On the Stability of the Malware Free Equilibrium in Cell Phones Networks with Spatial Dynamics

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***Abstract*— Recent outbreaks of virus and worm attacks targeted at cell phones have bought to the forefront the seriousness of the security threat to this increasingly popular means of communication. In this paper, we develop an analytic framework for modeling the dynamics of malware propagation in networks of mobile smart phones. We then characterize the conditions under which the network may reach a malware free equilibrium and derive the necessary conditions for its global asymptotic stability. The model accounts for malware transfers through a number of communication paradigms including the Internet, the telephone network, Bluetooth and WiFi in addition to accounting for the mobility of cell phones and the impact of the heterogeneous environments they pass through on the malware dynamics.**

I. Introduction

While malware such as worms and viruses have been prevalent in the Internet for more than a decade, such attacks have recently been reported in cell phones. Proof-of-concept worms for smart phones like `cabir` [4] as well as malicious code such as the `skulls` [6] and `mosquito` [5] trojans have recently been reported. In August 2005, mobile phones at the world athletics championship held at Helsinki's Olympic stadium were compromised by a virus attack [12]. With the growing popularity and prevalence of advanced cell phones with a myriad of communicational capabilities, such threats are extremely important and capable of causing extensive damage.

The communication capabilities of the new generation of cell phones can be broadly grouped into three categories **(1)** access to the telecom network through technologies like (GSM) and code division multiple access (CDMA) **(2)** access to the Internet which may occur either via accessing the telecom network or by using Bluetooth or wireless local area networking (WLAN) interfaces and **(3)** communication with other smart phones in its physical vicinity through Bluetooth, infra red interfaces etc. Consequently, the possible ways in which malware may spread in these devices are **(1)** malware downloads from the Internet and peer to peer networks (for example the `skulls` and `mosquito` trojans [5], [6]), **(2)** phone to phone spread which results when a compromised phone sends the malware to other phones either by random dialing or dialing the numbers in the address book (for example using mechanisms similar to the `timfonica` virus in Spain, 2000 and the `commwarrior` and `mabir` worms) and **(3)** phone to phone or computer to phone spread through Bluetooth or WLAN interface (for example the `cabir` worm [4]). Node mobility and the resulting variations in the number of other devices in the vicinity of a phone also affects the propagation of the malware, specially those that spread through the Bluetooth or WLAN interfaces.

New generation smart phones are largely vulnerable to, and can act as the catalysts for the spread of mobile viruses and thus are important from a practical perspective. There exists a slew of modeling work characterizing numerous aspects of worm spread, [10], [9], [11] to name a few, but seldom has the setting been a wireless environment. Also, unlike our model, existing work only considers static nodes. In this paper we consider the combined effect of all the three spreading mechanisms mentioned above and develop a model to characterize the spread of malware in networks of smart phones. The model also accounts for the mobility in cell phones and accommodates the effect of heterogeneity in the locality on the spreading dynamics. Compared to existing epidemic models for malware propagation in computer networks which assume static hosts, this is one of the key features of our model. The model is then used to derive the necessary conditions for the existence of a malware free equilibrium and conditions for its global asymptotic stability.

The rest of the paper is organized as follows: Section II presents the analytical framework and the model is further analyzed in Section III. We present numerical results and sensitivity analysis in Section IV and finally, Section V presents the concluding remarks.

II. Modeling Framework

The model developed in this section is based on a compartmental epidemic model with four classes. Unlike existing literature on epidemic models for computer networks, our model captures the effect of node mobility and its effect on the rate of malware spread when nodes move through heterogeneous environments with different conditions. At any given point in time, a cell phone is in one of the following four classes: susceptible, exposed, infected and recovered. Initially all phones belong to the susceptible phase and stay there until they come in contact with the malware. Once the malware propagates into a susceptible cell phone, the phone moves to the exposed state which corresponds to the latent period of an infection. In our case this corresponds to the case when a malware is sent to a phone which is currently turned off. The phones then stay in the infected state until they are either patched or quarantined upon which they move to the recovered state and stay there.

In our model, phones may enter or leave the network over time. However, the birth and death rates are the same and the total population at any given instant is assumed to be a constant. This assumption is based on the fact that the time for a fast worm to spread can be considered to be quite small compared to the rate at which the cell phone population in a country or city changes. Also, we assume that the time taken to download the malware is quite small and may be considered instantaneous. This assumption may be justified by noting the small size of most worms as well as the increasingly high data rates achieved by the new generation of smart phones.

A. Model for Spatial Dynamics

Due to the inherent mobility associated with cell phone users, over the course of a day, the phone may move through many locations such as residential or office areas, each with different phone densities and number of infected phones. We classify each possible location that a cell phone may visit as one of $\mathcal{P}$ patches or regions [3]. Each patch is characterized by its own infection rate and visitation probabilities. Thus an airport and a small stadium, with roughly the same number of phones and where phone users stay for roughly equal times are treated as belonging to the same patch. Similarly, two residential areas in opposite sides of a city or in two different cities may be classified into the same patch if they have approximately the same population density. This reduces the number of equations in our model to $4\mathcal{P}$. We denote the rate of travel from patch q to patch p by m_{pq}. The rate of change in the susceptible (S_p), exposed (E_p), infectious (I_p) and recovered (R_p) populations in patch p, $1 \leq p \leq \mathcal{P}$ due to *only* the movements between the patches is given by

$$\frac{dX_p}{dt} = \sum_{q=1}^{\mathcal{P}} m_{pq} X_q - \sum_{q=1}^{\mathcal{P}} m_{qp} X_p \tag{1}$$

where $X_p \in \{S_p, E_p, I_p, R_p\}$ and we actually have four sets of equations. With this basic model for the movement of cell phones between geographical regions, we now incorporate the effect of the other factors on the malware propagation dynamics.

B. Incorporating Infection Mechanisms

We first consider the spread of the malware due to downloads from the Internet. Given that a cell phone is on and in patch p (which happens with probability p_{on}^p and is derived in Equation (8)), we denote the probability that an arbitrary cell phone in patch p downloads the malware from the Internet at time t by $\gamma_p(t)$. Also, $\gamma_p(t)$ is a decreasing function of time since users are less likely to download a malware with time because of factors like awareness and publicity etc.

Now, only the susceptible cell phone population may download the malware from the Internet and the number of such downloads per second is proportional to the susceptible population in the patch. Also, since the downloads take a very small amount of time, the susceptible cell phones move directly to the infected phase. The rate of change in the populations of the four classes due to downloads from the Internet is then

$$\frac{dS_p}{dt} = -p_{on}^p \gamma_p(t) S_p \qquad \frac{dE_p}{dt} = 0 \tag{2}$$

$$\frac{dI_p}{dt} = p_{on}^p \gamma_p(t) S_p \qquad \frac{dR_p}{dt} = 0 \tag{3}$$

Now consider the spread of the malware through Bluetooth, infra red or WLAN interfaces when susceptible phones come in the physical vicinity of infected phones. We denote by β_p the rate at which a cell phone in patch p tries to infect other phones through these interfaces. Again, since only phones currently turned on may be infected with this mechanism and the malware transfer between two devices is considered instantaneous, the susceptible population directly moves to the infectious state. The contributions to the rate of change of populations of the four classes in this case are given by

$$\frac{dS_p}{dt} = -p_{on}^p \beta_p S_p \frac{I_p}{N_p} \qquad \frac{dE_p}{dt} = 0 \tag{4}$$

$$\frac{dI_p}{dt} = p_{on}^p \beta_p S_p \frac{I_p}{N_p} \qquad \frac{dR_p}{dt} = 0 \tag{5}$$

Finally, we consider the case where the malware may spread when a compromised phone randomly or selectively dials other numbers and transfers the malware through MMS or SMS. The dialed number may be in any of the $\mathcal{P}$ patches and thus a phone in one patch may infect a phone in another patch. The rate of such infections is proportional to the strength of the infectious population in the patch and given by I_p/N_p for patch p. We denote the rate at which a compromised phone tries to dial other numbers by α. Also, some of the randomly dialed numbers or out-dated numbers in the address book may be non-existent and thus all infection attempts will not be successful. We denote by ρ the probability that a dialed number is non-existent. Finally, some of the dialed cell phones may be switched off and in these cases, we assume that the malware gets queued up in the base station and is delivered once the phone is switched on. For this spreading mechanism, we thus have

$$\frac{dS_p}{dt} = -\sum_{i=1}^{\mathcal{P}} \alpha(1-\rho) S_p \frac{I_i}{N_i} \qquad \frac{dI_p}{dt} = 0 \tag{6}$$

$$\frac{dE_p}{dt} = \sum_{i=1}^{\mathcal{P}} \alpha(1-\rho) S_p \frac{I_i}{N_i} \qquad \frac{dR_p}{dt} = 0 \tag{7}$$

Note that in the equations above, all phones infected through random or selected dialing pass through the exposed state, even though the phones that are turned on get infected immediately. This does not result in any inaccuracies because in Section II-C, we evaluate and incorporate the estimated time that a phone spends in the exposed state in patch p, $1/\epsilon_p$, based on whether it was turned on or not when it was infected.

C. Combined Model

We now combine the various contributions along with the arrival and departures of cell phones to complete the model. First, we note that while new phones may join only in the susceptible phase, cell phone users may decide to quit the network permanently while they are in any of the four states. With the average phone lifetime in patch p denoted by $1/d_p$, the rate of population change due to the joining of new phones and departure of old ones is $-d_pE_p$, $-d_pI_p$ and $-d_pR_p$ for the exposed, infected and recovered classes and $d_pN_p - d_pS_p$ for the susceptible state. Note that the birth term of d_pN_p is devised to keep the total cell phone population constant.

The average time spent by an arbitrary cell phone in the exposed phase in patch p is denoted by $1/\epsilon_p$. With $1/\lambda^p_{on}$ and $1/\lambda^p_{off}$ denoting the average on and off times of a cell phone in patch p, we have

$$p^p_{on} = \frac{1/\lambda^p_{on}}{1/\lambda^p_{on} + 1/\lambda^p_{off}} = \frac{\lambda^p_{off}}{\lambda^p_{on} + \lambda^p_{off}} \quad (8)$$

$$(9)$$

and $p^p_{off} = 1 - p^p_{on}$. The expected duration of the exposed state in patch p, $1/\epsilon_p = E[\text{latent period}|\text{on}]p^p_{on} + E[\text{latent period}|\text{off}]p^p_{off}$, is then given by

$$\frac{1}{\epsilon_p} = \frac{\lambda^p_{on}}{\lambda^p_{off}(\lambda^p_{on} + \lambda^p_{off})} \quad (10)$$

Exposed cell phones in patch p leave the exposed state at a rate of ϵ_pE_p and thus enter the infected state at the same rate. Finally, with $1/\delta_p$ denoting the average time spent by a cell phone in patch p in the infected state, infected phones leave the infected state at a rate of δ_pI_p and enter the recovered phase at the same rate.

Combining the models of the previous two subsections with the contributions to the population change rates described above, we obtain the following equations which complete our model for malware propagation in cell phones:

$$\frac{dS_p}{dt} = d_p(N_p - S_p) - p^p_{on}\gamma_p(t)S_p - p^p_{on}\beta_pS_p\frac{I_p}{N_p} - \sum_{i=1}^{\mathcal{P}}\alpha(1-\rho)S_p\frac{I_i}{N_i} + \sum_{q=1}^{\mathcal{P}}m_{pq}S_q - \sum_{q=1}^{\mathcal{P}}m_{qp}S_p \quad (11)$$

$$\frac{dE_p}{dt} = \sum_{i=1}^{\mathcal{P}}\alpha(1-\rho)S_p\frac{I_i}{N_i} - (d_p+\epsilon_p)E_p + \sum_{q=1}^{\mathcal{P}}m_{pq}E_q - \sum_{q=1}^{\mathcal{P}}m_{qp}E_p \quad (12)$$

$$\frac{dI_p}{dt} = p^p_{on}\gamma_p(t)S_p + p^p_{on}\beta_pS_p\frac{I_p}{N_p} - (d_p+\delta_p)I_p + \epsilon_pE_p + \sum_{q=1}^{\mathcal{P}}m_{pq}I_q - \sum_{q=1}^{\mathcal{P}}m_{qp}I_p \quad (13)$$

$$\frac{dR_p}{dt} = \delta_pI_p - d_pR_p + \sum_{q=1}^{\mathcal{P}}m_{pq}R_q - \sum_{q=1}^{\mathcal{P}}m_{qp}R_p \quad (14)$$

where we have $N_p = S_p + E_p + I_p + R_p$, $\sum_{p=1}^{\mathcal{P}} N_p = C$, $N_p > 0$ and $S_p, E_p, I_p, R_p \geq 0$ at $t = 0$.

III. Model Analysis

In this section we analyze the model presented in the previous section to obtain the necessary conditions for the global asymptotic stability of the malware free equilibrium.

A. Mobility Model

We start with with mobility model and in the following, derive some properties governing the equilibrium. By adding Equation (11), (12), (13) and (14) and noting that $N_p = S_p + E_p + I_p + R_p$ we obtain the following equation for the population of cell phones in patch p

$$\frac{dN_p}{dt} = \sum_{q=1}^{\mathcal{P}}m_{pq}N_q - \sum_{q=1}^{\mathcal{P}}m_{qp}N_p \quad (15)$$

With the array $N = [N_1, \cdots, N_p]^t$ comprising of the cell phone population in each patch, we have

$$\frac{dN}{dt} = MN \quad (16)$$

where the mobility matrix M is given by

$$M = \begin{bmatrix} -\sum_{q=1}^{\mathcal{P}} m_{q1} & m_{12} & \cdots & m_{1\mathcal{P}} \\ m_{21} & -\sum_{q=1}^{\mathcal{P}} m_{q2} & \cdots & m_{2\mathcal{P}} \\ \vdots & \vdots & \ddots & \vdots \\ m_{\mathcal{P}\infty} & m_{\mathcal{P}} & \cdots & -\sum_{q=1}^{\mathcal{P}} m_{q\mathcal{P}} \end{bmatrix} \quad (17)$$

We then have the following claim about the propagation of the endemic equilibrium from one patch to another:

Claim 1 *If the system described by Equations (11)-(14) is at an equilibrium and the malware is in endemic equilibrium in patch p, then the malware is also at an endemic equilibrium at all patches that have access to patch p. In particular, if M is irreducible, the system is at an endemic equilibrium in all patches.*

Proof: Without loss of generality, assume $p = 1$ and since this patch is in endemic equilibrium, $E_1 + I_1 > 0$. For any other patch q, $q \neq 1$, adding Equations (12) and (13) we have

$$\frac{d}{dt}(E_q + I_q) = \sum_{i=1}^{\mathcal{P}}\alpha(1-\rho)S_q\frac{I_i}{N_i} - d_q(E_p + I_q) - \delta_qI_q + p^q_{on}\gamma_qS_q + p^q_{on}\beta_qS_q\frac{I_q}{N_q} + \sum_{r=1}^{\mathcal{P}}m_{qr}(E_r + I_r) - \sum_{r=1}^{\mathcal{P}}m_{rq}(E_q + R_q)$$

We prove the claim based on contradiction. Assume that $E_q = I_q = 0$ and $m_{q1} > 0$ i.e. patch q is malware free even though it has direct access to patch p. Then, the equation above reduces to

$$0 = \sum_{i=1}^{\mathcal{P}}\alpha(1-\rho)S_q\frac{I_i}{N_i} + p^q_{on}\gamma_qS_q + \sum_{r=1}^{\mathcal{P}}m_{qr}(E_r + I_r) \quad (18)$$

Since all quantities in the equation above are non-negative, we have

$$0 = \sum_{r=1}^{\mathcal{P}} m_{qr}(E_r + I_r) \tag{19}$$

which in turn implies $E_1+I_1 = 0$ meaning patch 1 is malware-free leading to a contradiction. Thus the malware in patch q is at an endemic equilibrium.

Above, we proved that any node with direct access to patch p will also be in an endemic equilibrium. Using the same set of arguments above, we can show that all patches that are directly connected to patch q will also be in endemic equilibrium. Thus patches not directly connected to p but at a distance of 2 are also in endemic equilibrium. By induction, all patches belonging to the same strongly connected component of the digraph as patch p will be at an endemic equilibrium. A sufficient condition for all patches to be in endemic equilibrium is then for M to be irreducible since then the entire digraph is strongly connected. ■

B. Malware Free Equilibrium

We now derive the conditions for the global stability of the malware free equilibrium. In [2] it has been proved that if M is irreducible then the mobility equation described in Equation (16) has a positive equilibrium which is asymptotically stable. The solution for this equilibrium given by

$$N_p = \hat{N}_p = \frac{C}{1 + \mathrm{I}^t_{\mathcal{P}-1}(M(p))^{-1} m_p} > 0 \tag{20}$$

where $M(p)$ denotes the matrix M with its p^{th} row and column deleted, $\mathrm{I}^t_{\mathcal{P}-1} = [1, \cdots, 1]$ a vector with $\mathcal{P}-1$ columns and m_p is a vector formed from the p^{th} column of M by deleting its p^{th} entry. The malware free equilibrium then exists in patch p with $\hat{S}_p = \hat{N}_p$ and $\hat{E}_p = \hat{I}_p = \hat{R}_p = 0$.

The local stability of the system depends on the basic reproduction number $\mathcal{R}_0$, which in turn depends on the system model. In [1] "next generation matrices" have been proposed to derive the basic reproduction numbers and we follow this approach for our model. In this method, the flow of individuals (cell phones in our case) between the states are written in the form of two vectors $\mathcal{F}$ and $\mathcal{V}$ which describe the inflow of new infected individuals and all other flows in the system, respectively. These vectors are then differentiated with respect to the state variables, evaluated at the malware free equilibrium, and only the part corresponding to the infected classes are then kept to form the matrices F and V, i.e.,

$$F = \left[\frac{\partial \mathcal{F}_i}{\partial x_j}(x_0)\right] \quad \text{and} \quad V = \left[\frac{\partial \mathcal{V}_j}{\partial x_j}(x_0)\right] \quad \text{with } 1 \le i,j \le m \tag{21}$$

where $\mathcal{F}_i$ and $\mathcal{V}_i$ are the i^{th} entries of $\mathcal{F}$ and $\mathcal{V}$, x_i is the i^{th} system state variable with $\dot{x}_i = \mathcal{F}_i(x) - \mathcal{V}_i(x)$, x_0 is the malware free equilibrium and m is the number of infectious states.

In our model, we have $m = 2\mathcal{P}$ corresponding to the E and I states in each patch. Ordering the infectious states according to the patches, i.e., $E_1, E_2, \cdots, E_{\mathcal{P}}, I_1, I_2, \cdots, I_{\mathcal{P}}$, from Equations (11)-(14) we have

$$\mathcal{F}_p = \begin{bmatrix} 0 \\ \sum_{i=1}^{\mathcal{P}} \alpha(1-\rho) S_p \frac{I_i}{N_i} \\ p^p_{on}\gamma_p(t) S_p + p^p_{on}\beta_p S_p \frac{I_p}{N_p} \\ 0 \end{bmatrix} \tag{22}$$

and

$$\mathcal{V}_p = \begin{bmatrix} \Big(-d_p(N_p - S_p) + p^p_{on}\gamma_p(t) S_p + p^p_{on}\beta_p S_p \frac{I_p}{N_p} + \\ \sum_{i=1}^{\mathcal{P}} \alpha(1-\rho) S_p \frac{I_i}{N_i} - \sum_{q=1}^{\mathcal{P}} m_{pq} S_q + \sum_{q=1}^{\mathcal{P}} m_{qp} S_p \Big) \\ (d_p + \epsilon_p) E_p - \sum_{q=1}^{\mathcal{P}} m_{pq} E_q + \sum_{q=1}^{\mathcal{P}} m_{qp} E_p \\ (d_p + \delta_p) I_p - \epsilon_p E_p - \sum_{q=1}^{\mathcal{P}} m_{pq} I_q + \sum_{q=1}^{\mathcal{P}} m_{qp} I_p \\ \delta_p I_p - d_p R_p + \sum_{q=1}^{\mathcal{P}} m_{pq} R_q - \sum_{q=1}^{\mathcal{P}} m_{qp} R_p \end{bmatrix} \tag{23}$$

Differentiating $\mathcal{F}$ and $\mathcal{V}$ with respect to $E_1, E_2, \cdots, E_{\mathcal{P}}, I_1, I_2, \cdots, I_{\mathcal{P}}$ and evaluating at the malware free equilibrium $\{\hat{S}_p, 0, 0, 0\}$ we have

$$F = \begin{bmatrix} \mathbf{0} & G \\ \mathbf{0} & H \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} A & \mathbf{0} \\ -C & B \end{bmatrix} \tag{24}$$

with $\mathbf{0}$ representing a $\mathcal{P} \times \mathcal{P}$ zero matrix and

$$G = \alpha(1-\rho) \begin{bmatrix} \frac{S_1}{N_1} & \frac{S_1}{N_2} & \cdots & \frac{S_1}{N_{\mathcal{P}}} \\ \frac{S_2}{N_1} & \frac{S_2}{N_2} & \cdots & \frac{S_2}{N_{\mathcal{P}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{S_{\mathcal{P}}}{N_1} & \frac{S_{\mathcal{P}}}{N_2} & \cdots & \frac{S_{\mathcal{P}}}{N_{\mathcal{P}}} \end{bmatrix} \tag{25}$$

$$H = p_{on} \begin{bmatrix} \beta_1 \frac{S_1}{N_1} & 0 & \cdots & 0 \\ 0 & \beta_2 \frac{S_2}{N_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \beta_{\mathcal{P}} \frac{S_{\mathcal{P}}}{N_{\mathcal{P}}} \end{bmatrix} \tag{26}$$

$$\tilde{M} = \begin{bmatrix} -\sum_{q\neq 1} m_{q1} & m_{12} & \cdots & m_{1p} \\ m_{21} & -\sum_{q\neq 2} m_{q2} & \cdots & m_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ m_{\mathcal{P}1} & m_{\mathcal{P}2} & \cdots & \sum_{q\neq \mathcal{P}} m_{q\mathcal{P}} \end{bmatrix} \tag{27}$$

$$A = \begin{bmatrix} d_1 + \epsilon_1 & 0 & \cdots & 0 \\ 0 & d_2 + \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{\mathcal{P}} + \epsilon_{\mathcal{P}} \end{bmatrix} - \tilde{M} \tag{28}$$

$$B = \begin{bmatrix} d_1 + \delta_1 & 0 & \cdots & 0 \\ 0 & d_2 + \delta_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{\mathcal{P}} + \delta_{\mathcal{P}} \end{bmatrix} - \tilde{M} \tag{29}$$

$$C = \begin{bmatrix} \epsilon_1 & 0 & \cdots & 0 \\ 0 & \epsilon_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \epsilon_{\mathcal{P}} \end{bmatrix} \tag{30}$$

We then have the following result for the global asymptotic stability of the malware free equilibrium:

Claim 2 *For the network of smart cell phones described in Equations (11)-(14), with $\mathcal{R}_0 = \rho(FV^{-1})$, the malware free equilibrium is globally asymptotically stable if $\mathcal{R}_0 < 1$ and equilibrium is unstable if $\mathcal{R}_0 > 1$. Here $\rho()$ denotes the spectral radius.*

Proof: The results of Theorem 2 of [1] are directly applicable here for showing the local stability (i.e. stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$). To prove the global stability, consider the non-autonomous system consisting of Equations (12), (13) and (14) with the substitution $S_p = N_p - E_p - I_p - R_p$. We then have

$$\frac{dE_p}{dt} = \sum_{i=1}^{\mathcal{P}} \alpha(1-\rho)(N_p - E_p - I_p - R_p)\frac{I_i}{N_i} -(d_p+\epsilon_p)E_p + \sum_{q=1}^{\mathcal{P}} m_{pq}E_q - \sum_{q=1}^{\mathcal{P}} m_{qp}E_p \quad (31)$$

$$\frac{dI_p}{dt} = \left[p_{on}^p \gamma_p(t) + p_{on}^p \beta_p \frac{I_p}{N_p}\right](N_p - E_p - I_p - R_p) -(d_p+\delta_p)I_p + \epsilon_p E_p + \sum_{q=1}^{\mathcal{P}} m_{pq}I_q - \sum_{q=1}^{\mathcal{P}} m_{qp}I_p \quad (32)$$

$$\frac{dR_p}{dt} = \delta_p I_p - d_p R_p + \sum_{q=1}^{\mathcal{P}} m_{pq}R_q - \sum_{q=1}^{\mathcal{P}} m_{qp}R_p \quad (33)$$

The system of equations above can be written as

$$x' = f(t, x) \quad (34)$$

where x is a $3\mathcal{P}$ vector consisting of E_p, I_p and R_p. Now, the malware free equilibrium of the cell phone network described in Equations (11)-(14) is given by $\{\hat{S}_p, 0, 0, 0\}$ which corresponds to the equilibrium $x = \{0, 0, 0\}$ in Equation (34). From Equation (20) we have $N_p(t) \to \hat{N}_p$ as $t \to \infty$ and substituting this value in Equations (31)-(33) we have

$$\frac{dE_p}{dt} = \sum_{i=1}^{\mathcal{P}} \alpha(1-\rho)(\hat{N}_p - E_p - I_p - R_p)\frac{I_i}{\hat{N}_i} -(d_p+\epsilon_p)E_p + \sum_{q=1}^{\mathcal{P}} m_{pq}E_q - \sum_{q=1}^{\mathcal{P}} m_{qp}E_p \quad (35)$$

$$\frac{dI_p}{dt} = \left[p_{on}^p \gamma_p(t) + p_{on}^p \beta_p \frac{I_p}{\hat{N}_p}\right](\hat{N}_p - E_p - I_p - R_p) -(d_p+\delta_p)I_p + \epsilon_p E_p + \sum_{q=1}^{\mathcal{P}} m_{pq}I_q - \sum_{q=1}^{\mathcal{P}} m_{qp}I_p \quad (36)$$

$$\frac{dR_p}{dt} = \delta_p I_p - d_p R_p + \sum_{q=1}^{\mathcal{P}} m_{pq}R_q - \sum_{q=1}^{\mathcal{P}} m_{qp}R_p \quad (37)$$

which is a set of three equations and three unknowns. Thus the systems of Equations (11)-(14) is asymptotically autonomous with limit equation

$$x' = g(x) \quad (38)$$

We now show that $x = 0$ is a globally asymptotically stable solution for the limit system above. Consider the linear system

$$x' = Lx \quad (39)$$

where x is a $3\mathcal{P}$ dimensional vector consisting of E_p, I_p and R_p. Substituting $\hat{N}_p/\hat{N}_i$ for S_p/N_i in L we obtain

$$\frac{dE_p}{dt} = \sum_{i=1}^{\mathcal{P}} \alpha(1-\rho)I_i\frac{\hat{N}_p}{\hat{N}_i} - (d_p+\epsilon_p)E_p + \sum_{q=1}^{\mathcal{P}} m_{pq}E_q - \sum_{q=1}^{\mathcal{P}} m_{qp}E_p \quad (40)$$

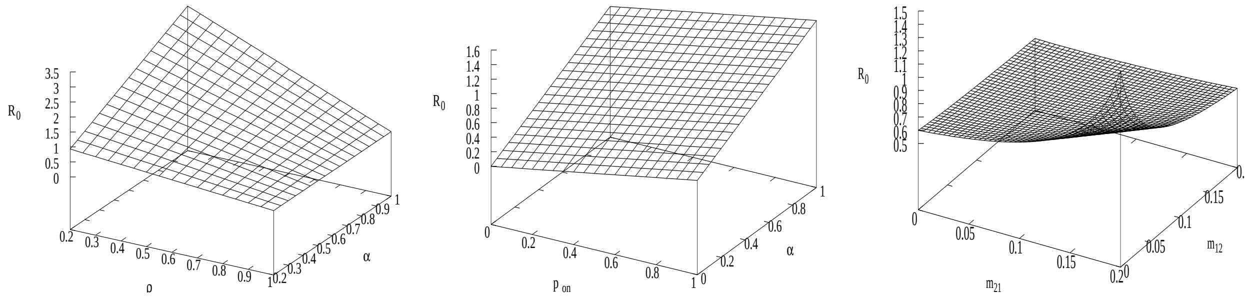
$$\frac{dI_p}{dt} = \left[p_{on}^p \gamma_p(t) + p_{on}^p \beta_p \frac{I_p}{\hat{N}_p}\right]\hat{N}_p - (d_p+\delta_p)I_p + \epsilon_p E_p + \sum_{q=1}^{\mathcal{P}} m_{pq}I_q - \sum_{q=1}^{\mathcal{P}} m_{qp}I_p \quad (41)$$

$$\frac{dR_p}{dt} = \delta_p I_p - d_p R_p + \sum_{q=1}^{\mathcal{P}} m_{pq}R_q - \sum_{q=1}^{\mathcal{P}} m_{qp}R_p \quad (42)$$

While Equations (37) and (42) are the same, comparing Equations (35) and (40) and Equations (36) and (41) we note that $g(x) \le Lx$ for all $x \in \mathbb{R}_+^{3\mathcal{P}}$. We also note that R_p does not appear in the Equations for E_p and I_p. Let $\tilde{x}$ be the part of the vector corresponding to the variables E_p and I_p and let $\tilde{L}$ be the corresponding sub-matrix of L. Now, the method of [1] for proving the local stability of the equilibrium can also be used to show the stability of the equilibrium point $\tilde{x} = 0$ for the subsystem $\tilde{x}' = \tilde{L}\tilde{x}$, with $\tilde{L} = F - V$. This implies that if $\mathcal{R}_0 < 1$, then the equilibrium point $\tilde{x} = 0$ of the subsystem $\tilde{x}' = \tilde{L}\tilde{x}$ is stable. When $\tilde{x} = 0$, Equation (14) becomes

$$\frac{dR}{dt} = (M - D)R \quad (43)$$

where M is the mobility matrix given in Equation (17), $R = (R_1, \cdots, R_{\mathcal{P}})^t$ and D is a diagonal matrix with the p^{th} diagonal entry equal to d_p. Now, it can be easily shown that $(-M)$ is a singular M-matrix. Using the result A_3 on page 179 of [13] it then follows that $-M + D$ is a non-singular M-matrix. Thus the equilibrium point $R = 0$ of this linear system in R is stable. This implies that the equilibrium point $x = 0$ of the system $x' = Lx$ described in Equation (39) is stable when $\mathcal{R}_0 < 1$. From a standard comparison theorem, such as Theorem 1.5.4 of [14], it then follows that $x = 0$ is a globally asymptotically stable equilibrium of the system $x' = g(x)$ in Equation (38). Now for $\mathcal{R}_0 < 1$, the linear system in Equation (40) and (41) has an unique malware free equilibrium since its coefficient matrix $F - V$ is non-singular. To complete the proof of global stability, one then only has to look at the result of Theorem 4.1 of [15] on asymptotically autonomous equations. ■



(a) $\mathcal{R}_0 = f(\alpha, \rho)$ (b) $\mathcal{R}_0 = f(\alpha, p_{on})$ (c) $\mathcal{R}_0 = f(m_{12}, m_{21})$

Fig. 1. Impact of various parameters on R_0

IV. Sensitivity Analysis

In this section we evaluate the model presented in the previous two sections in order to explore the impact of various parameters on the dynamics of malware propagation. To easily isolate the effects of various parameters, we consider a simple scenario where the mobility of the cell phones is limited to two patches.

In Fig. 1 we show the impact of the various parameters on the basic reproduction number, $\mathcal{R}_0$. The basic parameters used for our results case are: $d_1 = d_2 = 0.25$, $\beta_1 = 0.51$, $\beta_2 = 0.48$, $\epsilon_1 = \epsilon_2 = 0.25$, $\delta_1 = \delta_2 = 0.25$, $N = 20000$, $p^1_{on} = p^2_{on} = 0.8$, $\mu = 0.8$, $\lambda = 50$, $\rho = 0.8$, $\alpha = 0.4$, $\delta = 0.01$, $r_1 = r_2 = 0.5$, $m_{12} = 0.1$ and $m_{21} = 1$. In each figure, the parameters listed above are used except for two parameters, indicated in the caption of each figure, which are varied to observe their effect on $\mathcal{R}_0$. We observe that α is more dominant as compared to p_{on} and ρ in terms of its effects on $\mathcal{R}_0$. This is evident from Figures 1(a) and 1(b), where the graph shows a faster increase in $\mathcal{R}_0$ for high α values even when the other corresponding parameter is numerically insignificant. This is intuitive too since a higher dialing rate increases the likelihood of contacting a susceptible cell phone. We also note from Fig. 1(c) that for our 2 patch wireless model parameters, the rate of travel from patch 1 to patch 2, m_{21}, has a bigger impact as compared to the rate of travel from patch 2 to patch 1, m_{12}. This is because in the parameters chosen here, the rate of infections from Bluetooth or WLAN interfaces is smaller in patch 1 than in patch 2 ($\beta_1 < \beta_2$).

V. Conclusion

In this paper we presented a model for the dynamics of malware propagation in networks of smart cell phones. Analysis for the impact of various spreading mechanisms and malware transfer though a number of communication interfaces as well as the impact of node mobility was presented. We then derived the necessary conditions for a malware free wireless network state as well as the conditions for the global asymptotic stability of the malware free equilibrium of the network.

References

[1] P. van den Driessche and J. Watmough, “Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission,” *Mathematical Biosciences,* vol. 180, pp. 29-48, 2002.

[2] J. Arnio, J. Davis, D. Hartley, R. Jordan, J. Miller and P. van den Driessche, “A multi-species epidemic model with spatial dynamics,” *Mathematical Medicine and Biology,* March 2005.

[3] L. Sattenspiel and K. Dietz, “A structured epidemic model incorporating geographic mobility among regions,” *Mathematical Biosciences,* pp. 71-91, vol. 128, 1995.

[4] Kaspersky Labs, “Viruses move to mobile phones,” 2004. `http://www.kaspersky.com/news?id=149499226`

[5] CNET News, “Mosquito software bites smart phones,” `http://news.com.com/Mosquito+software+bites+smart+phones/2100-1039_3-5308164.html?tag=nl`, August 2004.

[6] CNET news, “Skulls program kills cell phone apps,” `http://news.com.com/Skulls+program+kills+cell+phone+apps/2100-7349_3-5460194.html?tag=nl`, November 2004.

[7] A. J. Ganesh, L. Massoulie and D. Towsley, “The Effect of Network Topology on the Spread of Epidemics,” *Proceedings of IEEE INFOCOM,* Miami, USA, March 2005

[8] Y. Wang, D. Chakrabarti, C. Wang and C. Faloutsos, “Epidemic spreading in real networks: An eigenvalue viewpoint,” *SRDS 2003,* (pages 25-34), Florence, Italy

[9] Z. Chen, L. Gao and K. Kwait, “Modeling the Spread of Active Worms,” *Proceedings of IEEE INFOCOM,* April 2003.

[10] M. Garetto, W. Gong and D. Towsley, “Modeling Malware Spreading Dynamics,” *Proceedings of IEEE INFOCOM,* April 2003.

[11] B. Stephenson and B. Sikdar, “A Quasi-species Approach for Modeling the Dynamics of Polymorphic Worms,” *Proceedings of IEEE INFOCOM,* Barcelona, Spain, April 2006.

[12] COMPUTERWORLD, “Mobile phone virus infects Helsinki championships,” `http://www.computerworld.com/securitytopics/security/virus/story/0,10801,103835,00.html`, August 2005.

[13] A. Berman and R. Plemmons, *Nonnegative Matrices in the Mathematical Sciences,* Academic Press, New York, 1979.

[14] V. Lakshmikantham, S. Leela and A. Martynyuk, *Stability Analysis of Nonlinear Systems,* Marcel Dekker, New york, 1989.

[15] H. Thieme, “Convergence results and a Poincarié-Bendixson trichotomy for asymptotically autonomous differential equations,” *Journal of Mathematical Biology,* pp. 755-763, vol. 30, 1992.