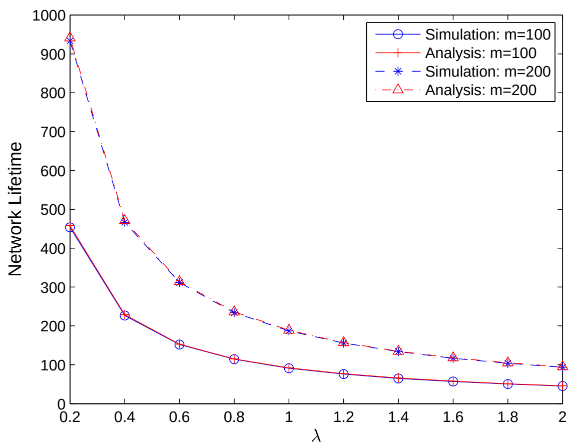


Fig. 3. Network Lifetime (time for first 20% nodes to run out of energy): Analysis versus simulation results.

sizes also for all the cases considered in this subsection and thus have been omitted.

B. Residual Energy Distribution

We next consider the model in Section III where each node has some capability to recharge its battery. For the results presented here we assume a simple model where a node generates a single unit of energy in a cycle with probability 0.25 and does not generate any energy with probability 0.75. The initial battery level of each sensor was kept at 100. In Fig. 4 we compare the analytic and simulation results for the number of sensors at different residual power levels after 25, 50 and 75 cycles of operation in a random network, with $\lambda = 1.0$. Results are presented for the case when there are 100 and 200 nodes in the network and $\lambda = 1.0$. Again we note the close match between the simulation and analytic results.

C. Effect of New Nodes

Next, we consider the scenario where new nodes are periodically added to the network to act as replacements for nodes that have fully exhausted their batteries. For random networks, the new nodes are randomly placed in the given area. Node additions are more appropriate in networks where nodes do not have the capability to recharge their batteries and for these results we assume that $\alpha_0 = 1$ and $\alpha_i = 0$ for $i > 0$. The residual battery power distribution in the sensors of a 100-node random network after 25 and 50 cycles of operation are shown in Fig. 5 for $\lambda = 1.0$ and we consider the cases where either one or three nodes are added to the network in each cycle. From the results we see that the addition of new nodes helps to maintain higher residual energy levels in the network. Again, the analytic results match closely with the simulation results.

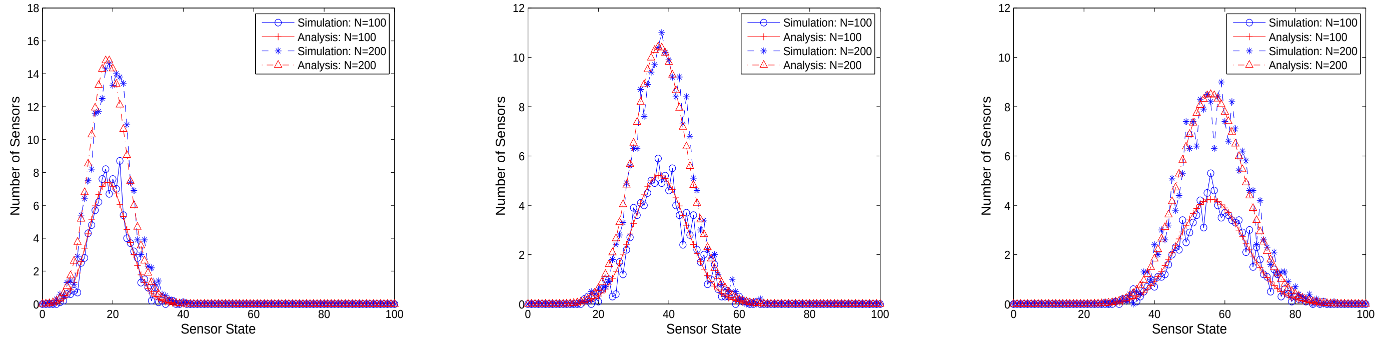
D. Sensitivity Analysis and Observations

In the next set of results, we use the proposed model to gain insights into the impact of various system parameters on the network lifetime. These results consider the time when 20% of the nodes run out of energy as the network lifetime. A random network with 100 nodes and $\lambda = 1$ is considered. We only show the analytic results since the simulation results match closely and our interest is only in the trends in the results.

Figure 6(a) shows the effect of the battery capacity and the addition of nodes on the network lifetime. For these results we do not consider rechargeable batteries. The first observation is that the lifetime increases linearly with the battery capacity. The second observation is that while increasing the number of nodes added in a cycle increases the lifetime, the lifetime quickly saturates. Also, the increase in lifetime is greater for larger battery capacities.

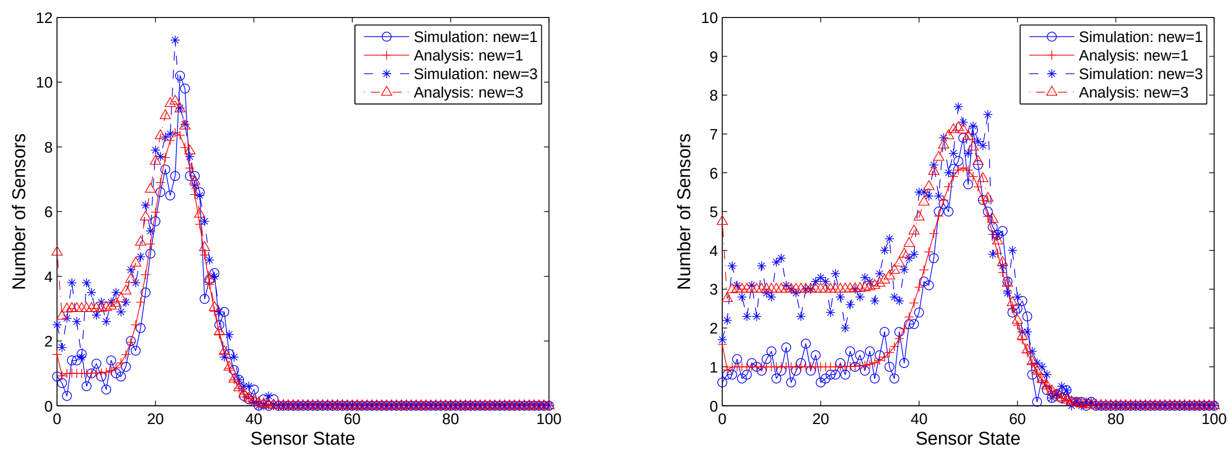
Figure 6(b) shows the effect of the battery capacity and the average recharge rates on the network lifetime. A two state recharge model is assumed where in each cycle either a node does not generate any energy (with probability α_0) or it generates a single unit of energy (with probability $\alpha_1 = 1 - \alpha_0$). The figure uses α_1 or the average recharge rates as the x-axis. Again, we observe that the network lifetime increases linearly with the battery capacity. Also, the lifetime increases exponentially with the recharge rates and the increase is larger for larger battery sizes.

Figure 6(c) shows the impact of the number of nodes and the loading factor at each node on the network lifetime. While increasing the number of nodes increases the traffic generated in the network, the total load in the network may not increase at the same rate if schemes such as in-network processing and aggregation are used. In the best case, the load at each node does not change (perfect compression or in-network processing). In most cases the load at each node will increase linearly with the network size and in applications based on flooding, the increase is faster than linear. To model the effect of loading, the x-axis of the figure considers the load at each node of a network with 100 nodes (denoted by λ_{100}) as the base and the load for a network with N nodes is modeled as $\lambda_N = \lambda_{100}(\frac{N}{100})^\gamma$ with γ being the loading factor. We use $\lambda_{100} = 1$. Thus $\gamma = 0$ corresponds to the case of perfect compression, $\gamma = 1$ corresponds to



(a) $t = 25$ cycles (b) $t = 50$ cycles (c) $t = 75$ cycles

Fig. 4. The residual power distribution at different times in a random network: Analytic versus simulation results.



(a) $t = 25$ cycles (b) $t = 50$ cycles

Fig. 5. Effect of node additions on the residual power distribution at different times in a random network: Analytic versus simulation results.

linear increase and $\gamma > 1$ is faster than linear increase. We observe that the lifetime increases exponentially as the in-network processing or compression increases. Also, the lifetime decreases exponentially with number of nodes in the network, with the decrease being faster for networks without in-network processing.

VIII. Conclusion

In this paper we have motivated the need and importance of analyzing the network lifetime as a function of time and energy consumption. Using the work on population dynamics as the basis, we developed a general model for evaluating the residual battery power levels in networks with and without battery recharging as well as networks with node additions. Expressions were derived for the network lifetime in the absence of battery recharging and the distribution and moments of the state occupancy of the sensors for the other cases. The impact of packet arrival rate at the sensor nodes and a sensor node's geographic location on the energy consumption was modeled.

Appendix

The appendix present the outline for obtaining the solution of Eqn. (10) as per the results of [14]. First consider the solution of the system in Eqn. (10) without the absorbing boundary condition. We first use the transformation $y = (x - x_0 - \mu t)/\sigma$ which when substituted in Eqn. (10) gives

$$\frac{\partial \varrho}{\partial t} = \frac{1}{2}\frac{\partial^2 \varrho}{\partial y^2} \tag{47}$$

and the process y has the initial condition $\varrho(y, 0|x_0) = \delta(y)$. Consider the moment generating function (MGF)

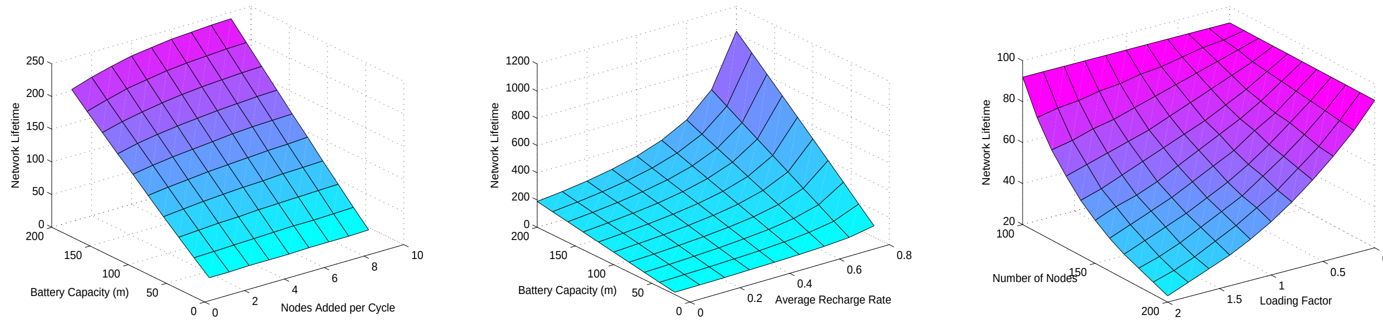
$$M(\theta, t) = \int_{-\infty}^{\infty} \varrho(y, t) e^{-\theta y} dy. \tag{48}$$

Then from Eqn. (47), $M(\theta, t)$ satisfies the equation

$$\frac{1}{2}\theta^2 M = \frac{\partial M}{\partial t} \tag{49}$$

with the initial condition $M(\theta, 0) = 1$. The solution of the partial differential equation above is given by

$$M(\theta, t) = e^{\frac{1}{2}\theta^2 t} \tag{50}$$



(a) m and node additions (b) m and recharge rate (c) Network size and loading factor

Fig. 6. Network lifetime as a function of various parameters in a random network.

which is the MGF of a normal distribution with zero mean and variance t. Thus it follows that $x = x_0 + \mu t + \sigma y$ also has a normal distribution and is given by

$$\varrho(x,t|x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} \tag{51}$$

To obtain a solution that also satisfies the boundary condition of Eqn. (11), we first note that for $x_0 > 0$ the linear transformation of Eqn. (51) given by

$$\varphi(x,t|x_0) = \varrho(x,t|x_0) + B\varrho(x,t|x_a) \tag{52}$$

where x_a is a constant also satisfies Eqn. (10) and its initial condition $\varrho(x,0|x_0) = \delta(x-x_0)$. We thus need to find suitable values of B and x_a such that the boundary condition in Eqn. (11) is also satisfied in order to complete the solution. The value of x_a can be obtained by considering the boundary at 0 as a mirror and placing a source at $x = -x_0$, the image of the initial condition in the mirror. This leads to the solution $\varphi(x,t) = \varrho(x,t|x_0) + B\varrho(x,t|-x_0)$. Then, using $B = -e^{-\frac{2\mu x_0}{\sigma^2}}$ we get the final solution

$$\varphi(x,t|x_0) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \left[e^{-\frac{(x-x_0-\mu t)^2}{2\sigma^2 t}} - e^{-\frac{2\mu x_0}{\sigma^2} - \frac{(x+x_0-\mu t)^2}{2\sigma^2 t}} \right]. \tag{53}$$

REFERENCES

[1] P. Leslie, "On the use of matrices in certain population mathematics," *Biometrika,* vol. 33, no. 3. pp, 183-212, November 1945.

[2] M. Stemm and R. Katz, "Measuring and Reducing Energy Consumption of Network Interfaces in Hand-held Devices," *IEICE Transactions on Communications,* vol. E80-B, no. 8, pp. 1125-1131, August 1997.

[3] D. Cook and S. Das, *Smart Environments: Technology, Protocols and Applications,* John Wiley, 2004.

[4] Y. Zhao, R. Govindan and D. Estrin, "Residual Energy Scans for Monitoring Wireless Sensor Networks," *Proc. IEEE WCNC,* pp. 356-362, Orlando, FL, March 2002.

[5] M. Bhardwaj and A. Chandrakasan, "Bounding the lifetime of sensor networks via optimal role assignments," *Proc. IEEE Infocom,* pp. 1587-1596, New York, NY, June 2002.

[6] C.-F. Chiasserini and M. Garetto, "Modeling the performance of wireless sensor networks," *Proc. IEEE Infocom,* pp. 220-231, Hong Kong, China, March 2004.

[7] Y. Chen and Q. Zhao, "On the lifetime of wireless sensor networks," *IEEE Communication Letters,* vol. 9, no. 11, pp. 976978, November 2005.

[8] H. Zhang and J. Hou, "On deriving the upper bound of lifetime for large sensor networks," *Proc. ACM Mobihoc,* pp. 121-132, Tokyo, Japan, May 2004.

[9] Y. Chen, C. Chuah and Q. Zhao, "Sensor placement for maximizing lifetime per unit cost in wireless sensor networks," *Proc. IEEE MILCOM,* pp. 1097-1102, Atlantic City, NJ, October 2005.

[10] S. Tuljapurkar, "An uncertain life: Demography in random environments," *Theoretical Population Biology,* vol. 35, no. 3, pp. 227-294, June 1989.

[11] R. Lande and S. Orzack, "Extinction dynamics of age-structured populations in a fluctuating environment," *Proceedings of the National Academy of Sciences,* vol. 85, no. 19, pp. 7418-7421, October 1988.

[12] S. Tuljapurkar and S. Orzack, "Population dynamics in variable environments I: Long-run growth rates and extinction," *Theoretical Population Biology,* vol. 18, no. 3, pp. 314-342, December 1980.

[13] E. Ewens, *Mathematical Population Genetics,* Springer, New York, 1979.

[14] D. Cox and H. Miller, *The theory of Stochastic Processes,* pp. 208-222, Chapman and Hall, London 1965.

[15] J. Paradiso and T. Starner, "Energy scavenging for mobile and wireless electronics," *IEEE Pervasive Computing,* vol. 4, no. 1, pp. 18-27, January-March 2005.

[16] M. Pereyma, "Overview of the Modern State of the Vibration Energy Harvesting Devices," *Proc. MEMSTECH,* pp.107-112, May 2007.

[17] H.-O. Georgii, *Stochastics: Introduction to Probability and Statistics,* pp. 283-289, de Gruyter, Berlin, 2008.

[18] V. Frost and B. Melamed, "Traffic modeling for telecommunications networks," *IEEE Communications Magazine,* vol. 32, no. 3, pp. 70-81, March 1994.

[19] S. Shah-Heydari and T. Le-Ngoc, "MMPP models for multimedia traffic," *Telecommunication Systems,* vol. 15, no. 3-4, pp. 273-293, December 2000.

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Krishna Ramachandran received his undergraduate degree in Mathematics and Computing from the Indian Institute of Technology, Kharagpur, India, in 2002 and a doctorate degree in Computer and Systems Engineering from Rensselaer Polytechnic Institute, Troy, NY, USA in 2007. The focus of his dissertation was on mathematically quantifying P2P network performance by formulating relevant analytical frameworks. His broad inerests include application of stochastic calculus and optimization to analyze problems arising in the fields of networking, finance and the automotive industry.

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Biplab Sikdar (S'98, M'02, SM'09) received the B. Tech degree in electronics and communication engineering from North Eastern Hill University, Shillong, India, the M. Tech degree in electrical engineering from Indian Institute of Technology, Kanpur and Ph.D in electrical engineering from Rensselaer Polytechnic Institute, Troy, NY, USA in 1996, 1998 and 2001, respectively. He is currently an Associate Professor in the Department of Electrical, Computer and Systems Engineering of Rensselaer Polytechnic Institute, Troy, NY, USA. His research interests include wireless MAC protocols, network routing and His research interests include wireless MAC protocols, network routing and multicast protocols, network security and queueing theory. Dr. Sikdar is a member of IEEE, Eta Kappa Nu and Tau Beta Pi and is an Associate Editor of the IEEE Transactions on Communications.